ANALYSIS 1:

Course and exercises with solutions

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Introduction

This course is intended primarily for all students who may need mathematical analysis for their higher education studies. Future students of computer science, technical sciences, economics, natural sciences.... will find here most tools and notions of calculus in analysis that they may need. The course is described in detail, theorems and propositions are demonstrated. All Exercises with corrections, will help student to integrate the concepts studied. This document contains the following 6 chapters in an easy-to-read style:

1. The first chapter of this presentation is devoted to the properties of the real numbers $\mathbb{R}$. (this is necessary for real analysis).
2. The second chapter deals with complex numbers $\mathbb{C}$.
3. The third chapter discusses sequences of real numbers and their properties.
4. The fourth chapter looks at real functions with one real variable, focusing on the notion of limits and continuity at a point.
5. The second-to-last chapter is devoted to differentiability, the Mean Value Theorem and its applications.
6. The last chapter covers the definitions and properties of the usual functions: logarithm functions, exponential functions, power functions, trigonometric functions, hyperbolic functions, inverse trigonometric functions, inverse hyperbolic functions.

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Contents

Chapter 1

The set of real numbers $\mathbb{R}$

1.1 Usual sets of numbers

Notations:

- The set of natural numbers is denoted by $\mathbb{N}$: $\mathbb{N} = \{0,1,2,3,....\}$

- We denote by $\mathbb{Z}$, the set of all integers, i.e. the set of all natural numbers and their opposites: $\mathbb{Z} = \{.... -2, -1,0,1,2,3....\}$

- We denote by $\mathbb{Q}$, the set of all rational numbers, which is the set of quotients $\frac{p}{q}$, where p and q are two integers, with a non-zero q: $\mathbb{Q} = \{\frac{p}{q}/p \in \mathbb{Z} \,\text{et}\, q \in \mathbb{Z}^*\}$

- The set of real numbers is denoted by $\mathbb{R}$. It contains rational and irrational numbers such that $\sqrt{2}$, π....

- The sets without 0 are respectively denoted by $\mathbb{N}^*$,$\mathbb{Z}^*$,$\mathbb{Q}^*$,$\mathbb{R}^*$

Remark:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

1.2 Axiomatic definitions of real numbers

The set of real numbers $\mathbb{R}$ has the following two operations

- $(x,y) \to x + y$
- $(x,y) \to x.y$

with an ordering relation $(x \leq y)$ or $(y \leq x)$ satisfying the following fifteen axioms :

1.2.1 Arithmetic axioms

A1. For any $x,y \in \mathbb{R}$; $x + y = y + x$ (commutativity of the addition)

A2. For any x,y et $z \in \mathbb{R}$; $(x + y) + z = x + (y + z)$ (associativity of the addition)

A3. There exists an element $0 \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$; $x + 0 = x$.

A4. For any $x \in \mathbb{R}$, there exists an element $-x \in \mathbb{R}$ such that $x + (-x) = 0$.

A5. For any $x,y \in \mathbb{R}$; $x.y = y.x$ (commutativity of the multiplication)

A6. For every x,y and $z \in \mathbb{R}$; $(x.y).z = x.(y.z)$ (associativity of the multiplication)

A7. There exists an element $1 \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$; $x.1 = x$

A8. For all $x \in \mathbb{R}^*$, There exists $x^{-1} \in \mathbb{R}^*$ such that: $x.x^{-1} = 1$

A9. For any x,y and $z \in \mathbb{R}$; $x.(y + z) = x.y + x.z$ (distributivity)

Remark:

1. Axioms (A1) and (A2) can be used to calculate the sum of three numbers x,y, et z expressed as $x + y + z$ and the symbol $\sum$ is used to denote the sum of n terms in the following way:
$$x_1 + x_2 + x_3 + ...x_n = \sum_{k=0}^{k=n} x_k$$
2. From axiom (A3) the neutral element 0 for addition in $\mathbb{R}$ is unique.
3. From axiom (A4) the additive inverse of a number x is unique and noted by $-x$.
4. Axioms (A5) and (A6) also allow us to calculate the product of three numbers x,y,et z clearly in the form $x.y.z$ and the symbol $\prod$ designates the product of n terms as follows:
$$x_1.x_2.x_3.....x_n = \prod_{k=1}^{k=n} x_k$$
5. From axiom (A7) the neutral element 1 for multiplication in $\mathbb{R}$ is unique.
6. By axiom (A8), the multiplicative inverse of a number $x \in \mathbb{R}^*$ is unique, denoted by $x^{-1} = \dfrac{1}{x}$.

1.2.2 Order axioms

A10. For all $x \in \mathbb{R}$ we have: $x \leq x$ (Reflexivity)

A11. For every $x,y \in \mathbb{R}$ we have: if $x \leq y$ and $y \leq x$ then $x = y$ (Antisymmetry)

A12. For any $x,y,z \in \mathbb{R}$ we have: if $x \leq y$ and $y \leq z$ then $x \leq z$ (Transitivity)

A13. Let $x,y \in \mathbb{R}$ we have: $x \leq y$ or $y \leq x$

A14. Consider $x,y,z \in \mathbb{R}$ we have: if $x \leq y$ then $(x + z \leq y + z)$ and $(x.z \leq y.z$ if $0 \leq z)$

Remark:

1. Axioms (A10), (A11), (A12) and (A13) express that $\leq$ is a totally ordered relation (see algebra course).

2. From the relationship (less than or equal to $\leq$) defined above, we can define its symmetrical relationship (greater than or equal to $\geq$) as follows:

$$\text{For all real numbers } x,y \in \mathbb{R};\ \ x \geq y \text{ if and only if } y \leq x.$$

   The relation $\geq$ is also a totally ordered relation on $\mathbb{R}$.

3. We define the relationship (strictly inferior $<$) by:

$$\text{For any } x,y \in \mathbb{R},\ x < y \text{ if and only if } (x \leq y) \text{ and } (x \neq y).$$

   and the relationship (strictly greater $>$) by:

$$\text{For every } x,y \in \mathbb{R},\ x > y \text{ if and only if: } (x \geq y) \text{ and } (x \neq y).$$

Before stating the least-upper-bound axiom (A15) sometimes called completeness property or supremum property, we need the following definitions

Definition 1.1: (Upper and lower bounds of a set)

Let E be a non-empty subset of $\mathbb{R}$ ($E \subset \mathbb{R}$).

- A subset E is said to be **bounded from above** (or right bounded) iff there exists $M \in \mathbb{R}$ such that:
$$\forall x \in E : x \leq M$$
  In this case, the real number M is called **an upper bound** of E.
  The set of all upper bounds of E is noted by: **Upper(E)**

- A subset E is said to be **bounded from below** (or left bounded) iff there exists $m \in \mathbb{R}$ such that:
$$\forall x \in E : m \leq x$$
  In this case, the real number m is called **a lower bound** of E. The set of all lower bounds of E is noted by: **Lower(E)**

- A subset E is said to be **bounded** iff there exists m and M such that: for any $x \in E$, $m \leq x \leq M$.

Examples:

1. $E =]4,5[$
   3 and 4 are both lower bounds of E since: for all $x \in E$, $4 \leq x$ et $3 \leq x$.
   5 and 6 are two upper bounds of E since: for any $x \in E$, $x \leq 5$ and $x \leq 6$.

2. $E = \{-2, -1, 0, 1, 4, 6\}$
   -2 is a lower bound of E since: for all $x \in E$, $-2 \leq x$.
   6 is an upper bound of E since: for every $x \in E$, $x \leq 6$.

Remark:

1. The upper and lower bound of a set E are not unique. In fact, in $\mathbb{R}$ the set E=]4,5[has an infinite number of lower and upper bounds.

2. The upper and lower bound of a set E may or may not belong to E. For example, if the set $E = \{-2, -1, 0, 1, 4, 6\}$, then -2 and -4 are both lower bounds of E, -2 belongs to E and -4 does not belong to E.

Définition 1.2: (Minimum and maximum of a set)

Let E be a non-empty subset of $\mathbb{R}$ ($E \subset \mathbb{R}$).

- The lower bound of E that belongs to E is called **the smallest element** or (**minimum** of E. This is denoted by $\min(E)$. In other words:

$$m = \min(E) \Leftrightarrow \begin{cases} m \in Lower(E). \\ \quad and \\ m \in E \end{cases} \Leftrightarrow m \in Lower(E) \cap E$$

- The upper bound of E that belongs to E is called the **greatest element** or **maximum** of E. This is denoted by $\max(E)$. In other words:

$$M = \max(E) \Leftrightarrow \begin{cases} M \in Upper(E). \\ \quad and \\ M \in E \end{cases} \Leftrightarrow M \in Upper(E) \cap E$$

Examples:

1. $E = [5,20]$
   Since 5 is a lower bound and belongs to E, then $\min(E) = 5$.
   $\max(E) = 20$ as 20 is an upper bound of E and 20 belongs to E.

2. $E =]0,6[$
   $\min(E)$ does not exist because there is no lower bound of E that belongs to E.
   $\max(E)$ does not exist, since there is no upper bound of E that belongs to E.

Remark:

1. If $\min(E)$ exists then it is unique.

2. If $\max(E)$ exists then it is unique.

Definition 1.3: (The least upper bound and the greatest lower bound)

- The greatest element in lower bounds of E is called the infimum of E and is noted by: $\inf(E)$. in other words :

$$m = \inf(E) \Leftrightarrow m = \max(Lower(E))$$

- The smallest element in upper bounds of E is called the supremum of E and is noted by: $\sup(E)$. in other words :

$$M = \sup(E) \Leftrightarrow M = \min(Upper(E))$$

Finally, we can state the least-upper-bound axiom as follows:

1.2.3 The least-upper-bound axiom

A15. For every non-empty subset $E \subset \mathbb{R}$ and bounded above, has a $\sup(E)$ in $\mathbb{R}$

Consequence:

For every non-empty subset $E \subset \mathbb{R}$ and bounded below, has a $\inf(E)$ in $\mathbb{R}$

1.3 Some fundamental properties of $\mathbb{R}$

The following properties are consequences of the preceding axioms

1.3.1 Inequalities

Let x,y,z,$t \in \mathbb{R}$ on a:

1. If $x \leq y$ then $x - z \leq y - z$
2. If $x \leq y$ then
$$\begin{cases} x.z \leq y.z & if\, z \geq 0 \\ x.z \geq y.z & if\, z \leq 0 \end{cases}$$
3. If $x \leq y$ then
$$\begin{cases} x^2 \leq y^2 & if\, 0 \leq x \leq y \\ y^2 \leq x^2 & if\, x \leq y \leq 0 \end{cases}$$
4. If $0 < x \leq y$ then $0 < \frac{1}{y} \leq \frac{1}{x}$
5. If $x \leq y < 0$ then $\frac{1}{y} \leq \frac{1}{x} < 0$
6. If $x \leq y$ with $x < 0$ and $y > 0$ then $\frac{1}{x} < \frac{1}{y}$
7. If $0 \leq x \leq 1$ then $0 \leq x^n \leq x^{n-1} \leq .... \leq x^2 \leq x \leq 1$ For all $n \in \mathbb{N}^*$
8. If $1 \leq x$ then $1 \leq x \leq x^2 \leq ...... \leq\leq x^n$ pour tout $n \in \mathbb{N}^*$
9. If $x \leq y$ and $z \leq t$ then $x + y \leq y + t$
10. If $x \leq y$ and $z \leq t$ with $x \geq 0$ and $z \geq 0$ then $x.z \leq y.t$

Definition1.4: (Absolute value)

The absolute value of a real x, denoted by $|x|$, is defined as follows:

$$|x| = \begin{cases} x & \text{si}\, x \geq 0 \\ -x & \text{si}\, x < 0 \end{cases}$$

1.3.2 Absolute value properties

Let $x,y \in \mathbb{R}$ we have:

1. $|x| \geq 0$ (The absolute value is always positive)
2. $|-x| = |x|$
3. $|x| \geq x$ et $|x| \geq -x$
4. $|x| = max(-x,x)$
5. $|x| = 0 \Leftrightarrow x = 0$
6. Let $\alpha \geq 0$ then: $|x| \leq \alpha \Leftrightarrow -\alpha \leq x \leq \alpha$
7. $|x.y| = |x|.|y|$
8. $|\frac{x}{y}| = \frac{|x|}{|y|}$ si $y \neq 0$
9. $||x| - |y|| \leq |x + y| \leq |x| + |y|$
10. $||x| - |y|| \leq |x - y| \leq |x| + |y|$

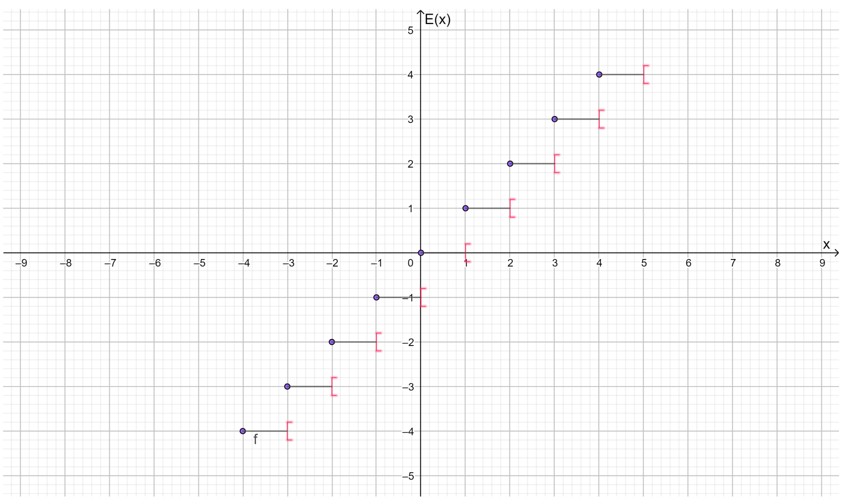
Définition 1.5: (Integer part of a real number)

Let x be a real number. The greatest integer less than or equal to x is called the integer part of x. We denote it by $E(x)$ or $\lfloor x \rfloor$.

