Chapter 3

Binary relations on a set

3.1 Equivalence relation

Definition 3.1. *Let $\mathcal{R}$ be a binary relation on E. $\mathcal{R}$ is an equivalence relation if:*

1. *$\mathcal{R}$is reflexive :*

$$\forall x \in E,\ x\mathcal{R}x$$

2. *$\mathcal{R}$ is symmetric :*

$$\forall x, y \in E,\ x\mathcal{R}y \Longleftrightarrow y\mathcal{R}x$$

3. *$\mathcal{R}$ is transitive :*

$$\forall x, y, z \in E,\ [x\mathcal{R}y \wedge y\mathcal{R}z] \Longrightarrow x\mathcal{R}z$$

Example 3.1. *We consider the following relation on $\mathbb{Z}$:*

$$\forall x, y \in \mathbb{Z},\ x\mathcal{R}y \Longleftrightarrow \exists k \in \mathbb{Z},\ x - y = 2k$$

it is an equivalence relation .

1. $\mathcal{R}$ *is reflexive : Let* $x \in \mathbb{Z}$, *we have*

$$x - x = 2 \times 0 \iff x\mathcal{R}x$$

Then, $\mathcal{R}$ *is reflexive.*

2. $\mathcal{R}$ *is symmetric :*

Let $x, y \in \mathbb{Z}$, *we have*

$$\begin{aligned} x\mathcal{R}y &\iff \exists k \in \mathbb{Z}, x - y = 2k \\ &\iff y - x = 2k' \ (k' = -k \in \mathbb{Z}) \\ &\iff y\mathcal{R}x \end{aligned}$$

Then, $\mathcal{R}$ *is symmetric.*

3. $\mathcal{R}$ *is transitive : Let* $x, y, z \in \mathbb{Z}$, *we have*

$$\begin{aligned} x\mathcal{R}y \wedge y\mathcal{R}z &\iff \begin{cases} \exists k \in \mathbb{Z},\ x - y = 2k.........(1) \\ \wedge \\ \exists k' \in \mathbb{Z},\ y - z = 2k'........(2) \end{cases} \\ (1) + (2) &\implies x - z = 2k'',\ (k'' = (k - k') \in \mathbb{Z}) \\ &\implies x\mathcal{R}z \end{aligned}$$

Then, $\mathcal{R}$ *is transitive.*

So $\mathcal{R}$ *is an equivalence relation .*

3.1.1 Equivalence class

Definition 3.2. *If* $\mathcal{R}$ *is an equivalence relation in a set* E, *the equivalence class of* $x \in E$ *is the set*

$$\dot{x} = \{y \in E \ / \ x\mathcal{R}y\}.$$

notation 3.1. *We denote by* $E/\mathcal{R}$ *(the set of quotients of* E *by* $\mathcal{R}$*) the set of equivalence classes of* $\mathcal{R}$

$$E/\mathcal{R} = \{\dot{x} \,/\, x \in E\}$$

Example 3.2. *In the previous example, give $\dot{x}$ and $E/\mathcal{R}$*

$$\begin{aligned}\dot{x} &= \{y \in \mathbb{Z} \,/\, x\mathcal{R}y\} \\ &= \{y \in \mathbb{Z} \,/\, x - y = 2k\} \\ &= \{x - 2k \,/\, k \in \mathbb{Z}\} \\ &= \{\dots, x-4, x-2, x, x+2, x+4, \dots\}\end{aligned}$$

$$\begin{aligned}\dot{0} &= \{y \in \mathbb{Z} \,/\, 0\mathcal{R}y\} \\ &= \{\dots, -4, -2, 0, 2, 4, \dots\}\end{aligned}$$

$$\begin{aligned}\dot{1} &= \{y \in \mathbb{Z} \,/\, 1\mathcal{R}y\} \\ &= \{\dots, -3, -1, 1, 3, \dots\} \\ \dot{2} &= \dot{0}\end{aligned}$$

$$\begin{aligned}\mathbb{Z}/\mathcal{R} &= \{\dot{x} \,/\, x \in \mathbb{Z}\}. \\ \mathbb{Z}/\mathcal{R} &= \{\dot{0}, \dot{1}\}.\end{aligned}$$

Proposition 3.1. *Let $\mathcal{R}$ be an equivalence relation in the set E. Then,*

- $\forall x \in E,\ \dot{x} \subset E.$
- $\forall x \in E,\ \dot{x} \neq \emptyset.$
- $\forall x, y \in E,\ x\mathcal{R}y \Longrightarrow \dot{x} = \dot{y}.$

3.2 Order relation

Definition 3.3. *Let $\mathcal{R}$ be a binary relation on E. It 's an order relation if:*

1. *$\mathcal{R}$ is reflexive :*

$$\forall x \in E,\ x\mathcal{R}x.$$

2. *$\mathcal{R}$ is anti symmetric :*

$$\forall x, y \in E,\ [x\mathcal{R}y \wedge y\mathcal{R}x] \Longrightarrow x = y.$$

3. $\mathcal{R}$ *is transitive :*

$$\forall x, y, z \in E,\ [x\mathcal{R}y \wedge y\mathcal{R}z] \Longrightarrow x\mathcal{R}z.$$

Definition 3.4. *Let $\mathcal{R}$ be an order on E.*

- *An order relation $\mathcal{R}$ on a set E is total if:*

$$\forall x, y \in E : x\mathcal{R}y \text{ ou } y\mathcal{R}x.$$

  It is also called $(E, \mathcal{R})$ a totally ordered set.

- *If the order $\mathcal{R}$ is not total, we say that $\mathcal{R}$ is a partial order.*

Example 3.3. *We equip $\mathbb{R}^2$ with the relation noted as $\mathcal{R}$ defined by:*

$$(x, y)\mathcal{R}(x', y') \Longrightarrow x \leqslant x' \text{ et } y \leqslant y'.$$

Demonstrate that $\mathcal{R}$ is order relation on $\mathbb{R}^2$. Is the order total?

1. $\mathcal{R}$ *is reflexive :*

$$\text{Let } (x, y) \in \mathbb{R}^2, \text{ we have } x \leqslant x \text{ and } y \leqslant y \Longrightarrow (x, y)\mathcal{R}(x, y)\ .$$

2. $\mathcal{R}$ *is anti symmetric :*

$$\text{Let } (x, y), (x', y') \in \mathbb{R}^2, \text{ we have } (x, y)\mathcal{R}(x', y') \text{ and } (x', y')\mathcal{R}(x, y), \text{ then we have both } x \leqslant x' \text{ and } x' \leqslant x \text{ then } x = x' \text{ and likewise } y = y'.$$

3. $\mathcal{R}$ *is transitive :*

$$\text{Let } (x, y), (x', y'), (x'', y'') \in \mathbb{R}^2, \text{ we have } (x, y)\mathcal{R}(x', y') \text{ and } (x', y')\mathcal{R}(x'', y''), \text{ then we have both } x \leqslant x' \leqslant x'' \text{ and } y \leqslant y' \leqslant y'' \text{ then } (x, y)\mathcal{R}(x'', y'').$$

So $\mathcal{R}$ is order relation .

The order is not total, because we cannot compare $(0, 1)$ and $(1, 0)$.