Chapter 2

Sets and applications

2.1 Set

Definition 2.1. *An ensemble, also known as a set, is a collection of objects. These objects are called the elements of the set.*

Example 2.1. 1. *We denote* $\mathbb{N}$ *by the set of natural numbers* $\mathbb{N} = \{0, 1, \ldots\}$.

2. *The set* $E = \{0, 1\}$.

Definition 2.2. *A set is said to be empty when it contains no elements and is denoted as* $\emptyset$ *or* $\{\}$.

Definition 2.3. *We call the cardinality of a set* E *the number of elements in* E *denoted by* $card(E)$ *or* $|E|$.

Remark 2.1. *The concept of cardinality does not apply to infinite sets, such as* $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ *and* $\mathbb{R}$.

Example 2.2. 1. *Consider the set* $E = \{0, 1\}$, *we have* $card(E) = 2$.

2. *Consider the set* $A = \{1, 2, 3, 4\}$, *we have* $card(A) = 4$.

3. *If* $A = \emptyset$, *then* $card(A) = 0$.

Definition 2.4. *The power set of E, denoted as $\mathcal{P}(E)$, is the set of all subsets that can formed from E, and we have $card(\mathcal{P}(E)) = 2^{card(E)}$.*

Example 2.3. *Let $E = \{1, 2\}$, the set $\mathcal{P}(E) = \{\emptyset, \{1\}, \{2\}, E\}$ and $card(\mathcal{P}(E)) = 2^2 = 4$.*

2.2 The Relationships between sets

2.2.1 Inclusion

Let A and B be two subsets of a set E. We say that A is included in B(A is a subset of B, or A is a part of B), and denote this as $A \subset B$, if all the elements of a set A are elements of set B.

$$A \subset B \iff (\forall x \in E, x \in A \Longrightarrow x \in B).$$

Example 2.4.
1. *We denote $\mathbb{R}$ as the set of real numbers. We have : $\mathbb{N} \subset \mathbb{R}$.*
2. *We denote $\mathbb{Z}$ as the set of integers, and $\mathbb{Q}$ as the set of rationals we have:*

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Remark 2.2.
- *We have $\emptyset \subset E$ et $E \subseteq E$.*
- *If A, B and C are subsets of E, then:*
  1. $A \not\subset B \iff (\exists x \in E : x \in A \wedge x \notin B)$.
  2. $(A \subset B \wedge B \subset C) \Longrightarrow (A \subset C)$.

2.2.2 Equality

Let's consider two sets, A and B which are subsets of E. We say that A and B are equal, denoted as $(A = B) \iff [(A \subset B) \wedge (B \subset A)]$, when $(\forall x \in A \iff x \in B)$. Otherwise, we state that they are distinct, also noted as $A \neq B$.

2.2.3 Union

Definition 2.5. *Let's consider two sets, A and B which are subsets of E. The union of A and B is the set of elements that are in A or B, and is denoted by :*

$$A \cup B = \{x \in E : x \in A \vee x \in B\}.$$

$$(x \in A \cup B) \iff (x \in A \vee x \in B).$$

Example 2.5. *Let $A = \{1, 3, 5\}$ and $B = \{1, x, y\}$. Then*

$$A \cup B = \{1, 3, 5, x, y\}.$$

Remark 2.3. $(x \notin A \cup B) \iff (x \notin A \wedge x \notin B)$.

2.2.4 Intersection

Definition 2.6. *Let's consider two sets, A and B which are subsets of E. The intersection of A and B is the set of elements that are both in A and in B, denoted by $A \cap B$.*

$$A \cap B = \{x \in E : x \in A \wedge x \in B\}.$$

$$(x \in A \cap B) \iff (x \in A \wedge x \in B).$$

Remark 2.4. $(x \notin A \cap B) \iff (x \notin A \vee x \notin B)$

Example 2.6. *Let $A = \{1, 3, 5\}$ and $B = \{1, x, y\}$. Then*

$$A \cap B = \{1\}.$$

Properties Let A, B and C be three sets :

1. $A \subset A \cup B$, $B \subset A \cup B$.

2. $A \cap B \subset A$, $A \cap B \subset B$.

3. $A \subset B \Rightarrow A \cup B = B$.

4. $A \subset B \Rightarrow A \cap B = A$.

5. $A \cap B \subset A \cup B$.

6. $A \cap A = A$, $A \cup A = A$.

7. $\emptyset \subset A$, $\emptyset \cap A = \emptyset$, $\emptyset \cup A = A$.

8. $A \cap B = B \cap A$ (commutativity of intersection).