Examples:

$\lfloor 11{,}12 \rfloor = 11$, $\lfloor \sqrt{3} \rfloor = 1$, $\lfloor -4{,}33 \rfloor = -5$, $\lfloor -7 \rfloor = -7$.

The following figure shows an integer function $f(x) = E(x)$



1.3.3 Properties of the integer part of a real number

1. For all $x \in \mathbb{R}$ we have: $\lfloor x \rfloor \leq x < \lfloor x+1 \rfloor$
2. For all $x \in \mathbb{R}$ we have: $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ with $n \in \mathbb{N}$
3. For all $x,y \in \mathbb{R}$ we have: $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$

Remark:

For any $x,y \in \mathbb{R}$ we have:

$$\lfloor x+y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor x \rfloor \\ \quad \text{or} \\ \lfloor x \rfloor + \lfloor y \rfloor + 1 \end{cases}$$

1.3.4 Characterization of the sup and inf

Theorem: Let $E \subset \mathbb{R}$ such that $E \neq \phi$ we have:

1. $M = \sup(E) \Leftrightarrow$
$$\begin{cases} \forall x \in E, x \leq M \\ \forall \varepsilon > 0, \exists x^* \in E, M - \varepsilon < x^* \end{cases}$$
2. $m = \inf(E) \Leftrightarrow$
$$\begin{cases} \forall x \in E, m \leq x \\ \forall \varepsilon > 0, \exists x^* \in E, x^* < m + \varepsilon \end{cases}$$

Remarks:

- If E has a maximum then $\sup(E) = \max(E)$
- If E admits a minimum then $\inf(E) = \min(E)$
- If $\inf(E) \in E$ then $\inf(E) = \min(E)$
- If $\sup(E) \in E$ then $\sup(E) = \max(E)$

1.3.5 Archimedean property

$\mathbb{R}$ satisfies the following Archimedean property:

$\forall a,b \in \mathbb{R}$ (with $b > 0$) then there exists $n \in \mathbb{N}$ such that: $bn > a$

1.3.6 Density of $\mathbb{Q}$ in $\mathbb{R}$

Between every two distinct real numbers a,b there exists a rational number q, i.e.:

$$\forall a,b \in \mathbb{R}\,(\text{avec}\, a < b), \exists q \in \mathbb{Q} \quad \text{tq:}\, a < q < b$$

In this case we say that $\mathbb{Q}$ is dense in $\mathbb{R}$.

1.4 Intervals in $\mathbb{R}$

Définition 1.6:(Interval)

Let I be a subset of $\mathbb{R}$
I is an interval in $\mathbb{R}$ if and only if:

$$\forall x,y \in I, \forall c \in \mathbb{R}; x \leq c \leq y \implies c \in I$$

Remarks:

1. The intersection of two intervals in $\mathbb{R}$ is an interval in $\mathbb{R}$.
2. The union of two intervals in $\mathbb{R}$ **not disjoint** is an interval in $\mathbb{R}$.
3. The union of two intervals in $\mathbb{R}$ **disjoint** is not an interval in $\mathbb{R}$.

Intervals in $\mathbb{R}$ can be classified into 9 kinds, as shown in the table below.
Let a,b be two real numbers such that $a < b$
.

Description	Definition	Notation
closed and bounded=segment	$\{x \in \mathbb{R}/a \leq x \leq b\}$	$[a,b]$
bounded and semi-open on the right	$\{x \in \mathbb{R}/a \leq x < b\}$	$[a,b[$
bounded and semi-open on the left	$\{x \in \mathbb{R}/a < x \leq b\}$	$]a,b]$
A bounded open	$\{x \in \mathbb{R}/a < x < b\}$	$]a,b[$
closed not bounded from above	$\{x \in \mathbb{R}/a \leq x\}$	$[a,+\infty[$
open not bounded from above	$\{x \in \mathbb{R}/a < x\}$	$]a,+\infty[$
closed not bounded from below	$\{x \in \mathbb{R}/x \leq b\}$	$]-\infty,b]$
open not bounded from below	$\{x \in \mathbb{R}/x < b\}$	$]-\infty,b[$
real line	$\mathbb{R}$	$]-\infty,+\infty[$

1.4.1 Carcterization of bounded parts in $\mathbb{R}$

Lemma:
Let E be a non-empty subset in $\mathbb{R}$, the following propositions are equivalent

1. E is bounded in $\mathbb{R}$.
2. There exists a bounded interval I in $\mathbb{R}$ such that: $E \subset I$
3. $\exists M \geq 0$ such that, $\forall x \in E, |x| \leq M$

Definition 1.7: Neighbourhood of a point

Let x be a real number. We say that $V \subset \mathbb{R}$ is a neighborhood of x if and only if there exists $\varepsilon > 0$ such that: $]x-\varepsilon, x+\varepsilon[\subset V$

Remark:

We say that $V \subset \mathbb{R}$ is a neighborhood of $+\infty$ (respectively $-\infty$) if and only if there exists $a \in \mathbb{R}$ such that: $]a,+\infty[\subset V$ (respectively $]-\infty,a[\subset V$)

Consequence:

Any non-empty interval I in $\mathbb{R}$ contains an infinite number of rationals.

Chapter's exercises with answers

Exercise 1

Given $x, y, z \in \mathbb{R},\ \alpha \in \mathbb{R}^{+}$, Prove the following inequalities:

1. $|x| \leq \alpha \Leftrightarrow -\alpha \leq x \leq \alpha$
2. $|x+y| \leq |x| + |y|$,
3. $||x| - |y|| \leq |x - y|$
4. $\sqrt{x^2+y^2} \leq |x| + |y|$
5. $\frac{1}{2}(x^2+y^2) \geq xy$
6. $xy + xz + yz \leq x^2 + y^2 + z^2$
7. $(\forall \epsilon > 0, |x| \leq \epsilon) \Longrightarrow x = 0$

Correction1

1. Let $x \in \mathbb{R},$ and $\alpha \geq 0$, We have

$$\begin{aligned} |x| \leq \alpha \iff & \quad 0 \leq |x| \leq \alpha \\ \iff & \quad |x|^2 \leq \alpha^2 \\ \iff & \quad x^2 \leq \alpha^2 \\ \iff & \quad x^2 - \alpha^2 \leq 0 \\ \iff & \quad (x-\alpha)(x+\alpha) \leq 0 \\ \iff & \quad x \in [-\alpha; \alpha] \\ \iff & \quad -\alpha \leq x \leq \alpha \end{aligned}$$

2. Let $x, y \in \mathbb{R}$, then obviously:

$$\begin{cases} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{cases}$$

By adding them together we find:

$$-(|x|+|y|) \leq x+y \leq |x|+|y|, \text{ ce qui implique que}$$

$$|x+y| \leq |x| + |y|$$

3. Let $x, y \in \mathbb{R}$; As we know that: $xy \leq |x||y|$, then:

$$\begin{aligned} xy \leq |x||y| \iff & \quad -2xy \geq -2|x||y| \\ \iff & \quad x^2+y^2-2xy \geq x^2+y^2-2|x||y| \\ \iff & \quad x^2+y^2-2xy \geq |x|^2+|y|^2-2|x||y| \\ \iff & \quad (x-y)^2 \geq (|x|-|y|)^2 \\ \iff & \quad \sqrt{(x-y)^2} \geq \sqrt{(|x|-|y|)^2} \\ \iff & \quad |x-y| \geq ||x|-|y||. \end{aligned}$$

4. Let $x, y \in \mathbb{R}$, we have:

$$\begin{aligned} 2|x||y| \geq 0 \iff & \quad x^2+y^2+2|x||y| \geq x^2+y^2 \\ \iff & \quad |x|^2+|y|^2+2|x||y| \geq x^2+y^2 \\ \iff & \quad (|x|+|y|)^2 \geq x^2+y^2 \\ \iff & \quad \sqrt{(|x|+|y|)^2} \geq \sqrt{x^2+y^2} \\ \iff & \quad |x|+|y| \geq \sqrt{x^2+y^2} \end{aligned}$$

5. Let $x, y \in \mathbb{R}$, we have:

$$\begin{aligned}(x-y)^2 \geq 0 &\iff x^2+y^2-2xy \geq 0\\ &\iff x^2+y^2 \geq 2xy\\ &\iff \frac{1}{2}(x^2+y^2) \geq xy.\end{aligned}$$

6. According to the previous question, we have :

$$\begin{cases} x^2/2 + y^2/2 & \geq xy\\ x^2/2 + z^2/2 & \geq xz\\ z^2/2 + y^2/2 & \geq zy \end{cases}$$

By adding them together we find:

$$xy + xz + yz \leq x^2 + y^2 + z^2$$

7. By Contradiction, let's suppose that: $\forall \epsilon > 0, |x| \leq \epsilon$ et $x \neq 0$ which implies, $\forall \epsilon > 0, 0 < |x| \leq \epsilon$.
For $\epsilon = \dfrac{|x|}{2}$, we find a contradiction $1 \leq \dfrac{1}{2}$

Exercise 2

Show that:

1. $\sqrt{3}$ is irrational
2. for all $(a, b) \in \mathbb{Q} \times \mathbb{Q}^*$, the numbers $a + b\sqrt{3}$ are irrational.
3. $\dfrac{\ln 3}{\ln 2}$ is irrational.

Correction2

1. By contradiction, let's suppose that: $\sqrt{3}$ is a rational number, then $\sqrt{3}$ can be written as: $\sqrt{3} = \dfrac{m}{n}$, with $m \in \mathbb{N}, n \in \mathbb{N}^*$ and the Greatest Common Factor $GCF(m, n) = 1$, which implies that

$$3n^2 = m^2 \tag{1}$$

which means that m^2 is divisible by 3.
Remark: m can only be written in one of the three following forms: $m \equiv 0[3]$,$m \equiv 1[3]$,$m \equiv 2[3]$ which gives $m^2 \equiv 0[3]$,$m^2 \equiv 1[3]$,$m^2 \equiv 2^2 = 1[3]$. The suitable hypothesis is 0nly the result of the first case . So :If m^2 is divisible by 3, then m is also divisible by 3 . As a result, there exists $k \in \mathbb{N} : m = 3k$. Substitute m in (1), we obtain: $n^2 = 3k^2$ which implies that n is divisible by 3 (contradiction with $GCF(m, n) = 1$). Therfore $\sqrt{3}$ is irrational number.

2. By Contradiction, we assume that there exist two numbers x, y with $(x, y) \in \mathbb{Q} \times \mathbb{Q}^*$ and $X = x + y\sqrt{3} \in \mathbb{Q}$ This hypothesis leads to a contradiction "$\sqrt{3} \in \mathbb{Q}$" because:

$$\begin{cases} X = x + y\sqrt{3} \in \mathbb{Q}\\ x \in \mathbb{Q} \end{cases} \Rightarrow \begin{cases} X = x + y\sqrt{3} \in \mathbb{Q}\\ -x \in \mathbb{Q} \end{cases} \Rightarrow \begin{cases} X - x = y\sqrt{3} \in \mathbb{Q}\\ \text{from hypot } y \in \mathbb{Q}^* \end{cases} \Rightarrow \begin{cases} y\sqrt{3} \in \mathbb{Q}\\ \dfrac{1}{y} \in \mathbb{Q}^* \end{cases}$$

$\Rightarrow y\sqrt{3} \times \dfrac{1}{y} = \sqrt{3} \in \mathbb{Q}$

3. By contradiction, let's suppose that: $\dfrac{\ln 3}{\ln 2}$ is a rational number, then we ca write $\dfrac{\ln 3}{\ln 2} = \dfrac{n}{m}$ with $n \in \mathbb{N}^*, m \in \mathbb{N}^*$, so $m \ln 3 = n \ln 2 \Rightarrow \ln 3^m = \ln 2^n$. Since ln is an injective function, it follows that: $3^m = 2^n \Rightarrow 3^m \equiv 0\,[2]$, contradiction with $3 \equiv 1\,[2] \Rightarrow 3^m \equiv 1\,[2]$. Finally $\dfrac{\ln 3}{\ln 2}$ is irrational.

Exercise 3

Justify whether the following assertions are true or false :

a. The sum, the product of two rational numbers, the reciprocal of a non-zero rational number is a rational number.

b. The sum or product of two irrational numbers is an irrational.

c. The sum of a rational number and an irrational number is an irrational.

d. The product of a rational number and an irrational number is an irrational.

