Chapter 1

Notions of logic

1.1 Logic

1.1.1 Assertion

Definition 1.1. *An assertion or proposition is a sentence that is either true or false, not both at the same time. We generally designate an assertion with a capital letter* $P, Q, R, \ldots$.

If an assertion P is true or false, we write

$$\begin{cases} T(\text{ or } 1) \text{ if it is True} \\ F(\text{ or } 0) \text{ if it is false} \end{cases}$$

The truth table summarizes the two possibility of P :

P
T
F

Example 1.1. *1)-* $1 + 2 = 3$.

2)- 3 is negative.

3)- What time is it ?

4)- $x + y = z$.

1) Is a true assertion and 2) is false assertion , 3) and 4) Are not assertions.

1.1.2 The logical connectors

From assertions $P, Q, R, \ldots,$ we can form assertions using the following logical connectors : **negation (no), conjunction(and), disjunction(or), implication, equivalence**. These connectors are defined by their truth table.

(1)- Negation $\ll$ *no* $\gg$

The negation of an assertion P is an assertion denoted $no(P)$ or $\overline{P}$, which is true if P is false, and false if P is true.

P	$\overline{P}$
T	F
F	T

Example 1.2. *1. We have $P : 1 + 2 = 3$ is true, then $\overline{P}$: $1 + 2 \neq 3$ is false.*

2. $P : f$ is positive function, then $\overline{P}$: f is not positive function.

(2)- The logical connector $\ll$ *and* $\gg$ (conjunction)

Let (P) and (Q) be two assertions.
The **Conjunction** of this two assertions is an assertion denoted by $(P \wedge Q)$ and reads $(P \; and \; Q)$, which is true when the assertions P and Q are both true, and false otherwise.
We summarize this, in a truth table :

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.3. *The assertion* $[(1+2=3) \wedge (3 \textit{ is negative })]$ *is false.*

(3)- The logical connector $\ll$ *or* $\gg$ (disjunction)

Let (P) and (Q) be two assertions.

The **disjunction** of this two assertions is an assertion denoted by $(P \vee Q)$ and reads $(P \textit{ or } Q)$, which is true when at least one of the two assertions P *or* Q are true, and false otherwise.

We summarize this, in a truth table :

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.4. *The assertion* $[(1+2=3) \vee (3 \textit{ is negative})]$ *is true.*

(4)- The implication $\ll \Longrightarrow \gg$

Let (P) and (Q) be two assertions.

The **implication** from P to Q is an assertion denoted $(P \Longrightarrow Q)$ reads $(P \textit{ implique } Q)$ or $(if P then Q)$. It is false when (P) is true and (Q) is false, and true otherwise. The mathematical definition of implication is $(\overline{P} \vee Q)$. Its truth table is therefore as follows :

P	Q	$\overline{P}$	$\overline{P} \vee Q (P \Longrightarrow Q)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Example 1.5. *1.* $0 \leq x \leq 9 \Longrightarrow \sqrt{x} \leq 3.$ *is true (take the square root).*

2. $\sin(\theta) = 0 \Longrightarrow \theta = 0$ *is false(for* $\theta = 2\pi$ *for example).*

Remark 1.1. 1. *In **the implication** (* $P \Longrightarrow Q$*):* P *is called **sufficient condition** and* Q *said **necessary condition** of implication.*

2. *In practice, if* P, Q *and* R *are three assertions, then :*

$$(P \Longrightarrow Q) \text{ and } (Q \Longrightarrow R) \text{ is written } (P \Longrightarrow Q \Longrightarrow R).$$

(5)- Equivalence ≪⟺≫

Equivalence is defined by :

$$(P \Longleftrightarrow Q) \text{ is the assertion } [(P \Longrightarrow Q) \text{ and } (Q \Longrightarrow P)].$$

We will say (P *equivalent to* Q) or (P *if and only if* Q). This assertion is true when $P$and Q are true or when $P$and Q are false. The truth table is :

P	Q	$\overline{P}$	$\overline{Q}$	$P \Longrightarrow Q(\overline{P} \vee Q)$	$Q \Longrightarrow P(\overline{Q} \vee P)$	$(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)(P \Longleftrightarrow Q)$
T	T	F	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Example 1.6. 1. $x + 2 = 0 \Longleftrightarrow x = -2.$ *is true.*

2. *For* $x, x' \in \mathbb{R}$, $x \cdot x' = 0 \Longleftrightarrow (x = 0 \text{ or } x' = 0)$ *is true.*

Remark 1.2.

