Chapter 4

Algebraic Structures

4.1 Laws of internal composition (L.I.C)

Definition 4.1. *Let E be a non-empty set. An internal composition law $*$ on E is a mapping from $E \times E$ to E associating every pair (a, b) in $E \times E$ with an element of E, denoted as $a * b$:*

$$\begin{array}{rcl} * : E \times E & \longrightarrow & E \\ (a, b) & \longmapsto & a * b \end{array}$$

Remark 4.1. *The internal composition law can be noted by $*$, $\perp, \ldots$, or other symbols.*

Example 4.1.
- *The standard operations constitute internal composition law on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\ldots$.*
- *Intersection and union constitute internal composition laws on the power set of the set E.*
- *Let $*$ be defined on $\mathbb{Q}$:*

$$a * b = \frac{a+b}{2}$$

Then $*$ is an internal composition law.

- *Let $*$ be defined on $\mathbb{R}$:*

$$a * b = \frac{1}{a+b}$$

Then $*$ is not an internal composition law, because $(-1, 1) \in \mathbb{R} \times \mathbb{R}$ does not have a defined image.

4.2 Properties of internal operations

4.2.1 Associativity

Definition 4.2. *we say that $*$ is associative if and only if :*

$$\forall (a, b, c) \in E^3 : \ a * (b * c) = (a * b) * c$$

Example 4.2. *Let $*$ be an internal composition law defined on $\mathbb{R}$ by :*

$$a * b = a + b - 1$$

We have

$$\begin{aligned}(a * b) * c &= (a + b - 1) + c - 1 \\ &= a + b + c - 2 ....(1) \\ a * (b * c) &= a + (b + c - 1) - 1 \\ &= a + b + c - 2 ....(2)\end{aligned}$$

when $(1) = (2)$, then $*$ is associative.

4.2.2 Commutativity

Definition 4.3. *we say that $*$ is commutative if and only if :*

$$\forall (a, b) \in E^2 : \ a * b = b * a$$

Example 4.3. *Let $*$ be an internal composition law defined on $\mathbb{R}$ by :*

$$a * b = a + b - 1$$

we have

$$\begin{aligned} a * b &= a + b - 1 \\ &= b + a - 1 \\ &= b * a \end{aligned}$$

Then $*$ is commutative.

4.2.3 Neutral element

Definition 4.4. *The law of internal composition $*$ admits a neutral element on set E if and only if :*

$$\exists e \in E, \forall a \in E : \ e * a = a * e = a.$$

Remark 4.2. *The neutral element, if it exists, is unique. Indeed let e' be another neutral element for $*$,then*

$$e' = e' * e = e * e' = e$$

Example 4.4. *Let $*$ be an internal composition law defined on $\mathbb{R}$ by :*

$$a * b = a + b - 1$$

we have

$$\begin{aligned} a * e = a &\implies a + e - 1 = a \\ &\implies e = 1 \end{aligned}$$

Then $e = 1$ is a neutral element .

4.2.4 Symmetric element

Definition 4.5. *We assume that E has a neutral element e for $*$. Let a and a' be two elements of E. We say that a' is symmetric to a (for the law $*$) if:*

$$\forall a \in E, \exists a' \in E : \ a * a' = a' * a = e.$$

Example 4.5. *Let $*$ be an internal composition law defined on $\mathbb{R}$ by :*

$$a * b = a + b - 1$$

we have

$$a * a' = 1 \implies a + a' - 1 = 1$$
$$\implies a' = (2 - a) \in \mathbb{R}$$

Then $a' = 2 - a$ is a symmertic element.

4.2.5 Distributivity

Definition 4.6. *Given two laws of internal composition $*$ et $\top$ defined on E.*

- *We say that the law $\top$ is left distributive with respect to the law $*$ if:*

$$\forall (a, b, c) \in E^3 : a \top (b * c) = (a \top b) * (a \top c)$$

- *We say that the law $\top$ is right distributive with respect to the law $*$ if:*

$$\forall (a, b, c) \in E^3 : \ (b * c) \top a = (b \top a) * (c \top a)$$

The law $\top$ is said to be distributive with respect to the law $*$ if it is both left and right distributive with respect to $*$.