Correction3

Let $X \in \mathbb{R}$, Recall that X is rational if and only if there exists $n \in \mathbb{Z}, m \in \mathbb{N}^* : X = \dfrac{n}{m}$.

a **True:** Let $X = \dfrac{n}{m}$, and $Y = \dfrac{p}{q}$ with $n, p \in \mathbb{Z},$ and $m, q \in \mathbb{N}^*$ two rational numbers, then:

$$\begin{cases} X + Y = \dfrac{n}{m} + \dfrac{p}{q} = \dfrac{qn + mp}{mq} \in \mathbb{Q}, \text{ because } qn + mp \in \mathbb{Z}, \text{ et } mq \in \mathbb{N}^* \\ X \times Y = \dfrac{n}{m} \times \dfrac{p}{q} = \dfrac{np}{mq} \in \mathbb{Q}, \text{ because } np \in \mathbb{Z}, \text{ and } mq \in \mathbb{N}^* \\ \text{For } n \neq 0, \text{ inverse of } X \text{ is } \dfrac{1}{X} = \dfrac{m}{n} \in \mathbb{Q} \end{cases}$$

b **False:** Let $x = \sqrt{3}$ (irrational number), $y = -\sqrt{3}$ (irrational number), but $x + y = 0 \in \mathbb{Q}$ et $xy = -3 \in \mathbb{Q}$.

c **True:** By cotradiction: let $x \in \mathbb{Q}$ and $y \in \mathbb{R}\backslash\mathbb{Q}$, suppose that $x + y \in \mathbb{Q}$

$$\begin{aligned} \text{Since } x \in \mathbb{Q} &\Longrightarrow -x \in \mathbb{Q} \\ &\Longrightarrow -x + (x + y) = y \in \mathbb{Q} \text{ Contradiction with } y \in y \in \mathbb{R}\backslash\mathbb{Q} \end{aligned}$$

d **False** As $0 \in \mathbb{Q}$ and $\sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$, but $0\sqrt{2} = 0 \in \mathbb{Q}$

e **True** Let $x \in \mathbb{Q}^*$ and $y \in \mathbb{R}\backslash\mathbb{Q}$, by contradiction, let's assume that $xy \in \mathbb{Q}$

$$\begin{aligned} x \in \mathbb{Q}^*, xy \in \mathbb{Q} &\Longrightarrow (1/x) \in \mathbb{Q}^* \wedge xy \in \mathbb{Q} \\ &\Longrightarrow (1/x) \times xy \in \mathbb{Q} \\ &\Longrightarrow y \in \mathbb{Q} \end{aligned}$$

Contradiction with $y \in \mathbb{R}\backslash\mathbb{Q}$

Exercise 4

Let $x, y \in \mathbb{R}$, show that:

1. $f(x) = E(x)$ is an increasing function
2. $E(x) + E(y) \leq E(x+y) \leq E(x) + E(y) + 1$
3. $\forall n \in \mathbb{N}^*$, $E(\frac{E(nx)}{n}) = E(x)$

Correction4

1. Let $x, y \in \mathbb{R}$ such that: $x < y$

$$\text{We have } E(y) \leq y < E(y) + 1, \text{ and } x < y \Rightarrow x < y < E(y) + 1$$
$$\Rightarrow E(x) \leq x < E(y) + 1$$
$$\Rightarrow E(x) < E(y) + 1$$

Since $E(x) \in \mathbb{Z}$, $E(y) \in \mathbb{Z}$; which implies that: $E(x) \leq E(y)$

2. Let $x, y \in \mathbb{R}$, we have:

$$\begin{cases} E(x) & \leq x < E(x) + 1 \\ E(y) & \leq y < E(y) + 1 \end{cases} \Rightarrow E(x) + E(y) \leq x + y < E(x) + E(x) + 2$$

We know that

$$\begin{cases} E(x+y) \text{ is the largest integer less than or equal to } x+y \\ E(x+y)+1 \text{ is the smallest integer strictly greater than } x+y \end{cases}$$

Then
$$E(x) + E(y) \leq E(x+y) \leq x + y < E(x+y) + 1 \leq E(x) + E(x) + 2$$

Consequently:
$$E(x) + E(y) \leq E(x+y) \leq E(x) + E(x) + 1$$

3. We have
$$\forall x \in \mathbb{R} : E(x) \leq x$$

We multiply both members of the inequality by $n \in \mathbb{N}^*$, we get: $nE(x) \leq nx$. Using the fact that $E(nx)$ is the largest integer less than or equal to nx we obtain:

$$nE(x) \leq E(nx) \Rightarrow E(x) \leq \frac{E(nx)}{n} \tag{2}$$

Using the fact that E is increasing function, the result (2) gives: $E(x) \leq E\left(\frac{E(nx)}{n}\right)$.
For the inverse we have:

$$E\left(\frac{E(nx)}{n}\right) \leq \frac{E(nx)}{n} \Rightarrow nE\left(\frac{E(nx)}{n}\right) \leq E(nx)$$
$$\Rightarrow nE\left(\frac{E(nx)}{n}\right) \leq nx$$
$$\Rightarrow E\left(\frac{E(nx)}{n}\right) \leq x$$

Using the fact that $E(x)$ is the largest integer less than or equal to x, we find

$$E\left(\frac{E\left(nx\right)}{n}\right) \leq E(x).$$

Consequently

$$E\left(\frac{E\left(nx\right)}{n}\right) = E(x)$$

Exercise 5

For each of the following sets, describe the set of all upper bounds for the set :

1. the set of odd integers;
2. $\left\{1 - \frac{1}{n} : n \in \mathbb{N}^*\right\}$;
3. $\{r \in \mathbb{Q} : r^3 < 8\}$;
4. $\{\sin x : x \in \mathbb{R}\}$

Correction5

1. $Upper(\mathbb{Z}) = \emptyset$
2. $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}^*\right\}$, it is clear that $[1, +\infty[\subset Upper(A)$
   Let us suppose that there exists $\alpha \in Upper(A) \wedge \alpha < 1$, according to Archimedes' property, there exists $n \in \mathbb{N}$ such that: $n > \dfrac{1}{1-\alpha}$. on the other hand:
   $$\begin{array}{rcl} n > \frac{1}{1-\alpha} & \Rightarrow & \frac{1}{n} < 1 - \alpha \\ & \Rightarrow & \frac{1}{n} + \alpha < 1 \\ & \Rightarrow & \alpha < 1 - \frac{1}{n} \end{array}$$
   contradiction with $\alpha \in Upper(A)$, as a result: $Upper(A) = [1, +\infty[$
3. A=$\{r \in \mathbb{Q} : r^3 < 8\}$=$\{r \in \mathbb{Q} : r < 2\} \Rightarrow [2, +\infty[\subset Upper(A)$
   Let us suppose that there exists $\alpha \in Upper(A) \wedge \alpha < 2$, from the density of $\mathbb{Q}$ in $\mathbb{R}$ there exists a rational number r such that $\alpha < r < 2$, contradiction with $\alpha \in Upper(A)$, as a result: $Upper(A) = [2, +\infty[$
4. $\{\sin x : x \in \mathbb{R}\} = [-1, 1] \Rightarrow Upper(A) = [1, +\infty[$

Exercise 6

For each of the sets in (1),(2),(3) of the preceding exercise, find the least upper bound of the set, if it exists.

Correction6

1. the sup does not exist
2. $\sup(A) = \min(Upper A) = 1$
3. $\sup(A) = \min(Upper A) = 2$
4. $\sup\{\sin x : x \in \mathbb{R}\} = \min(Upper A) = 1$

Exercise 7

Let A, B be two non-empty bounded parts of $\mathbb{R}$. Show that:

1. the subset $-A = \{-x,\ x \in A\}$ is bounded.
2. $\sup(-A) = -\inf(A)$
3. $\inf(-A) = -\sup(A)$
4. Si $A \subset B$, alors:
$$\begin{cases} \sup(A) & \leq \sup(B) \\ \inf(B) & \leq \inf(A) \end{cases}$$
5. $\sup(A \cup B) = \max(\sup(A), \sup(B))$
6. $\inf(A \cup B) = \min(\inf(A), \inf(B))$

Correction7

Let A be a non-empty subset of $\mathbb{R}$.

1.
$$\begin{aligned} A \text{ is bounded } &\Leftrightarrow A \text{ is bounded from below } \wedge A \text{ is bounded from above} \\ &\Leftrightarrow \exists(\alpha, \beta) \in \mathbb{R}^2 : \forall x \in A : \alpha \leq x \leq \beta \end{aligned}$$
As a result
$$\begin{aligned} \exists(\alpha, \beta) \in \mathbb{R}^2 : \forall x \in A : \alpha \leq x \leq \beta &\Leftrightarrow \exists(\alpha, \beta) \in \mathbb{R}^2 : \forall y = -x \in (-A) : -\beta \leq y \leq -\alpha \\ &\Leftrightarrow -A \text{ is bounded from above and below} \\ &\Leftrightarrow -A \text{ is bounded} \\ &\Leftrightarrow \text{ the existence of} \sup(-A) \text{ and } \inf(-A). \end{aligned}$$

2. According to Sup's Theorem we have
$$\begin{aligned} \sup(-A) = a &\Leftrightarrow \begin{cases} \forall x \in A & : -x \leq \alpha \\ \forall \epsilon > 0, \exists x_\epsilon \in A & : a - \epsilon < -x_\epsilon \leq a \end{cases} \\ &\Leftrightarrow \begin{cases} \forall x \in A & : x \geq -\alpha \\ \forall \epsilon > 0, \exists x_\epsilon \in A & : -a \leq x_\epsilon < -a + \epsilon \end{cases} \\ &\Leftrightarrow -a = \inf A \end{aligned}$$

3. According to the previous question (2)
$$\inf(-A) = -\sup -(-A) = -\sup A$$

4. (a) We have
$$\forall x \in B : x \leq \sup B$$
since $A \subset B$:
$$\forall x \in A : x \leq \sup B$$
hence $\sup B$ is an upper bound for the set A which implies that $\sup A \leq \sup B$

(b) We have

$$\forall x \in B : x \geq \inf B$$

since $A \subset B$:

$$\forall x \in A : x \geq \inf B$$

hence $\inf B$ is a lower bound for the set A which implies that $\inf A \geq \inf B$

5. From (4.a), we have:

$$\begin{cases} A \subset A \cup B \\ B \subset A \cup B \end{cases} \Rightarrow \begin{cases} \sup A \leq \sup(A \cup B) \\ \sup B \leq \sup(A \cup B) \end{cases} \Rightarrow \max(\sup A, \sup B) \leq \sup(A \cup B)$$

On the other hand

$$\forall x \in A \cup B : \begin{cases} x & \leq \sup A \\ \vee & \\ x & \leq \sup B \end{cases} \Rightarrow x \leq \max(\sup A, \sup B)$$

hence $\max(\sup A, \sup B)$ is a an upper bound of the set $A \cup B$ which implies that:

$$\sup(A \cup B) \leq \max(\sup A, \sup B)$$

Consequently

$$\sup(A \cup B) = \max(\sup A, \sup B)$$

6. From (4.b)

$$\begin{cases} A \subset A \cup B \\ B \subset A \cup B \end{cases} \Rightarrow \begin{cases} \inf A \geq \inf(A \cup B) \\ \inf B \geq \inf(A \cup B) \end{cases} \Rightarrow \min(\inf A, \inf B) \geq \inf(A \cup B)$$

In addition

$$\forall x \in A \cup B : \begin{cases} x & \geq \inf A \\ \vee & \\ x & \geq \inf B \end{cases} \Rightarrow x \geq \min(\inf A, \inf B)$$

hence $\min(\inf A, \inf B)$ is a lower bound of the set $A \cup B$ which implies that $\inf(A \cup B) \geq \min(\inf A, \inf B)$. Consequently

$$\inf(A \cup B) = \min(\inf A, \inf B)$$

Exercise 8

Determine (if they exist) $\sup, \inf, \max, \min$ of the following sets :

1. $A = [1, 2] \cap \mathbb{Q}$
2. $B = [1, 2[\cap \mathbb{Q}$
3. $C = \left\{ v_n = \frac{1}{n+1},\ n \in \mathbb{N} \right\}$
4. $D = \{x \in \mathbb{R} :\ x^2 \leq 3\}$
5. $E = \{x \in \mathbb{R} :\ |x| > 1\}$
6. $F = \{x \in \mathbb{R} :\ |x^2 - 1| > 1\}$

Correction8

1. It is clear that $A = [1,2] \cap \mathbb{Q} \subset [1,2]$ which implies that $\forall x \in A : 1 \leq x \leq 2$, consequently A is bounded, so the existence of the sup and inf is evident. Now we will compute them

$$\begin{cases} 2 \in A \\ 2 \text{ is an upper bound} \end{cases} \Rightarrow \sup A = \max A = 2$$

$$\begin{cases} 1 \in A \\ 1 \text{ is a lower bound} \end{cases} \Rightarrow \inf A = \min A = 1$$

2. We have $A = [1,2[\cap\mathbb{Q} \subset [1,2[$ which means that $\forall x \in A : 1 \leq x < 2$, consequently A is bounded, so the sup and inf exist. Now let's compute them:

   (a) the same reasoning with 1 gives: l'inf $A = \min A = 1$.

   (b) To compute the sup(A), we have: $Upper(A) = [2, +\infty[$ is the set of all upper bounds of A and as $\inf([2, +\infty[) = 2$ which shows that $\sup A = 2$ and since $2 \notin A$, so the max doesn't exist.

3. We have:

$$\forall n \in \mathbb{N} : n \geq 0 \Rightarrow n+1 \geq 1 \Rightarrow 0 < \frac{1}{n+1} \leq 1 \Rightarrow \forall n \in \mathbb{N} : 0 < v_n \leq 1. \tag{3}$$

   This implies that C is bounded, hence the sup and inf exist. Now let's compute them:

   (a) For $n = 0$, we find $v_0 = 1$, and since 1 is an upper bound of the set C, consequently $\max(C) = 1 = \sup(C)$.