9. $A \cup B = B \cup A$. (commutativity of union).

10. $(A \cap B) \cap C = A \cap (B \cap C)$ (associativity of intersection).

11. $(A \cup B) \cup C = A \cup (B \cup C)$(associativity of union).

12. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$(distributivity of intersection with respect to union).

13. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$(distributivity of union with respect of intersection).

2.2.5 Complement

Definition 2.7. *Let A be a subset of E. The complement of A in E, denoted as A^c, $C_E A$ or $\overline{A}$ is defined as:*

$$A^c = \{x \in E / x \notin A\}.$$

Remark 2.5. *1.* $x \in A^c \Longleftrightarrow x \notin A$.

2. $A \cup A^c = E$.

3. $A \cap A^c = \emptyset$.

4. If E and A they are finite, we have:

$$card(A^c) = card(E) - card(A).$$

Example 2.7. *Let $E = \{1, 2, 3, 5\}$ and $A = \{3, 5\}$, then $A^c = \{1, 2\}$, and $card(A^c) = card(E) - card(A) = 2$.*

2.2.6 Set difference - Symmetric difference

Definition 2.8. *Let A and B be two sets in E. The set difference of A and B is the set of elements in A that are not in B, denoted as $A \setminus B$ or $A - B$ and read as A minus B*

$$A \setminus B = A \cap B^c = \{x \in E : x \in A \text{ and } x \notin B\}.$$

Proposition 2.1.

1. $A \setminus B = \varnothing \Leftrightarrow A \subset B$.
2. *If A and B are finite we have:* $card(A \setminus B) = card(A) - card(A \cap B)$.

Definition 2.9. *Let E be a non-empty set and $A, B \subset E$, the symmetric difference between two sets, A and B, is the set of elements that belong to $A \setminus B$ or $B \setminus A$ noted $A \triangle B$*

$$\begin{aligned} A \triangle B &= (A \setminus B) \cup (B \setminus A) \\ &= (A \cup B) \setminus (B \cap A) \\ &= (A \cap B^c) \cup (A^c \cap B) \end{aligned}$$

$$x \in A \triangle B \Longleftrightarrow \{x \in (A \setminus B) \vee x \in (B \setminus A)\}.$$

Proposition 2.2.

1. $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.
2. $A \triangle B = \varnothing \Leftrightarrow (A \cup B) \setminus (A \cap B) = \varnothing \Leftrightarrow A = B$.
3. *If A and B are finite, we have:* $card(A \triangle B) = card(A) + card(B) - 2card(A \cap B)$.

Properties Let A and B be two subsets of a set E, we have

1. $(A^c)^c = A$.
2. $(A \cap B)^c = A^c \cup B^c$ Morgan's law.
3. $(A \cup B)^c = A^c \cap B^c$ Morgan's law.
4. $(A \subset B) \Longleftrightarrow (B^c \subset A^c)$.
5. $A \setminus A = \emptyset$.
6. $A \setminus \emptyset = A$.

2.2.7 Cartesian product

Definition 2.10. *Let A, B be two sets. The cartesian product of A and B is the set of pairs such that $a \in A$ and $b \in B$. This set will be denoted by $A \times B$*

$$A \times B = \{(a, b) \,/\, a \in A \ et \ b \in B\}.$$

Example 2.8. 1. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$.

2. $[0, 1] \times \mathbb{R} = \{(x, y) \mid 0 \leqslant x \leqslant 1, \ y \in \mathbb{R}\}$

Remark 2.6. 1. *More generally, if $A_1, A_2, \ldots, A_n$ are n sets,*

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \,/\, a_i \in A_i, \ i = 1, 2, \ldots, n\}.$$

The set $A_1 \times A_1 \times \ldots \times A_1$ is also denoted as $\prod_{i=1}^{n} A_i$ and $(a_1, a_2, \ldots, a_n)$ is called n-tuple of $A_1 \times A_1 \times \ldots \times A_1$.

2. *If $A_1 = A_2 = \ldots = A_n$, we denote*

$$\begin{aligned} A_1 \times A_1 \times \ldots \times A_1 &= A \times A \times \ldots \times A \\ &= A^n. \end{aligned}$$

Example 2.9. *Let $E = \{1, 2, 3, 5, 8, 9, x, y\}$, $A = \{1, 2, 3\}$ and $B = \{1, 2, 9\}$*

1. - $A \subset E$ *and* $B \subset E$.
   - $A \not\subset B$ *because* $(3 \in A) \wedge (3 \notin B)$.
   - $B \not\subset A$ *because* $(9 \in B) \wedge (9 \notin A)$.
2. - $A \cap B = \{1, 2\}$.
   - $A \cup B = \{1, 2, 3, 9\}$.
3. - $A \setminus B = \{3\}$.
   - $B \setminus A = \{9\}$.
4. $A \vartriangle B = \{3, 9\}$.
5. $A \times B = \{(1, 1), (1, 2), (1, 9), (2, 1), (2, 2), (2, 9), (3, 1), (3, 2), (3, 9)\}$.