In practice, if P, Q *and* R *are three assertions, then :*

$$(P \Longleftrightarrow Q) \text{ and } (Q \Longleftrightarrow R) \text{ is noted } (P \Longleftrightarrow Q \Longleftrightarrow R).$$

Proposition 1.1. *Let P, Q and R three assertions.*

We have the following equivalences:

1. $(P \wedge Q) \Longleftrightarrow (Q \wedge P)$ *(commutativity of and).*

2. $(P \vee Q) \Longleftrightarrow (Q \vee P)$ *(commutativity of or).*

3. $(\overline{P \wedge Q}) \Longleftrightarrow (\overline{P} \vee \overline{Q})$*(morgan's laws).*

4. $\overline{(P \vee Q)} \Longleftrightarrow (\overline{P} \wedge \overline{Q})$ *(morgan's laws).*

5. $\overline{\overline{P}} \Longleftrightarrow P.$

6. $(P \wedge P) \Longleftrightarrow P.$

7. $(P \vee P) \Longleftrightarrow P.$

8. $[(P \wedge Q) \wedge R] \Longleftrightarrow [P \wedge (Q \wedge R)]$*(associativity of and).*

9. $[(P \vee Q) \vee R] \Longleftrightarrow [P \vee (Q \vee R)]$ *(associativity of or).*

10. $[(P \wedge Q) \vee R] \Longleftrightarrow [(P \vee R) \wedge (Q \vee R)]$ *(distributiveness of and with respect to or).*

11. $[(P \vee Q) \wedge R] \Longleftrightarrow [(P \wedge R) \vee (Q \wedge R)]$ *distributiveness of or with respect to and).*

12. $\overline{(P \Longrightarrow Q)} \Longleftrightarrow (P \wedge \overline{Q}).$

13. $(P \Longrightarrow Q) \Longleftrightarrow (\overline{Q} \Longrightarrow \overline{P})$ *(principe of contraposition).*

14. $(P \Longleftrightarrow Q) \Longleftrightarrow (P \Longrightarrow Q \wedge Q \Longrightarrow P).$

preuve 1.1. *We prove proposition (12).*

P	Q	$\overline{Q}$	$P \Longrightarrow Q$	$\overline{P \Longrightarrow Q}$	$P \wedge \overline{Q}$	$\overline{(P \Longrightarrow Q)} \Longleftrightarrow (P \wedge \overline{Q})$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

1.1.3 Quantifiers

In mathematic, we often use expressions of the form : " for everything,...", whatever...", "there exists at least...", " there exist only one...".
This expressions specify how the elements of a set can satisfy a certain property. These expressions are called **quantifiers.**

There are two types of quantifiers :

(1)- The universal quantifier $\ll \forall \gg$

The expression " for all x *of* E such that $P(x)$ is written mathematically "

$$\forall x \in E,\ P(x)$$

To express that the assertion $P(x)$ is true for all elements x *of* E.

Example 1.7. $P(x)$: *The function f is zero for all $x \in \mathbb{R}$ becomes :*

$$P(x) : \forall x \in \mathbb{R},\ f(x) = 0.$$

(2)-The existential quantifier $\ll \exists \gg$

The expression " There existe x of E such that $P(x)$ " is written mathematically "$\exists x \in E,$ $P(x)$" to express that the assertion $P(x)$ is true for at least one x *of* E.

Example 1.8. $P(x)$: *The function f vanishes at x_0 becomes :*

$$P(x) : \exists x_0 \in \mathbb{R},\ f(x_0) = 0.$$

Remark 1.3. *The expression " There exists a unique x of E such that $P(x)$ " i.e a unique x, written mathematically "$\exists! x \in E,\ P(x)$" to express that the assertion $P(x)$ is true for a unique value x of E.*

Remark 1.4. *We can construct assertions with several quantifiers. In this cas, we will take care of the order of these quantifiers, for example the two logical sentences*

$$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}, x + y > 0 \quad \text{and} \quad \exists y \in \mathbb{R},\ \forall x \in \mathbb{R}, x + y > 0$$

are different. The first is true because y can depend on $x(y = 1 - x)$. On the other hand the second is false $(x = -y)$.

- The negation of quantifiers

Let $P(x)$ be a proposition,

1. The negation of $\forall x \in E, P(x)$ is: $\exists x \in E, \overline{P(x)}$.

2. The negation of $\exists x \in E, P(x)$ is: $\forall x \in E, \overline{P(x)}$.

Example 1.9. *1. The negation of* $(\forall x \in [1,+\infty[,\ x^2 \geq 1)$ *is:* $(\exists x \in [1,+\infty[,\ x^2 < 1)$.

2. The negation of $(\forall x \in \mathbb{R},\ \exists y > 0\text{: } x+y > 0)$ *is:* $(\exists x \in \mathbb{R}, \forall y > 0\text{: } x+y \leq 0)$.

1.2 Reasoning methods

To show that $(P \Longrightarrow Q)$ is true , we can use the following classical reasoning methods:

1.2.1 Direct Reasoning

We assume that P is true and we prove that Q is also true.