   (b) From the inequality (3), 0 is a lower bound of the set C, which shows that

$$\inf(C) \geq 0$$

   Suppose that $\inf(C) > 0$, by applying Archimedes' theorem with $a = 1$ and $b = \inf(C) > 0$, we obtain the existence of $k \in \mathbb{N}$ such that:

$$1 < (k+1)(\inf(C)) \Rightarrow \frac{1}{k+1} < \inf(C) \Rightarrow v_k < \inf(C) \leq v_n, \forall n \in \mathbb{N}$$

   Contradiction with $n = k$. $\{v_k < \inf(C) \leq v_k\}$ As a result $\inf(C) = 0$

4. We have

$$\begin{aligned} x \in D &\Leftrightarrow x^2 \leq 3 \\ &\Leftrightarrow x^2 - 3 \leq 0 \\ &\Leftrightarrow (x-3)(x+3) \leq 0 \\ &\Leftrightarrow x \in [-3, 3]. \end{aligned}$$

   which implies that

$$\begin{cases} \inf(D) = \min(D) & = -3 \\ \sup(D) = \max(D) & = 3 \end{cases}$$

5. We have

$$\begin{aligned} x \in E &\Leftrightarrow |x| > 1(\text{ as the function } z \to z^2 \text{ is increasing on } \mathbb{R}^+, \text{ hence}) \\ &\Leftrightarrow x^2 > 1 \\ &\Leftrightarrow x^2 - 1 > 0 \\ &\Leftrightarrow (x-1)(x+1) > 0 \\ &\Leftrightarrow x \in]-\infty, -1[\cup]1, +\infty[\end{aligned}$$

   This implies that E is neither bounded from above nor from below, so E is not bounded.

6. We have

$$\begin{array}{lll} x \in E & \Leftrightarrow & |x^2-1| > 1 \text{ as the function } x \to x^2 \text{ is increasing, hence} \\ & \Leftrightarrow & (x^2-1)^2 > 1 \\ & \Leftrightarrow & (x^2-1)^2 - 1 > 0 \\ & \Leftrightarrow & (x^2-1-1)(x^2-1+1) > 0 \\ & \Leftrightarrow & (x^2-2)\,x^2 > 0 \\ & \Leftrightarrow & (x^2-2) > 0 \\ & \Leftrightarrow & \left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right) > 0 \\ & \Leftrightarrow & x \in]-\infty, -\sqrt{2}[\cup]\sqrt{2}, +\infty[\end{array}$$

This implies that E is neither bounded from above nor from below, so E is not bounded.

Exercise 9

Let $a,\ b \in \mathbb{Q}$, with $a < b$. show that :

$$\exists c \in \mathbb{Q} :\ a < c < b$$

Correction9

We have

$$\left\{ \begin{array}{c} a \in \mathbb{Q}, b \in \mathbb{Q} \\ a < b \end{array} \right. \Rightarrow \left\{ \begin{array}{c} a \in \mathbb{Q}, b \in \mathbb{Q} \\ a + a < b + a \\ a + b < b + b \end{array} \right. \Rightarrow \left\{ \begin{array}{c} a \in \mathbb{Q}, b \in \mathbb{Q} \\ a < \dfrac{b+a}{2} \\ \dfrac{b+a}{2} < b \end{array} \right. \Rightarrow \left\{ \begin{array}{c} a \in \mathbb{Q}, b \in \mathbb{Q} \\ a < c = \dfrac{b+a}{2} < b \end{array} \right. \Rightarrow \left\{ \begin{array}{c} c \in \mathbb{Q} \\ a < c < b \end{array} \right.$$

Which shows the existence of $c \in \mathbb{Q} : a < c < b$

Chapter 2

Complex numbers

2.1 Algebraic form:

the set of complex numbers is created as an extension of the set of real numbers, containing in particular an imaginary number denoted i. This new number, combined with the real numbers, is the basis of the complex numbers. The appearance of these numbers has simplified the resolution of many physical problems. In particular, electronics and electrical engineering make extensive use of complex numbers.

Definition 2.1

Every number z can be uniquely written in the algebraic form $z = x + iy$ where $x,y \in \mathbb{R}$ and $i^2 = -1$. This is called the algebraic form of the complex number z. The real x is called the "real part of z" and is written $Re(z)$. The real y is called the "imaginary part of z" and is denoted by $Im(z)$. If $y = 0$ then $z \in \mathbb{R}$ is a real number, if $x = 0$ then z is a pure imaginary.

Example 2.1

1. $2 + 4i$ is a complex number whose real part is 2 and imaginary part is 4.
2. $\pi + \sqrt{2}i$ is a complex number which real part is π and imaginary part is $\sqrt{2}$.
3. 3i is pure imaginary.

Complex numbers follow the same rules as the four operations on real numbers (addition, subtraction, multiplication and division).

1. Equality: Two complex numbers $z = x + iy$ and $z^{prime} = a + ib$ are equal if (x, y) = (a, b). pay attention: there is no inequality in $\mathbb{C}$.

2. addition, multiplication: Let be two complex numbers $z = x + iy$ and $z' = a + ib$.

$$\begin{cases} z + z' = (x + a) + i(y + b) \\ zz' = (xa - yb) + i(xb + ay). \end{cases}$$

Definition 2.2: Conjugate Complex Numbers

Let $z = x+iy$ be any complex number. The complex number $x-iy$ is called the complex conjugate of z , and is denoted by $\bar{z}$.

Example 2.2

For example, the conjugate of $z = 2 - 3i$ est $\overline{z} = 2 + 3i$

Proposition 2.1

Let z and z' be two complex numbers then:

1. $\overline{\overline{z}} = z$
2. $\overline{z + z'} = \overline{z} + \overline{z'}$
3. $\overline{z.z'} = \overline{z}.\overline{z'}$
4. $z + \overline{z} = 2 \times Re(z)$
5. $z - \overline{z} = 2i \times Im(z)$
6. $\overline{(\frac{z}{z'})} = \frac{\overline{z}}{\overline{z'}}$ with $z' \neq 0$

2.2 Trigonometric Form of Complex Numbers :

Definition 2.3: Modulus of Complex Number

For any complex number $z = x + iy$, the real number $r = |z|$, defined by:

$$r = |z| = \sqrt{x^2 + y^2}$$

is called the modulus of z.

Example 2.3

The modulus of the complex number $z = 2 - \sqrt{2}i$ is $|z| = \sqrt{2^2 + (\sqrt{2})^2} = \sqrt{6}$.

Proposition 2.2

Let z and z' be two complex numbers then:

1. $|z| \geq 0$
2. $|z| = |\overline{z}|$
3. $z.\overline{z} = |z|^2$
4. $|z| = 0 \Longleftrightarrow z = 0$
5. $|z.z'| = |z||z'|$
6. $|Re(z)| \leq |z|$ and $|Im(z)| \leq |z|$
7. $|z^n| = |z|^n$, $n \in \mathbb{N}$
8. $|z + z'| \leq |z| + |z'|$ et $||z| - |z'|| \leq |z - z'|$

Definition 2.4: Argument of Complex Number

Every non-zero complex number $z = x + iy$ can be written in the form $z = |z|(\frac{x}{|z|} + \frac{y}{|z|}i)$,The argument of the complex number z, denoted $(arg(z))$, is the real number $\theta \in [0,2\pi[$ defined by :

$$\begin{cases} \cos\theta = \dfrac{x}{|z|} \\ \sin\theta = \dfrac{y}{|z|}. \end{cases}$$

where $|z|$ is the modulus of the complex number z .

Example 2.4

for $z = 1 + \sqrt{3}i$, we have:

$$\begin{aligned} z &= 1 + \sqrt{3}i \\ &= 2(\frac{1}{2} + \frac{\sqrt{3}}{2}) \\ &= 2(\cos\frac{\pi}{3} + \sin\frac{\pi}{3}) \end{aligned}$$

in this example $\arg(z) = \theta = \frac{\pi}{3}$ and $r = |z| = 2$

Proposition 2.3

Let z and z' be two complex numbers then:

1. $\arg(z.z') = \arg(z) + \arg(z')$
2. $\arg(\overline{z}) = -\arg(z)$
3. $\arg(\frac{1}{z}) = -\arg(z)$

Theorem 2.1: Trigonometric

Any non-null complex number z can be written as:

$$z = r(\cos\theta + i\sin\theta) \text{ with } r = |z| \text{ et } \theta = \arg(z) + 2k\pi,\ k \in \mathbb{Z}$$

2.3 Exponential Form

Remark 2.1

From proposition (2.3) and since the product of two exponentials is equal to the exponential of the sum. For this reason, we introduce the following notation:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Definition 2.5

Any non-zero complex number z can be written in exponential form

$$z = |z|e^{i\arg(z)}$$

Example 2.5

1. $e^{i\dfrac{\pi}{4}} = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}$

2. For $z = 1 + i\sqrt{3}$, we have:

$$\begin{aligned} z &= 1 + i\sqrt{3} \\ &= 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) \\ &= 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) \\ &= 2e^{i\dfrac{\pi}{3}} \end{aligned}$$

2.4 De Moiver's Theorem and Euler's Formula

De Moiver's Formula

For any real number θ and for any integer n:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Or

$$(e^{i\theta})^n = e^{in\theta}$$

Euler's Formulas

for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$ we have

1. $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

2. $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

3. $\cos nx = \dfrac{e^{inx} + e^{-inx}}{2}$

4. $\sin nx = \dfrac{e^{inx} - e^{-inx}}{2i}$

Linearization of trigonometric polynomials

It consists in transforming the powers $\cos^n(x)$, $\sin^n(x)$ into sums and multiples of expressions of the type $\sin(kx)$ and $\cos(k.x)$. To do this, we use Euler's formulas and Newton's binomial $(a + b)^n$.

Example 2.6

From Euler's formula

$$\boxed{\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)}$$

We have:

1.
$$\begin{cases} (a+b)^2 = & a^2 + b^2 + 2ab \\ \cos^2(x) = & \frac{1}{4}\left(e^{i2x} + e^{-i2x} + 2\right) \end{cases} \Rightarrow \cos^2(x) = \frac{1}{4}\left(2\cos(2x) + 2\right)$$

which implies that:

$$\boxed{\cos^2(x) = \frac{1}{2}\left(\cos(2x) + 1\right)}$$

2.
$$\begin{cases} (a+b)^3 = & a^3 + b^3 + 3a^2b + 3ab^2 \\ \cos^3(x) = & \frac{1}{8}\left(e^{i3x} + e^{-i3x} + 3e^{ix} + 3e^{-ix}\right) \end{cases} \Rightarrow \cos^3(x) = \frac{1}{8}\left(2\cos(3x) + 6\cos(x)\right)$$

which implies that:

$$\boxed{\cos^3(x) = \frac{1}{4}\left(\cos(3x) + 3\cos(x)\right)}$$

3.
$$\begin{cases} (a+b)^4 = & a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ \cos^4(x) = & \frac{1}{16}\left(e^{i4x} + e^{-i4x} + 4e^{i2x} + 4e^{-i2x} + 6\right) \end{cases}$$

$$\Rightarrow \cos^4(x) = \frac{1}{16}\left(2\cos(4x) + 8\cos(2x) + 6\right)$$

which implies that:

$$\boxed{\cos^4(x) = \frac{1}{8}\left(\cos(4x) + 4\cos(2x) + 3\right)}$$

4.
$$\begin{cases} (a+b)^5 = & a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\ \cos^5(x) = & \frac{1}{32}\left(e^{i5x} + e^{-i5x} + 5(e^{i3x} + e^{-i3x}) + 10(e^{ix} + e^{-ix})\right) \end{cases}$$

which implies that:

$$\boxed{\cos^5(x) = \frac{1}{16}\left(\cos(5x) + 5\cos(3x) + 10\cos(x)\right)}$$

5.
$$\begin{cases} (a+b)^6 = & a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \\ \cos^6(x) = & \frac{1}{64}\left(e^{i6x} + e^{-i6x} + 6(e^{i4x} + e^{-i4x}) + 15(e^{i2x} + e^{-i2x}) + 20\right) \end{cases}$$

which implies that:

$$\cos^6(x) = \frac{1}{32}\left(\cos(6x) + 6\cos(4x) + 15\cos(2x) + 10\right)$$

2.5 Square roots

Definition 2.6

The square root of a complex number a is any number b whose square is a

Example 2.7

a. $1+i$ and $-1-i$ are the square roots of $2i$ because $(1+i)^2 = (-1-i)^2 = 2i$.

b. -4 has two opposite square roots : $+2i, -2i$.

Remark 2.2

To determine the square roots of $z = x + iy$ it is sometimes simpler to proceed by identification, i.e. to find the real numbers α and β such that $(x+iy) = (\alpha+i\beta)^2$ we obtain:

$$\begin{cases} \alpha^2 - \beta^2 & = x \\ 2\alpha\beta & = y \\ \alpha^2 + \beta^2 & = \sqrt{x^2+y^2} \end{cases}$$

Definition 2.7

Let $n \in \mathbb{N}^*$, $a \in \mathbb{C}$. The complex number z such that $z^n = a$ is called an n-th root of a.

Example 2.8

$a = 2, b = -1 - i\sqrt{3}, c = -1 + i\sqrt{3}$: these are the cubic roots of 8 in $\mathbb{C}$, also known as the third roots of 8.