Example 4.6. *Let $*$ be an internal composition law defined on $\mathbb{R}$ by :*

$$a * b = a + b - 1,$$

and Let $\top$ be an internal composition law defined on $\mathbb{R}$ by :

$$a \top b = a + b - ab$$

Then the law $\top$ is said to be distributive with respect to the law $*$. When $\top$ is commutative, it is then demonstrated that $\top$ is left distributive with respect to the law $*$.

$$\begin{aligned} a \top (b * c) &= a \top (b + c - 1) \\ &= 2a + b + c - ab - ac - 1 \text{............}(1) \\ (a \top b) * (a \top c) &= (a + b - ab) * (a + c - ac) \\ &= 2a + b + c - ab - ac - 1 \text{............}(2) \end{aligned}$$

When (1) = (2), then the law $\top$ is distributive with respect to the law $*$.

4.3 Stability

Definition 4.7. *Let E be a set equipped with an internal law. A subset F of E is said to be stable for this internal law if and only if :*

$$\forall a, b \in F : \; a * b \in F$$

Example 4.7. *$\mathbb{N}$ is a subset of $\mathbb{R}$ stable for internal composition laws $+$ and $\times$.*

4.4 Group

Definition 4.8. *Let the internal composition law be defined on a set G, we say that the pair $(G, *)$ is a group if:*

1. *The law $*$ is associative*

$$\forall (a, b, c) \in G^3 : a * (b * c) = (a * b) * c$$

2. *There exists a neutral element e*

$$\exists e \in G, \, \forall a \in G : \; e * a = a * e = a.$$

3. *Every element in G has a symmetric element*

$$\forall a \in G, \, \exists a' \in E : \; a' * a = a * a' = e$$

It is also said that the set G has a group structure for the law $*$.

Example 4.8.
1. *$(\mathbb{N}, \times)$ not a group.*
2. *$(\mathbb{Z}, +)$ is a group.*
3. *$(\mathbb{Z}, \times)$ not a group.*
4. *$(\mathbb{R}, +)$ is a group.*

4.4.1 Subgroup

Definition 4.9. *Let* $(G,*)$ *. a non-empty subset* H *of* G *is a subgroup of* G *if :*

$$\begin{cases} \forall (a,b) \in H \times H & \Longrightarrow \quad a * b \in H.........(1) \\ \forall a \in H & \Longrightarrow \quad a' \in H........(2) \end{cases}$$

Example 4.9. *Let* $(\mathbb{Z},+)$ *be a group, then* $3\mathbb{Z}$ *is a subgroup of* $\mathbb{Z}$*.*

We have :

$$\begin{aligned} 3\mathbb{Z} &= \{3z/z \in \mathbb{Z}\} \\ &= \{\ldots,-6,-3,0,3,6,\ldots\} \end{aligned}$$

1. *Let* $a,b \in 3\mathbb{Z}$*, then* $\exists z_1 \in \mathbb{Z}$ *such that* $a = 3z_1$ *and* $\exists z_2 \in \mathbb{Z}$ *such that* $b = 3z_2$*, so* $a+b = 3(z_1+z_2) \in 3\mathbb{Z}$*.*

2. *Let* $a \in 3\mathbb{Z}$*, then* $-a = -3z_1 = 3(-z_1) \in 3\mathbb{Z}$*.*

For (1) *and* (2)*, then* $3\mathbb{Z}$ *is a subgroup of* $\mathbb{Z}$*.*

Theorem 4.1. *Let* H *be a non-empty subset of a group* G*, then* H *is a subgroup of* G *if and only if :*

$$\forall (a,b) \in H \times H \Longrightarrow a * b' \in H.$$