2.2.8 Cardinal of a finite set

Definition 2.11. *The number of elements in a finite set is called the cardinal of A. This number is denoted by $Card(A)$ or $|A|$.*

Example 2.10.

1. *If $A = \{1, 2, 3, 4\}$, then $Card(A) = 4$.*

2. *If $A = \emptyset$, then $Card(A) = 0$.*

Remark 2.7. *The concepts of cardinality does not apply to infinite sets, for example $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ et $\mathbb{R}$.*

2.3 Application

Let E and F be two sets.

Definition 2.12. *An application $f : E \longrightarrow F$ is defined for each element $x \in E$, a unique element of F noted $f(x)$, where E is the domain set and F is the codomain set.*

Example 2.11.

1.
$$\begin{aligned} f \ : \ & \mathbb{R} \longrightarrow \mathbb{R} \\ & x \longmapsto f(x) = x \end{aligned}$$

   f is an application

2.
$$\begin{aligned} g \ : \ & \mathbb{N} \longrightarrow \mathbb{N} \\ & n \longmapsto g(n) = n - 1. \end{aligned}$$

   g is not an application

Remark 2.8. 1. *The graph of* $f : E \longrightarrow F$ *is*

$$\Gamma_f = \{(x; y) \in E \times F \,/\, y = f(x)\}.$$

2. *Let* $f : E \longrightarrow F$ *and* $g : G \longrightarrow H$ *two applications.* $f = g$ *if and only if* $E = G$ *and* $F = H$ *and* $\forall x \in E, f(x) = g(x)$.

3. *Let* $f : E \longrightarrow F$ *an application. Let's fix* $y \in F$*, every element* $x \in E$ *such that :* $y = f(x)$ *is a pre-image*

notation 2.1. 1. *We denote* $\mathcal{F}(E, F)$ *as the set of all applications from* E *to* F.

2. *We denote id the identity application.*

$$\begin{aligned} id \;:\; & E \longrightarrow E \\ & x \longmapsto id(x) = x \end{aligned}$$

2.4 Direct and reciprocal(inverse) images

Let E and F be two sets.

Definition 2.13. *(Direct image) Let* $A \subset E$ *and* $f : E \longrightarrow F$*, The direct image of* A *by* f *is the set:*

$$f(A) = \{f(x) \,/\, x \in A\} \subset F.$$

Definition 2.14. *(Inverse image) Let* $B \subset F$ *and* $f : E \longrightarrow F$*, The inverse image of* B *by* f *is the set:*

$$f^{-1}(B) = \{x \in E \,/\, f(x) \in B\} \subset E.$$

Example 2.12. 1. *Let* f *an application*

$$\begin{aligned} f \;:\; & \mathbb{N} \longrightarrow \mathbb{N} \\ & n \longmapsto f(x) = 2n + 1 \end{aligned}$$

Let $A = \{0, 1, 2\}$, *then*

$$\begin{aligned} f(A) &= \{f(n) \,/\, n \in A\} \\ &= \{f(0), f(1), f(2)\} \\ &= \{1, 3, 5\}. \end{aligned}$$

Let $B = \{5\}$, *then*

$$\begin{aligned} f^{-1}(B) &= \{x \in E \,/\, f(x) \in B\} \\ &= \{x \in E \,/\, f(x) = 5\} \\ &= \{2\}. \end{aligned}$$

Properties 2.1. *Let* $f : E \longrightarrow F$ *be an application. Let* A_1 *and* A_2 *be two subsets of E. Then,*

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

2. $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

3. $A_1 \subset A_2 \Longrightarrow f(A_1) \subset f(A_2)$.

4. $A_1 \subset f^{-1}(f(A_1))$.

Let B_1 *and* B_2 *be two subsets of F.*

1. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

2. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

3. $B_1 \subset B_2 \Longrightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$.