Chapter's exercises with answers

Exercise 1

Put the following complex numbers into algebraic form:

1. $\frac{1}{i}$
2. $\frac{2+i}{5-i}$
3. $\frac{i}{i+1}$
4. $\frac{1}{2-i}$
5. $\frac{1}{2+i}+\frac{1}{2-i}$
6. $\frac{1}{1+i}+\frac{i}{1+i}$

Correction 1

1. $\frac{1}{i}=\frac{1\times i}{i\times i}=\frac{i}{i^2}=\frac{i}{-1}=-i$
2. $\frac{2+i}{5-i}=\frac{(2+i)(5+i)}{(5-i)(5+i)}=\frac{10+2i+5i+i^2}{5^2-i^2}=\frac{9+7i}{26}=\frac{9}{26}+\frac{7}{26}i$
3. $\frac{i}{i+1}=\frac{i(i-1)}{(i+1)(i-1)}=\frac{i^2-i}{i^2-1}=\frac{-1-i}{-2}=\frac{1+i}{2}=\frac{1}{2}+\frac{1}{2}i$
4. $\frac{1}{2-i}=\frac{2+i}{(2-i)(2+i)}=\frac{2+i}{2^2-i^2}=\frac{2+i}{5}=\frac{2}{5}+\frac{1}{5}i$
5. $\frac{1}{2+i}=\frac{2-i}{(2+i)(2-i)}=\frac{2-i}{2^2-i^2}=\frac{2-i}{5}=\frac{2}{5}-\frac{1}{5}i$, and from (4) we have :

$$\begin{cases}\frac{1}{2-i}=\frac{2}{5}+\frac{1}{5}i\\ \frac{1}{2+i}=\frac{2}{5}-\frac{1}{5}i\end{cases}\Rightarrow\frac{1}{2-i}+\frac{1}{2+i}=\frac{2}{5}+\frac{1}{5}i+\frac{2}{5}-\frac{1}{5}i=\frac{4}{5}$$

6. $\frac{1}{1+i}=\frac{(1-i)}{(1+i)(1-i)}=\frac{1-i}{1^2-i^2}=\frac{1-i}{2}=\frac{1}{2}-\frac{1}{2}i$, and from (3) we have :

$$\begin{cases}\frac{1}{1+i}=\frac{1}{2}-\frac{1}{2}i\\ \frac{i}{i+1}=\frac{1}{2}+\frac{1}{2}i\end{cases}\Rightarrow\frac{1}{1+i}+\frac{i}{i+1}=\frac{1}{2}-\frac{1}{2}i+\frac{1}{2}+\frac{1}{2}i=\frac{1}{2}+\frac{1}{2}=1$$

Exercise 2

Determine the modulus and an argument for each complex number :

1. -2
2. $3i$
3. $2e^{-2i}$
4. $-1+i\sqrt{3}$
5. $\frac{1+i}{\sqrt{3}-1}$
6. $(\sqrt{3}-i)^9$

Correction 2

1. $|-2| = |-2+0i| = \sqrt{(-2)^2+0^2} = \sqrt{4} = 2$
2. $|3i| = |0+3i| = \sqrt{(0)^2+3^2} = \sqrt{9} = 3$
3. $$|2e^{-2i}| = |2|\,|e^{-2i}| \quad \text{and} \quad \text{as } \forall x \in \mathbb{R} : |e^{xi}| = 1$$
$$\Rightarrow \quad |2e^{-2i}| = |2| = |2+0i| = \sqrt{2^2+0^2} = 2$$
4. $\left|-1+i\sqrt{3}\right| = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{4} = 2$
5. $$\begin{aligned}\left|\frac{1+i}{\sqrt{3}-i}\right| &= \frac{|1+i|}{|\sqrt{3}-i|} \\ &= \frac{\sqrt{1^2+1^2}}{\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}} \\ &= \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}\end{aligned}$$
6. $$\left|\sqrt{3}-i\right| = \sqrt{3+1} = 2 \quad \Rightarrow \quad \left|(\sqrt{3}-i)^9\right| = 2^9$$

Exercise 3

Find the points of the complex plane which satisfy the following conditions.

1. $|z| \leq 2$
2. $z + \overline{z} = 1$
3. $|z - 3 + 5i| = 2$

Correction 3

1. $\{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \leq 2\} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 4\}$.
2. $\{(x,y) \in \mathbb{R}^2 : 2x = 1\} = \{(x,y) \in \mathbb{R}^2 : x = \frac{1}{2}\}$
3. $\{(x,y) \in \mathbb{R}^2 : \sqrt{(x-3)^2+(y+5)^2} = 2\} = \{(x,y) \in \mathbb{R}^2 : (x-3)^2+(y+5)^2 = 2\}$

Exercise 4

Let $\alpha \in \mathbb{R}$. Express $\cos 5\alpha$ as a function of $\cos\alpha$, then $\sin 5\alpha$ as a function of $\sin\alpha$, give the value of $\cos\frac{\pi}{10}$.

Correction 4

1. According to Newton's binomial we have:

$$\begin{aligned}
(\cos a + i\sin a)^5 &= \textstyle\sum_{k=0}^{k=5} \mathrm{C}_5^K (\cos a)^k (i\sin a)^{5-k} \\
&= \mathrm{C}_5^0(\cos a)^0(i\sin a)^5 + \mathrm{C}_5^1(\cos a)^1(i\sin a)^4 + \mathrm{C}_5^2(\cos a)^2(i\sin a)^3 \\
&+ \mathrm{C}_5^3(\cos a)^3(i\sin a)^2 + \mathrm{C}_5^4(\cos a)^4(i\sin a)^1 + \mathrm{C}_5^5(\cos a)^5(i\sin a)^0 \\
&= i\sin^5 a + 5\cos a\sin^4 a - 10i\cos^2 a\sin^3 a \qquad (1)\\
&- 10\cos^3 a\sin^2 a + 5i\cos^4 a\sin a + \cos^5 a \\
&= \cos^5 a - 10\cos^3 a\sin^2 a + +5\cos a\sin^4 a \\
&+ i[5\cos^4 a\sin a - 10\cos^2 a\sin^3 a + \sin^5 a]
\end{aligned}$$

2. On the other hand, according to Moiver we have:

$$(\cos a + i\sin a)^5 = \cos 5a + i\sin 5a \qquad (2)$$

3. From (1) and (2), we find:

$$\begin{cases} \cos 5a &= \cos^5 a - 10\cos^3 a\sin^2 a + +5\cos a\sin^4 a \\ \sin 5a &= 5\cos^4 a\sin a - 10\cos^2 a\sin^3 a + \sin^5 a \end{cases} \qquad (3)$$

4. From the first formula in (3), we have:

$$\begin{cases} \cos 5a &= \cos^5 a - 10\cos^3 a\sin^2 a + +5\cos a\sin^4 a \\ &= \cos^5 a - 10\cos^3 a(1-\cos^2 a) + 5\cos a(1-\cos^2 a)^2 \\ &= \cos^5 a - 10\cos^3 a + 10\cos^5 a + 5\cos a(1+\cos^4 a - 2\cos^2 a) \\ &= 16\cos^5 a - 20\cos^3 a + 5\cos a \end{cases}$$

5. For $a = \dfrac{\pi}{10}$, which implies that :

$$\begin{cases} \cos 5\dfrac{\pi}{10} &= \cos\dfrac{\pi}{2} = 0 \\ &\Rightarrow 16\cos^5(\dfrac{\pi}{10}) - 20\cos^3(\dfrac{\pi}{10}) + 5\cos(\dfrac{\pi}{10}) = 0 \\ &\Rightarrow \cos(\dfrac{\pi}{10})[16\cos^4(\dfrac{\pi}{10}) - 20\cos^2(\dfrac{\pi}{10}) + 5] = 0 \end{cases}$$

If we put $x = \cos^2(\frac{\pi}{10})$, we get: $16x^2 - 20x + 5 = 0$.

$$\begin{aligned} 16x^2 - 20x + 5 = 0 &\Leftrightarrow x = \tfrac{1}{8}(5 - \sqrt{5}) \vee x = \tfrac{1}{8}(5 + \sqrt{5}) \\ &\Leftrightarrow x \simeq 0,345 \vee x \simeq 0,904 \\ &\Leftrightarrow \sqrt{x} \simeq 0,587 < \cos(\frac{\pi}{4}) \vee \sqrt{x} \simeq 0,950 \\ \text{We know } \cos \searrow \text{ on } [0, \frac{\pi}{2}] &\Rightarrow \cos(\frac{\pi}{10}) > \cos(\frac{\pi}{4}) \end{aligned}$$

Therefore: $\cos(\frac{\pi}{10}) = \sqrt{\frac{1}{8}(5 + \sqrt{5})} \simeq 0,950$

Exercise 5

Find the square roots of the following complex numbers:

1. 5+i
2. 6-8i
3. $4\sqrt{3}$+i

Correction 5

$$\begin{aligned} (x+iy) \text{ is a square root of } (a+ib) &\Leftrightarrow (x+iy)^2 = a+ib \\ &\Leftrightarrow x^2 - y^2 + 2ixy = a + ib \wedge |x+iy|^2 = |a+ib| \\ &\Leftrightarrow \begin{cases} x^2 - y^2 & = a \\ 2xy & = b \\ x^2 + y^2 = \sqrt{a^2 + b^2} \end{cases} \end{aligned}$$

1.

For $z = 5 + i$, We have: $(x+iy)^2 = 5 + i$

$$\begin{aligned} &\Leftrightarrow x^2 - y^2 + 2ixy = 5 + i \wedge |x+iy|^2 = |a+ib| \\ &\Leftrightarrow \begin{cases} x^2 - y^2 & = 5 \\ 2xy & = 1 \Rightarrow (x > 0 \wedge y > 0) \vee (x < 0 \wedge y < 0) \\ x^2 + y^2 = \sqrt{25+1} & = \sqrt{26} \end{cases} \\ &\Leftrightarrow \begin{cases} x^2 & = \frac{5+\sqrt{26}}{2} \\ y^2 & = \frac{1}{4x^2} = \frac{1}{4}\frac{2}{5+\sqrt{26}} = \frac{1}{2(5+\sqrt{26})} \end{cases} \\ &\Leftrightarrow (x,y) = (\sqrt{\frac{5+\sqrt{26}}{2}}, \sqrt{\frac{5+\sqrt{26}}{2}}) \vee (x,y) = (-\sqrt{\frac{5+\sqrt{26}}{2}}, -\sqrt{\frac{5+\sqrt{26}}{2}}) \end{aligned}$$

2.

For $z = 6 - 8i$ We have: $(x+iy)^2 = 6 - 8i$

$$\Leftrightarrow \quad x^2 - y^2 + 2ixy = 6 - 8i \wedge |x+iy|^2 = |6-8i|$$

$$\Leftrightarrow \quad \begin{cases} x^2 - y^2 & = 6 \\ 2xy & = -8 \Rightarrow (x > 0 \wedge y < 0) \vee (x < 0 \wedge y > 0) \\ x^2 + y^2 = \sqrt{64+36} & = \sqrt{100} = 10 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x^2 & = \frac{10+6}{2} = 8 \\ y^2 & = \dfrac{64}{4x^2} = \dfrac{16}{x^2} = 2 \end{cases}$$

$$\Leftrightarrow \quad (x,y) = (\sqrt{8}, -\sqrt{2}) \vee (x,y) = (-\sqrt{8}, \sqrt{2})$$

3.

For $z = 4\sqrt{3} + i$ We have: $(x+iy)^2 = 4\sqrt{3} + i$

$$\Leftrightarrow \quad x^2 - y^2 + 2ixy = 4\sqrt{3} + i \wedge |x+iy|^2 = \left|4\sqrt{3} + i\right|$$

$$\Leftrightarrow \quad \begin{cases} x^2 - y^2 & = 4\sqrt{3} \\ 2xy & = 1 \Rightarrow (x > 0 \wedge y > 0) \vee (x < 0 \wedge y < 0) \\ x^2 + y^2 = \sqrt{48+1} & = \sqrt{49} = 7 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x^2 & = \dfrac{4\sqrt{3}+7}{2} \\ y^2 & = \dfrac{1}{4x^2} = \dfrac{1}{4}\dfrac{2}{(4\sqrt{3}+7)} = \dfrac{1}{2(4\sqrt{3}+7)} \end{cases}$$

$$\Leftrightarrow \quad (x,y) = \left(\sqrt{\frac{4\sqrt{3}+7}{2}}, \sqrt{\frac{1}{2(4\sqrt{3}+7)}}\right)$$

$$\vee \quad (x,y) = \left(-\sqrt{\frac{4\sqrt{3}+7}{2}}, -\sqrt{\frac{1}{2(4\sqrt{3}+7)}}\right)$$

Exercise 6

Solve the following equation in $\mathbb{C}$:

1. $z^2 + (2-2i)z = 3i + 1$
2. $z^3 = \dfrac{1+i}{\sqrt{2}}$
3. $z^6 = 27i$

Correction 6

1. $z^2+(2-2i)z=3i+1 \Leftrightarrow z^2+(2-2i)z-(3i+1)=0$

$\triangle=(2-2i)^2+4(3i+1)=i+1$, we are looking for the square roots of $\triangle$.