Example 1.10. *Let us show that for* $n \in \mathbb{N}$ *if* n *is even* $\Longrightarrow n^2$ *is even. We assume that* n *is even, i.e.,* $\exists k \in \mathbb{N},\ n = 2k$ *then*

$$n.n = 2(2k^2) \Longrightarrow n^2 = 2k',$$

We pose $k' = 2k^2 \in \mathbb{N}$ *thus* $\exists k' \in \mathbb{N},\ n^2 = 2k',\ n^2$ *is even, hence the result.*

1.2.2 Case by case reasoning

If you want to check $\forall x \in E : P(x)$. We show $\forall x \in A : P(x)$ and $\forall x \in \overline{A} : P(x)$ where A part of E.

Example 1.11. *Demonstrate that* $\forall n \in \mathbb{N} \Longrightarrow \frac{n(n+1)}{2} \in \mathbb{N}$.

cas 1 : n *is even,* $\exists k \in \mathbb{N}$ *such that* $n = 2k \Longrightarrow \frac{2k(2k+1)}{2} = k(2k+1) \in \mathbb{N}$.

cas 2 : n *is odd,* $\exists k \in \mathbb{N}$ *such that* $n = 2k+1 \Longrightarrow \frac{(2k+1)(2k+1)+1}{2} = \frac{(2k+1)(2k+2)}{2} = (2k+1)(2k+1) \in \mathbb{N}$.

Conclusion in any case $\forall n \in \mathbb{N} \Longrightarrow \frac{n(n+1)}{2} \in \mathbb{N}$.

1.2.3 Reasoning by the contrapositive

Knowing that $(P \Longrightarrow Q) \Longleftrightarrow (\overline{Q} \Longrightarrow \overline{P})$, to show that $(P \Longrightarrow Q)$ we use the contrapositive, that's to say it is enough to show that $\overline{Q} \Longrightarrow \overline{P}$ directly, we assume that $\overline{Q}$ is true and we show that $\overline{P}$ is true.

Example 1.12. *Let $n \in \mathbb{N}$. Show that n^2 is even $\Longrightarrow n$ is even .*

We assume that n is not even. We want to show that n^2 is not even. n is not even, it is odd then $\exists k \in \mathbb{N}$: $n = 2k+1 \Longrightarrow n^2 = 2l+1$ et $l = 2k^2+2k \in \mathbb{N}$. and then n^2 is not even. By contrapositive this is equivalent to if n^2 is even $\Longrightarrow n$ is even.

1.2.4 Reasoning by the absurd

To show that an assertion R is true by the absurd, we assume that $\overline{R}$ is true and we show that we then obtain a contradiction. Thus, to show by the absurd, the implication $P \Longrightarrow Q$, we assume both P is true and that Q is false (i.e., $P \Longrightarrow Q$ is false) and we look for a contradiction.

Example 1.13. *Let n be a natural number. Let show by the absurd that if $3n+2$ is odd $\Longrightarrow$ n is odd.*

Suppose that $3n+2$ is odd and n is even.

n is even then $\exists k \in \mathbb{N}$: $n = 2k \Longrightarrow 3n+2$ is even, we thus obtain that $3n+2$ is even and $3n+2$ is odd, contradiction.

1.2.5 Counter example

To show that a proposition is false, it is enough to give what is called a counter example, that is to say a particular case for which the proposition is false.

Example 1.14. *The proposition (n is an even number) $\Longrightarrow$ (n^2+1 is even), false because for $n = 2$, $4+1 = 5$ is not even, it is a counter-example.*

1.2.6 Reasoning by recurrence

To show that $\forall n \in \mathbb{N}, n \geq n_0, P(n)$ is true, we follow three steps:

1. Initialization: We show that $P(n_0)$ is true.

2. Heredity: We assume that $P(n)$ is true for $n \geq n_0$ and demonstrate that $P(n+1)$ is true.

3. Conclusion: By the principe of recurrence $\forall n \in \mathbb{N}, P(n)$ is true.

Example 1.15. *Show that $\forall n \in \mathbb{N} : 2^n \geq n$.*

Let us denote by $P(n)$ the following assertion $\forall n \in \mathbb{N} : 2^n \geq n$. We will prove by recurrence that $P(n)$ is true.

1. *Initialization: $P(0) : 2^0 = 1 \geq 0$ is true.*

2. *Heredity: Suppose that $P(n)$ is true. We will show that $P(n+1)$ is also true.*

   We have:

$$\begin{aligned} 2^{n+1} &= 2^n \times 2 \\ &= 2^n + 2^n \\ &> n + 2^n (\text{ we have } 2^n \geq n) \\ &> n + 1 (\text{ we have } 2^n \geq 1) \end{aligned}$$

3. *Conclusion: by the Principe of recurrence $P(n)$ is true $\forall n \in \mathbb{N} : 2^n \geq n$.*