2.5 Injection

Definition 2.15. *Let* $f : E \longrightarrow F$ *be an application. We say that f is injective if every element of F has at most one pre-image, i.e.,*

$$\forall x, x' \in E : f(x) = f(x') \Longrightarrow x = x'.$$

Or

$$\forall x, x' \in E : x \neq x' \Longrightarrow f(x) \neq f(x').$$

Example 2.13. *1.*

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$
$$n \longmapsto 2n+1$$

f is injective because:

$$\begin{aligned} \forall n, n' \in E : f(n) = f(n') &\Longrightarrow 2n+1 = 2n'+1 \\ &\Longrightarrow 2n = 2n' \\ &\Longrightarrow n = n'. \end{aligned}$$

2.

$$g : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto 5x+3$$

g is injective because:

$$\begin{aligned} \forall x, x' \in E : g(x) = g(x') &\Longrightarrow 5x+3 = 5x'+3 \\ &\Longrightarrow 5x = 5x' \\ &\Longrightarrow x = x'. \end{aligned}$$

2.6 Surjection

Definition 2.16. *Let $f : E \longrightarrow F$ be an application. We say that f is surjective if every element of F at least he has a pre-image, i.e.,*

$$\forall y \in F, \exists x \in E : f(x) = y.$$

Example 2.14. *1.*

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$
$$n \longmapsto 2n+1$$

f is not surjective, indeed if we assume that it is surjective , that is

$$\begin{aligned} \forall y \in \mathbb{N}, \exists n \in \mathbb{N} : f(n) = y &\Longrightarrow 2n+1 = y \\ &\Longrightarrow n = \tfrac{y-1}{2} \notin \mathbb{N} \textit{ contradiction }. \end{aligned}$$

2.

$$g : \mathbb{R} \longrightarrow \mathbb{R}$$
$$n \longmapsto 5x + 3$$

g is surjective because:

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : g(x) = y \implies 5x + 3 = y$$
$$\implies x = \frac{y-3}{5} \in \mathbb{R}.$$

2.7 Bijection

Definition 2.17. *Let $f : E \longrightarrow F$ be an application. We say that f is bijective if it is both surjective and injective,*

$$\forall y \in F, \exists!\, x \in E : f(x) = y.$$

Meaning that every element in F has a unique pre-image by f.

Example 2.15. *1.*

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$
$$n \longmapsto 2n + 1$$

f is not bijective because it is not surjective.

2. g is bijective.

2.8 The composition of applications

Definition 2.18. *Let E, F, G three sets and f, g be two applications such that:*

$$E \xrightarrow{f} F \xrightarrow{g} G$$

One can deduce an application from E to G, denoted as $g \circ f$ and called the composition of f and g, by

$$\forall x \in E, (g \circ f)(x) = g(f(x)).$$

Example 2.16. *Let*

$$f \;:\; \mathbb{R} \longrightarrow \mathbb{R}^+$$
$$x \longmapsto x^2 + 1$$

and

$$g \;:\; \mathbb{R}^+ \longrightarrow \mathbb{R}$$
$$x \longmapsto \sqrt{x},$$

then

$$g \circ f \;:\; \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \sqrt{x^2 + 1}.$$

Proposition 2.3. *Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be two applications.*

1. *The composition of two injections is an injection, i.e,*

   (If f and g are injective, then $g \circ f$ is injective).

2. *The composition of two surjections is an surjection, i.e,*

   If f and g are surjective, then $g \circ f$ is surjective).

3. *The composition of two bijections is bijection, i.e,*

   (If f and g are bijective, $g \circ f$ is bijective).

4. *If f and g are bijective. Then*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Proof. 1. Let's assume that f and g are injective, and let's show that $g \circ f$ is injective.

$$\forall x_1, x_2 \in E,\ (g \circ f)(x_1) = (g \circ f)(x_2)$$

Since g is injective, we will have:

$$g(f(x_1)) = g(f(x_2)) \Longrightarrow f(x_1) = f(x_2),$$

Since g is injective, thus :

$$(g \circ f)(x_1) = (g \circ f)(x_2) \Longrightarrow x_1 = x_2,$$

then $g \circ f$ is injective.

□

Proposition 2.4. *1. If $g \circ f$ is injective, then f is injective.*

2. If $g \circ f$ is surjective, then f is surjective.

3. If $g \circ f$ is bijective, then f is injective and g is surjective.

Remark 2.9. *When an application f is bijective, it means that the inverse application f^{-1} exists, and f^{-1} is also bijective from F to E and $(f^{-1})^{-1} = f$.*

Proposition 2.5. *If $f : E \longrightarrow F$ is a bijection, then*

$$f^{-1} \circ f = Id_E \text{ and } f \circ f^{-1} = Id_F.$$