For $z=1+i$, We have: $(x+iy)^2=1+i$

$$\Leftrightarrow \quad x^2-y^2+2ixy=1+i \wedge |x+iy|^2=|1+i|$$

$$\Leftrightarrow \quad \begin{cases} x^2-y^2 & =1 \\ 2xy & =1 \Rightarrow (x>0 \wedge y>0) \vee (x<0 \wedge y<0) \\ x^2+y^2=\sqrt{1+1} & =\sqrt{2} \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x^2 & =\dfrac{1+\sqrt{2}}{2} \\ y^2 & =\dfrac{1}{4x^2}=\dfrac{1}{4}\dfrac{2}{(1+\sqrt{2})}=\dfrac{1}{2(1+\sqrt{2})} \end{cases}$$

$$\Leftrightarrow \quad (x,y)=(\sqrt{\frac{1+\sqrt{2}}{2}},\sqrt{\frac{1}{2(1+\sqrt{2})}})$$

$$\vee \quad (x,y)=(-\sqrt{\frac{1+\sqrt{2}}{2}},-\sqrt{\frac{1}{2(1+\sqrt{2})}})$$

$$\text{Hence} \quad \triangle=\left(\sqrt{\frac{1+\sqrt{2}}{2}}+i\sqrt{\frac{1}{2(1+\sqrt{2})}}\right)^2$$

$$, \quad z_1=\frac{-(2-2i)-\left(\sqrt{\frac{1+\sqrt{2}}{2}}+i\sqrt{\frac{1}{2(1+\sqrt{2})}}\right)}{2}$$

$$, \quad z_2=\frac{-(2-2i)-\left(\sqrt{\frac{1+\sqrt{2}}{2}}+i\sqrt{\frac{1}{2(1+\sqrt{2})}}\right)}{2}$$

2.

$$z^3=\frac{1+i}{\sqrt{2}} \quad \Rightarrow \quad |z^3|=|z|^3=\sqrt{\tfrac{1}{2}+\tfrac{1}{2}}=1$$
$$\Rightarrow \quad |z|=1$$

Which implies the existence of $\theta \in \mathbb{R}$ such that: $z = \cos\theta + i\sin\theta$, hence

$$\begin{aligned}
z = \cos\theta + i\sin\theta \quad &\Rightarrow \quad z^3 = (\cos\theta + i\sin\theta)^3 \\
&\Rightarrow \quad z^3 = \cos 3\theta + i\sin 3\theta \wedge z^3 = \frac{1+i}{\sqrt{2}} \\
& \quad\;\; \frac{1+i}{\sqrt{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \\
&\Rightarrow \quad \cos 3\theta + i\sin 3\theta = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \\
&\Rightarrow \quad 3\theta = \frac{\pi}{4} + 2k\pi \\
&\Rightarrow \quad \theta = \frac{\pi}{12} + \frac{2}{3}k\pi
\end{aligned}$$

For $k = 0 \Rightarrow \theta = \frac{\pi}{12} \Rightarrow z_1 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$,
For $k = 1 \Rightarrow \theta = \frac{9\pi}{12} \Rightarrow z_2 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}$,
For $k = 2 \Rightarrow \theta = \frac{17\pi}{12} \Rightarrow z_3 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}$,

3.

$$\begin{aligned}
z^6 = 27i \quad &= \quad |z^6| = |27i| = 27 \\
&= \quad |z|^6 = 27 = (\sqrt{3})^6 \\
&= \quad |z| = \sqrt{3}
\end{aligned}$$

Which implies the existence of $\theta \in \mathbb{R}$ such that: $z = \sqrt{3}(\cos\theta + i\sin\theta)$, d'où

$$\begin{aligned}
z = \sqrt{3}(\cos\theta + i\sin\theta) \quad &\Rightarrow \quad z^6 = 27(\cos\theta + i\sin\theta)^6 \\
&\Rightarrow \quad z^6 = 27(\cos 6\theta + i\sin 6\theta) \wedge z^6 = 27i \\
& \quad\;\; i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\
&\Rightarrow \quad \cos 6\theta + i\sin 6\theta = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\
&\Rightarrow \quad 6\theta = \frac{\pi}{2} + 2k\pi \\
&\Rightarrow \quad \theta = \frac{\pi}{12} + \frac{1}{3}k\pi
\end{aligned}$$

For $k = 0 \Rightarrow \theta = \frac{\pi}{12} \Rightarrow z_1 = \sqrt{3}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$,
For $k = 1 \Rightarrow \theta = \frac{5\pi}{12} \Rightarrow z_2 = \sqrt{3}(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$,
For $k = 2 \Rightarrow \theta = \frac{7\pi}{12} \Rightarrow z_3 = \sqrt{3}(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12})$,
For $k = 3 \Rightarrow \theta = \frac{13\pi}{12} \Rightarrow z_4 = \sqrt{3}(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12})$,

For $k=4 \Rightarrow \theta = \frac{17\pi}{12} \Rightarrow z_5 = \sqrt{3}(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12})$,
For $k=5 \Rightarrow \theta = \frac{21\pi}{12} \Rightarrow z_6 = \sqrt{3}(\cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12})$,

Exercise 7

1. For all $x \in \mathbb{R}$, compute the following sums using the exponential form of a complex number:

$$A = \cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x + \cos 6x + \cos 7x$$
$$B = \sin x + \sin 2x + \sin 3x + \sin 4x + \sin 5x + \sin 6x + \sin 7x$$

2. by using Euler formulas, linearize: $\cos x^3, \sin x^3, \ \cos x^3 \sin x^4$.

Correction 7

1. Let's put, $z = e^{ix} = \cos x + i\sin x$ with $x \neq 2k\pi, k \in \mathbb{Z}$ then:

$$\begin{array}{rl} z = \cos x + \sin x & \Rightarrow \ \forall n \in \mathbb{C} : z^n = \cos(nx) + i\sin(nx) \\ & \Rightarrow \ \sum_{k=1}^{k=n} z^n = \sum_{k=1}^{k=n} \cos(nx) + i\sum_{k=1}^{k=n} \sin(nx) \\ & \Rightarrow \ z\frac{z^n-1}{z-1} = \sum_{k=1}^{k=n} \cos(nx) + i\sum_{k=1}^{k=n} \sin(nx) \end{array}$$

On the other hand, we have:

$$\begin{aligned} z\frac{z^n-1}{z-1} &= e^{ix}\frac{e^{inx}-1}{e^{ix}-1} = \frac{e^{i(n+1)x}-e^{ix}}{e^{ix}-1} \\ &= \frac{(e^{i(n+1)x}-e^{ix})(e^{-ix}-1)}{(e^{ix}-1)(e^{-ix}-1)} = \frac{e^{inx}-e^{i(n+1)x}-1+e^{ix}}{2-(e^{ix}+e^{-ix})} \\ &= \frac{\cos(nx)+i\sin(nx)-\cos(n+1)x-i\sin(n+1)x+\cos x+i\sin(x)-1}{2(1-\cos x)} \\ &= \frac{\cos(nx)-\cos(n+1)x+\cos(x)-1}{2(1-\cos x)} + i\frac{\sin(nx)-\sin(n+1)x+\sin(x)}{2(1-\cos x)} \end{aligned}$$

Which implies that:

$$\begin{cases} \sum_{k=1}^{k=n} \cos(nx) = & \frac{\cos(nx)-\cos(n+1)x+\cos(x)-1}{2(1-\cos x)} \\ \sum_{k=1}^{k=n} \sin(nx) = & \frac{\sin(nx)-\sin(n+1)x+\sin(x)}{2(1-\cos x)} \end{cases}$$

2. From Euler's formula $\boxed{\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)} \ \wedge \ \boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$, w have:

$$\begin{cases} (a+b)^3 = & a^3 + b^3 + 3a^2b + 3ab^2 \\ \cos^3(x) = & \frac{1}{8}\left(e^{i3x} + e^{-i3x} + 3e^{ix} + 3e^{-ix}\right) \\ \sin^3(x) = & \frac{i}{8}(e^{i3x} - e^{-i3x} - 3e^{ix} + 3e^{-ix}) \end{cases} \Rightarrow \begin{cases} \cos^3(x) = & \frac{1}{8}\left(2\cos(3x) + 6\cos(x)\right) \\ \\ \sin^3(x) = & \frac{1}{4}\left(3\sin x - \sin(3x)\right) \end{cases}$$

For the last expression, we proceed as follows:

$$\begin{aligned} \cos x^3 \sin x^4 = \ & \frac{1}{128} \quad (e^{ix} + e^{-ix})^3(e^{ix} - e^{-ix})^4 \\ & = \frac{1}{128}(e^{i3x} + e^{-i3x} + 3e^{ix} + 3e^{-ix})(e^{i4x} + e^{-i4x} - 4e^{i2x} - 4e^{-i2x} + 6) \\ = \ & \frac{1}{128} \quad ([e^{i7x} + e^{-i7x}] - [e^{i5x} + e^{-i5x}] - 3[e^{i3x} + e^{-i3x}] + 3[e^{ix} + e^{-ix}]) \\ & = \frac{1}{128}(2\cos 7x - 2\cos 5x - 6\cos 3x + 6\cos x) \end{aligned}$$

Chapter 3

Sequences of real numbers

3.1 Definitions and examples

Definition 1

A sequence of real numbers is a real-valued function whose domain is the set of natural numbers $\mathbb{N}$ or an infinite subset $\mathcal{N}_1 \subset \mathbb{N}$ to the real numbers i.e:

$$\begin{array}{rcl} u : \mathbb{N} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & u(n) \end{array} \quad \text{or} \quad \begin{array}{rcl} u : \mathcal{N}_1 & \longrightarrow & \mathbb{R} \\ n & \longmapsto & u(n) \end{array}$$

Notations:

- For $n \in \mathbb{N}$, $u(n)$ is denoted by u_n and is called the general term or n-th term of the sequence.
- The sequence u is denoted by $(u_n)_{n\in\mathbb{N}}$ or $(u_n)_{n\in\mathcal{N}_1}$.

Example 1

1. The sequence $(u_n)_{n\in\mathbb{N}^*}$ defined by: $u_n = \dfrac{1}{n}$, starts with $u_1 = 1$, and $u_2 = \dfrac{1}{2}$, $u_3 = \dfrac{1}{3}$,......

2. The recurrent sequence defined by: $\begin{cases} u_1 = 1 \\ u_n = 1 + \dfrac{1}{u_{n-1}} \end{cases}$ starts with $u_1 = 1$, and $u_2 = 2$, $u_3 = \dfrac{3}{2}$,.....

Remark

The ways in which a sequence can be defined.

- By an explicit definition of the general term of the sequence (u_n) i.e.: Express u_n in terms of n. For example, $u_n = \dfrac{2n+1}{n+7}$.
- By a recurrence formula, i.e. a relationship that links any term in the sequence to the one that precedes it. In this case, to calculate u_n, you need to calculate all the terms that precede it. For example :

$$\begin{cases} u_0 = 2 \\ u_{n+1} = 3u_n - 1 \end{cases}$$

3.2 Bounded sequences

Definition 2

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.

- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded from above iff: $\exists M \in \mathbb{R}, \forall n \in \mathbb{N}; u_n \leq M$
- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded from below iff: $\exists m \in \mathbb{R}, \forall n \in \mathbb{N}; m \leq u_n$
- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded iff: it is bounded from above and bounded from below which means :
$$\exists M \in \mathbb{R}_+, \forall n \in \mathbb{N};\ |u_n| \leq M$$

3.3 Increasing and decreasing sequences

Definition 3

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence

- $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence iff: $\forall n \in \mathbb{N}; u_n \leq u_{n+1}$
- $(u_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence iff: $\forall n \in \mathbb{N}; u_n < u_{n+1}$
- $(u_n)_{n\in\mathbb{N}}$ is a decreasing sequence iff: $\forall n \in \mathbb{N}; u_n \geq u_{n+1}$
- $(u_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence iff: $\forall n \in \mathbb{N}; u_n > u_{n+1}$
- $(u_n)_{n\in\mathbb{N}}$ is monotonic if it is increasing or decreasing.
- $(u_n)_n$ is strictly monotonic if it is strictly increasing or strictly decreasing.
- $(u_n)_{n\in\mathbb{N}}$ is a constant sequence iff $\forall n \in \mathbb{N};\ u_{n+1} = u_n$

3.4 Finite and infinite limit of a numerical sequence

Definition 4: Convergent sequences

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence. We say that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to l iff:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 \implies |u_n - l| \leq \varepsilon$$

In this case, we say that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent to the limit l and we note $\lim\limits_{n\to+\infty} u_n = l$

Remark

$$|u_n - l| \leq \varepsilon \Leftrightarrow l - \varepsilon \leq u_n \leq l + \varepsilon \Leftrightarrow u_n \in [l - \varepsilon, l + \varepsilon]$$

The above definition means that for any strictly positive real ε, there exists an integer n_0 (rank) such that: all terms $u_{n_0}, u_{n_0+1}, u_{n_0+2} \dots$ are in the interval $[l - \varepsilon, l + \varepsilon]$.

Example 2

- The sequence $u_n = \dfrac{n}{n+1}$ converges to 1

Using the definition of convergence, we show that $\lim\limits_{n \to +\infty} u_n = 1$

Let $\varepsilon > 0$ we have:

$$\begin{aligned} &|u_n - 1| \leq \varepsilon \\ \Leftrightarrow\ &|\frac{n}{n+1} - 1| \leq \varepsilon \\ \Leftrightarrow\ &|\frac{n}{n+1} - 1| \leq \varepsilon \\ \Leftrightarrow\ &|1 - \frac{1}{n+1} - 1| \leq \varepsilon \\ \Leftrightarrow\ &\frac{1}{n+1} \leq \varepsilon \\ \Leftrightarrow\ &\frac{1}{\varepsilon} - 1 \leq n \end{aligned}$$

By setting $n_0 = \lfloor \frac{1}{\varepsilon} \rfloor > \frac{1}{\varepsilon} - 1$, we obtain :

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}\, (n_0 = \lfloor \frac{1}{\varepsilon} \rfloor), \forall n \in \mathbb{N}; n \geq n_0 \implies |u_n - 1| \leq \varepsilon$

$\implies (u_n)_{n \in \mathbb{N}}$ converges to $l = 1$

Using Maple, we get the following graph:

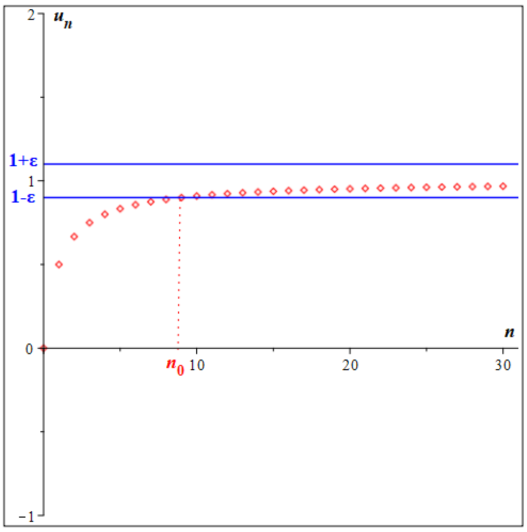


Figure 3.1: $\varepsilon = 0.1$

Definition 5

1. We say that the sequence $(u_n)_{n\in\mathbb{N}}$ tends to $+\infty$ as n tends to infinity and we note $\lim\limits_{n\to+\infty} u_n = +\infty$ iff:
$$\forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 \implies u_n \geq A$$

2. We say that the sequence $(u_n)_{n\in\mathbb{N}}$ tends to $-\infty$ as n tends to infinity and we note $\lim\limits_{n\to+\infty} u_n = -\infty$ iff:
$$\forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 \implies u_n \leq -A$$

Example 3

- Let be the following sequences :
$$\begin{cases} u_n = 2n+1 \\ v_n = -3n+4 \end{cases}$$

We show that $\lim\limits_{n\to+\infty} u_n = +\infty$ and $\lim\limits_{n\to+\infty} v_n = -\infty$

1. Let $A > 0$ we have:
$$\begin{aligned} & u_n \geq A \\ \Leftrightarrow\ & 2n+1 \geq A \\ \Leftrightarrow\ & 2n \geq A-1 \\ \Leftrightarrow\ & 2n \geq \frac{A-1}{2} \end{aligned}$$

Let's put $n_0 = [\frac{A-1}{2}]+1 > \frac{A-1}{2}$
$\implies (\forall A > 0, \exists n_0 \in \mathbb{N}\,(n_0 = [\frac{A-1}{2}]+1), \forall n \in \mathbb{N}; n \geq n_0 \implies u_n \geq A)$

2. The same method used for the sequence $(v_n)_{n\in\mathbb{N}}$

Definition 6: divergent sequences

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that the sequence $(u_n)_{n\in\mathbb{N}}$ is divergent if it is not convergent, i.e

$$\forall l \in \mathbb{R}, \exists \varepsilon > 0, \forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}; (n \geq n_0) \wedge (|u_n - l| > \varepsilon)$$

Remark

here are two types of divergence

1. Divergence of infinite type: in this case the sequence converges to $+\infty$ or $-\infty$. For example the sequence with general term $u_n = 2n+4$.

2. Divergence of type limit does not exist: in this case the sequence has no finite or infinite limit.

For example, the sequence with general term $u_n = (-1)^n$

Proof:
We will show that the sequence $(-1)^n$ does not have a finite or infinite limit.

1. By contradiction, suppose that: $\lim_{n\to+\infty} (-1)^n = l / l \in \mathbb{R}$. According to the convergence definition with $\varepsilon = \frac{1}{4}$ we get:

$$\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 \implies u_n \in [l - \frac{1}{4}, l + \frac{1}{4}]$$
$$\implies -1, 1 \in [l - \frac{1}{4}, l + \frac{1}{4}]$$
$$\implies \begin{cases} l - \frac{1}{4} \leq 1 \leq l + \frac{1}{4} \\ l - \frac{1}{4} \leq -1 \leq l + \frac{1}{4} \end{cases}$$
$$\implies \begin{cases} l - \frac{1}{4} \leq 1 \leq l + \frac{1}{4} \\ -l - \frac{1}{4} \leq 1 \leq -l + \frac{1}{4} \end{cases}$$
$$\implies \left\{ -\frac{1}{2} \leq 2 \leq \frac{1}{2} \right.$$

It's a contradiction.

2. By contradiction, suppose that: $\lim_{n\to+\infty} (-1)^n = +\infty$. According to the convergence definition with $A = 4$ we get:

$$\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 \implies u_n \geq 4$$
$$\implies u_n \in [4, +\infty[\implies -1, 1 \in [4, +\infty[$$

It's a contradiction.

3. We use the same method for the case: $\lim_{n\to+\infty} (-1)^n = -\infty$

Proposition 1:

If a sequence of real numbers $(u_n)_{n\in\mathbb{N}}$ has a limit, then this limit is unique.

Proof:

By contradiction

Suppose that:; $\begin{cases} \lim_{n\to+\infty} u_n = l_1 \\ \lim_{n\to+\infty} u_n = l_2 \end{cases}$

Taking $\varepsilon = \frac{|l_1 - l_2|}{4}$ with $l_1 \neq l_2$ which implies

$$\begin{cases} \exists n_1 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_1 \implies |u_n - l_1| \leq \varepsilon \\ \exists n_2 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_2 \implies |u_n - l_2| \leq \varepsilon \end{cases}$$

Putting $n_0 = \max(n_1, n_2)$

$$\Longrightarrow (\forall n \in \mathbb{N}; n \geq n_0 \Longrightarrow |u_n - l_1| + |u_n - l_2| \leq 2\varepsilon)$$

With $n \geq n_0$ we get

$$\begin{aligned} |l_1 - l_2| \leq |u_n - l_1| + |u_n - l_2| \leq 2\varepsilon \\ \Longrightarrow |1_1 - l_2| \leq 2\varepsilon \\ \Longrightarrow \frac{|1_1 - l_2|}{4} \leq \frac{\varepsilon}{2} \\ \Longrightarrow \varepsilon \leq \frac{\varepsilon}{2} \qquad \text{it's a contradiction} \end{aligned}$$

Proposition 2

If $(u_n)_{n\in\mathbb{N}}$ is a convergent sequence, then $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence.

Proof:

We'll show the following implication:

$$(u_n)_{n\in\mathbb{N}} \text{ is a convergent sequence} \Longrightarrow (u_n)_{n\in\mathbb{N}} \text{ is bounded}$$

Suppose that $(u_n)_{n\in\mathbb{N}}$ is convergent, then for $\varepsilon = 1$ we have:

$$\begin{aligned} \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 \Longrightarrow |u_n - l| \leq 1 \\ \Longrightarrow m = l - 1 \leq u_n \leq l + 1 = M \end{aligned}$$

So the set $\{u_{n_0}, u_{n_0+1}, ...\}$ is bounded.
On the other hand $A = \{u_0 ....., u_{n_0-2}, u_{n_0-1}\}$ is bounded (because $Card(A) < +\infty$). Then the set of values of (u_n) is: $\{u_0 ....., u_{n_0-2}, u_{n_0-1}, u_{n_0}, u_{n_0+1}, ...\}$ is bounded, this means (u_n) is bounded.

3.5 Finding Limits: Properties of Limits

Theorem 1

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ two convergent sequences with: $\lim\limits_{n\to+\infty} u_n = l$ and $\lim\limits_{n\to+\infty} v_n = l'$. The properties of limits are summarized as follows:

1. $\lim\limits_{n\to+\infty} \lambda u_n = \lambda l$ with $\lambda \in \mathbb{R}$

2. $\lim\limits_{n\to+\infty} (u_n + v_n) = l + l'$

3. $\lim\limits_{n\to+\infty} u_n v_n = ll'$

4. If $u_n \neq 0$ for $n \geq n_0$ and $l \neq 0$ then $\lim\limits_{n\to+\infty} \frac{1}{u_n} = \frac{1}{l}$

5. If $v_n \neq 0$ for $n \geq n_0$ and $l' \neq 0$ then $\lim\limits_{n\to+\infty} \frac{u_n}{v_n} = \frac{l}{l'}$

Remark

$\lim_{n \to +\infty} u_n = l \implies \lim_{n \to +\infty} |u_n| = |l|$. **Be careful** the reverse is not true. For example, if we take the sequence $u_n = (-1)^n$ we have $\lim_{n \to +\infty} |u_n| = 1$ but $\lim_{n \to +\infty} u_n$ doesn't exist.

Proposition 3: Infinite limit's operations

Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ two sequences with: $\lim_{n \to +\infty} u_n = +\infty$ and $\lim_{n \to +\infty} v_n = +\infty$ then:

1. $\lim_{n \to +\infty} (u_n + v_n) = +\infty$

2. If $\forall n \geq n_0, u_n \neq 0$ then $\lim_{n \to +\infty} \frac{1}{u_n} = 0$

3.6 Limits and inequalities

Theorem 2

1. Let $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ be two convergent sequences, then:

$$\text{If } \exists n_0 \in \mathbb{N}, \forall n \geq n_0;\ u_n \leq v_n \quad \text{this implies} \lim_{n \to +\infty} u_n \leq \lim_{n \to +\infty} v_n$$

2. If, we have $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ two sequences which verify:- $\begin{cases} \exists n_0 \in \mathbb{N}, \forall n \geq n_0;\ u_n \leq v_n \\ \quad \text{and} \\ \lim_{n \to +\infty} u_n = +\infty \end{cases}$

   this implies $\lim_{n \to +\infty} v_n = +\infty$

3. Squeeze Theorem : If $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ three sequences with:

$$\begin{cases} \exists n_0 \in \mathbb{N}, \forall n \geq n_0;\ u_n \leq v_n \leq w_n \\ \quad \text{and} \\ \lim_{n \to +\infty} u_n = \lim_{n \to +\infty} w_n = l \end{cases}$$

   then the sequence $(v_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n \to +\infty} v_n = l$

3.7 Convergence theorems

Theorem 3: Convergence of monotonic sequences

- If a sequence of real numbers is increasing and bounded from above, then it converges.
- If a sequence of real numbers is decreasing and bounded from below, then it converges.

Example 4

Let $(u_n)_{n\in\mathbb{N}}$ be a numerical sequence defined by: $\begin{cases} u_0 = 1 \\ u_{n+1} = \dfrac{1+u_n^2}{2} \end{cases}$.

1. Prove that $\forall n \in \mathbb{N};\ u_n \leq 1$

2. Deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent.

- by using proof by induction, we have for $n = 0$, $u_0 = \dfrac{1}{2} \leq 1$ so the proposition is true. Let's assume that the proposition is true for $k \in \{1,\dots,n\}$ and we'll show that $u_{n+1} \leq 1$. According to the assumption we have:

$$u_n \leq 1 \implies u_n^2 \leq 1 \implies 1+u_n^2 \leq 2 \implies \frac{1+u_n^2}{2} \leq 1 \implies u_{n+1} \leq 1$$

So, assertion $\forall n \in \mathbb{N};\ u_n \leq 1$ is true.

- On a $\forall n \in \mathbb{N}; u_{n+1} - u_n = \dfrac{1+u_n^2}{2} - u_n = \dfrac{(u_n-1)^2}{2} \geq 0$

- Since $(u_n)_{n\in\mathbb{N}}$ is increasing and bounded from above so $(u_n)_{n\in\mathbb{N}}$ is convergent.

Definition 7: Adjacent sequences

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two real sequences. We say that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent iff:

$$\begin{cases} (u_n)_{n\in\mathbb{N}} \quad \text{is increasing} \\ \quad\quad \text{and} \\ (v_n)_{n\in\mathbb{N}} \quad \text{is decreasing} \\ \quad\quad \text{and} \\ \lim\limits_{n\to+\infty} (u_n - v_n) = 0 \end{cases}$$

Theorem 4:

If the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent then they converge to the same limit.

Example 5

The sequences $u_n = \sum\limits_{k=1}^{n} \dfrac{1}{k^2}$ and $v_n = u_n + \dfrac{2}{n}$ are adjacent :

- $u_{n+1} - u_n = \sum\limits_{k=1}^{n+1} \dfrac{1}{k^2} - \sum\limits_{k=1}^{n} \dfrac{1}{k^2} = \dfrac{1}{(n+1)^2} \geq 0 \implies (u_n)_{n\in\mathbb{N}}$ is increasing

- $v_{n+1} - v_n = \dfrac{1}{(n+1)^2} + \dfrac{2}{n+1} - \dfrac{2}{n} = -\dfrac{(n+2)}{n(n+1)^2} \leq 0 \implies (v_n)_{n\in\mathbb{N}}$ is decreasing

- $\lim_{n\to+\infty}(u_n - v_n) = \lim_{n\to+\infty}(-\frac{2}{n}) = 0$

Therefore the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are convergent to the same limits.

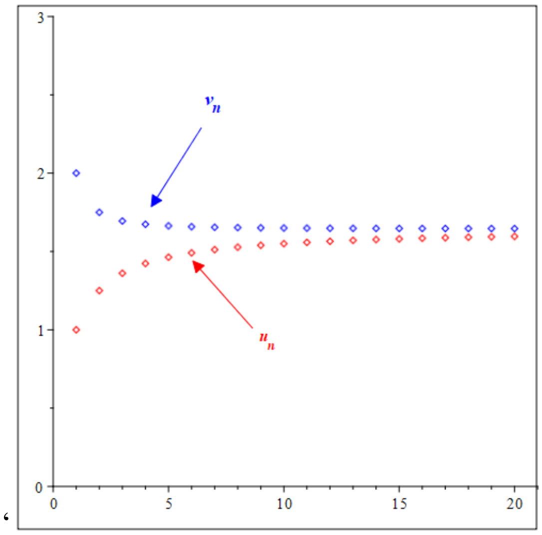


Figure 3.2: (u_n) and (v_n) are adjacent

Definition 8: Cauchy sequence

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.
$(u_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence in $\mathbb{R}$ iff:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p,q \in \mathbb{N};\ p,q \geq n_0 \implies |u_p - u_q| \leq \varepsilon$$

Remark

$|u_p - u_q| \leq \varepsilon \Leftrightarrow$ the distance between u_p, and u_q is less than ε.
So the definition above means that:- for any strictly positive real ε, there exists n_0 (rank), such that the distance between each two terms u_p,u_q (with $p,q \geq n_0$) is less than ε.

Using Maple, we obtain the following graph of a Cauchy sequence:

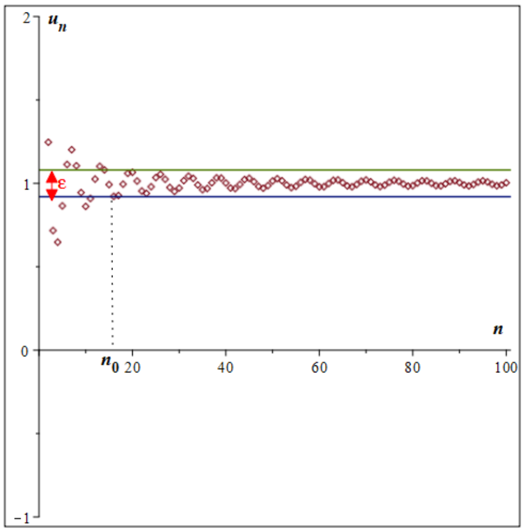


Figure 3.3: $u_n = \dfrac{\cos(n)+\sin(n)+n}{n}$, $\varepsilon = 0.08$

Example 6

- $u_n = \dfrac{1}{n}$ is a Cauchy sequence

Let $p,q \in \mathbb{N}^*$ with $p \leq q$ then we have:

$$|u_p - u_q| = \left|\frac{1}{p} - \frac{1}{q}\right| \leq \left|\frac{1}{p}\right| + \left|\frac{1}{q}\right| \quad \text{according to the triangular inequality}$$

$$\Longrightarrow |u_p - u_q| \leq \frac{2}{p} \quad (\text{ because: } \quad \frac{1}{q} \leq \frac{1}{p})$$

Let $\varepsilon > 0$, we put $n_0 = \left[\dfrac{2}{\varepsilon}\right] + 1 > \dfrac{2}{\varepsilon}$

So, $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}^*, \forall p,q \in \mathbb{N}^*; p,q \geq n_0 \Longrightarrow |u_p - u_q| \leq \varepsilon$

Theorem 5:

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence then:
$(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence $\Longleftrightarrow$ $(u_n)_{n\in\mathbb{N}}$ is convergent

3.8 Subsequence

Definition 9

The sequence $(u_{\phi(n)})_{n\in\mathbb{N}}$ is a subsequence of the sequence $(u_n)_{n\in\mathbb{N}}$ if $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ is a strictly increasing sequence of of natural numbers.

Example 7

1. $u_n = (-1)^n \longrightarrow \begin{cases} u_{2n} = (-1)^{2n} = 1 \\ \\ u_{2n+1} = (-1)^{2n+1} = -1 \end{cases}$

   $(u_{2n})_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ are subsequences taken from $(u_n)_{n\in\mathbb{N}}$

2. $v_n = \cos(\frac{n\pi}{3}) \longrightarrow v_{3n} = \cos(n\pi) = (-1)^n$

   $(v_{3n})_{n\in\mathbb{N}}$ is a sub-sequence of $(v_n)_{n\in\mathbb{N}}$

Proposition 4:

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers:

1. If $\lim_{n\to+\infty} u_n = l$, then for any subsequence $(u_{\phi(n)})_{n\in\mathbb{N}}$; $\lim_{n\to+\infty} u_{\phi(n)} = l$
2. If $(u_n)_{n\in\mathbb{N}}$ admits a divergent subsequence then $(u_n)_{n\in\mathbb{N}}$ is divergent
3. If $(u_n)_{n\in\mathbb{N}}$ has two subsequences converging to distinct limits then $(u_n)_{n\in\mathbb{N}}$ is divergent.

Example 8

the sequence with general term $u_n = (-1)^n$ is divergent:
We have:

$$\begin{cases} u_{2n} = 1 \\ \quad \text{and} \\ u_{2n+1} = -1 \end{cases} \Longrightarrow \begin{cases} \lim_{n\to+\infty} u_{2n} = 1 \\ \quad \text{and} \\ \lim_{n\to+\infty} u_{2n+1} = -1 \end{cases}$$

So,$(u_{2n})_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ are two subsequences of $(u_n)_{n\in\mathbb{N}}$ which converge to distinct limits, therefore $(u_n)_{n\in\mathbb{N}}$ is divergent.

Theorem 6: Bolzano-Weierstrass Property

Every bounded sequence has a convergent sub-sequence.

Definition 9: Cluster Points of the sequence

A cluster Point of a numerical sequence $(u_n)_{n\in\mathbb{N}}$ is any scalar which is the limit of a subsequence of $(u_n)_{n\in\mathbb{N}}$.

Example 9

- Let's consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by: $u_n = \cos(n\frac{\pi}{2})$

$$\begin{cases} u_{4n} = \cos(2n\pi) = 1 \implies \lim_{n\to+\infty} u_{4n} = 1 \\ u_{4n+1} = \cos(\frac{\pi}{2}) = 0 \implies \lim_{n\to+\infty} u_{4n+1} = 0 \\ u_{4n+2} = \cos(\pi) = -1 \implies \lim_{n\to+\infty} u_{4n+2} = -1 \\ u_{4n+3} = \cos(3\frac{\pi}{2}) = 0 \implies \lim_{n\to+\infty} u_{4n+3} = 0 \end{cases}$$

So the sequence $(u_n)_{n\in\mathbb{N}}$ is divergent. The numbers $1, -1, 0$ are the cluster points of the sequence $(u_n)_{n\in\mathbb{N}}$.

3.9 Limit inferior and limit superior

Definition 10

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.
Denoting by S = The set of cluster points of the sequence $(u_n)_{n\in\mathbb{N}}$.
We define the limit superior (resp. inferior) of $(u_n)_{n\in\mathbb{N}}$ as

$$\begin{cases} \limsup u_n = \sup S \\ \\ \liminf u_n = \inf S \end{cases}$$

Example 10

Let $(u_n)_{n\in\mathbb{N}}$ defined by: $u_n = (-1)^n$
The set of all cluster points of the sequence $(u_n)_{n\in\mathbb{N}}$ is $S = \{1, -1\}$
so, $\limsup u_n = 1$, and $\liminf u_n = -1$

Chapter's exercises with answers

Exercise 1

Let x_n be a sequence of real numbers such that $\lim_{n\to+\infty} x_n^2 = 0$. Show that $\lim_{n\to+\infty} x_n = 0$.

Correction 1

According to the definition, the $\lim_{n\to+\infty} x_n^2 = 0$ expression can be interpreted as follows: For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that:

$$\forall n \in \mathbb{N}, n \geq 0 \implies x_n^2 < \epsilon^2 \implies \sqrt{x_n^2} < \sqrt{\epsilon^2} \implies |x_n| < \epsilon$$

Which shows that

$$\lim_{n\to+\infty} x_n = 0$$

Exercise 2

Let's consider the sequences (x_n), (y_n), (a_n), (b_n), (u_n), and (v_n) of real numbers such that:

1. $(\forall n \in \mathbb{N},\ x_n \leq 2 \text{ et } y_n \leq 3) \text{ et } x_n + y_n \to 5$
2. $(\forall n \in \mathbb{N},\ 0 \leq u_n \leq 1 \text{ et } 0 \leq v_n \leq 1) \text{ et } u_n.v_n \to 1$
3. $a_n^2 + b_n^2 + a_n b_n \to 0$

Show that the sequences defined above are convergent.

Correction 2

1. We have:

$$\begin{cases} x_n \leq 2 \\ y_n \leq 3 \end{cases} \Leftrightarrow \begin{cases} 0 \leq 2 - x_n \\ 0 \leq 3 - y_n \end{cases}$$

We add $(2 - x_n)$ to the second inequality and $(3 - x_n)$ to the first inequality, we find:

$$\begin{cases} 2 - x_n \leq 2 - x_n + 3 - y_n \\ 3 - y_n \leq 3 - y_n + 2 - x_n \end{cases} \Leftrightarrow \begin{cases} 2 - x_n \leq 5 - (x_n + y_n) \\ 3 - y_n \leq 5 - (y_n + x_n) \end{cases} \Leftrightarrow \begin{cases} 0 \leq 2 - x_n \leq 5 - (x_n + y_n) \\ 0 \leq 3 - y_n \leq 5 - (y_n + x_n) \end{cases}$$

Using the squeeze theorem and the fact that $x_n + y_n \to 5$, we get :

$$\begin{cases} 0 \leq \lim_{n\to+\infty} (2 - x_n) \leq \lim_{n\to+\infty} [5 - (x_n + y_n)] \\ 0 \leq \lim_{n\to+\infty} (3 - y_n) \leq \lim_{n\to+\infty} [5 - (y_n + x_n)] \end{cases}$$

So

$$\lim_{n\to+\infty} x_n = 2 \text{ et } \lim_{n\to+\infty} y_n = 3$$

2. Since,

$$\forall n \in \mathbb{N}, \begin{cases} 0 & \leq v_n \leq 1 \\ 0 & \leq v_n \leq 1 \end{cases}$$

We multiply the first inequality by $v_n; (0 \leq v_n)$ and the second by $u_n; (0 \leq u_n)$, we find:

$$\forall n \in \mathbb{N}, \begin{cases} v_n u_n & \leq v_n \\ u_n v_n & \leq u_n \end{cases} \Leftrightarrow \forall n \in \mathbb{N}, \begin{cases} v_n u_n & \leq v_n \leq 1 \\ u_n v_n & \leq u_n \leq 1 \end{cases}$$

Using the squeeze theorem and the fact that $v_n u_n \to 1$, we get

$$\begin{cases} 1 \leq \lim\limits_{n\to+\infty} v_n \leq 1 \\ 1 \leq \lim\limits_{n\to+\infty} u_n \leq 1 \end{cases}$$

So

$$\lim_{x\to+\infty} u_n = 1 \text{ et } \lim_{x\to+\infty} v_n = 1$$

3. We have

$$\begin{aligned} \lim_{n\to+\infty} a_n^2 + b_n^2 + a_n b_n = 0 &\Leftrightarrow \lim_{n\to+\infty} 2(a_n^2 + b_n^2 + a_n b_n) = 0 \\ &\Leftrightarrow \lim_{n\to+\infty} 2a_n^2 + 2b_n^2 + 2a_n b_n = 0 \\ &\Leftrightarrow \lim_{n\to+\infty} a_n^2 + b_n^2 + (a_n^2 + b_n^2 + 2a_n b_n) = 0 \\ &\Leftrightarrow \lim_{n\to+\infty} a_n^2 + b_n^2 + (a_n + b_n)^2 = 0. \end{aligned}$$

On the other hand, we have:

$$\begin{cases} 0 \leq a_n^2 \leq a_n^2 + b_n^2 + (a_n + b_n)^2 \\ 0 \leq b_n^2 \leq a_n^2 + b_n^2 + (a_n + b_n)^2 \end{cases} \Leftrightarrow \begin{cases} 0 \leq a_n^2 \leq a_n^2 + b_n^2 + (a_n + b_n)^2 \\ 0 \leq b_n^2 \leq a_n^2 + b_n^2 + (a_n + b_n)^2 \end{cases}$$

Which implies that:

$$\begin{cases} 0 \leq \lim\limits_{n\to+\infty} a_n^2 \leq \lim\limits_{n\to+\infty} [a_n^2 + b_n^2 + (a_n + b_n)^2] \\ 0 \leq \lim\limits_{n\to+\infty} b_n^2 \leq \lim\limits_{n\to+\infty} [a_n^2 + b_n^2 + (a_n + b_n)^2] \end{cases} \Leftrightarrow \begin{cases} 0 \leq \lim\limits_{n\to+\infty} a_n^2 \leq 0 \\ 0 \leq \lim\limits_{n\to+\infty} b_n^2 \leq 0 \end{cases} \Leftrightarrow \begin{cases} \lim\limits_{n\to+\infty} a_n^2 = 0 \\ \lim\limits_{n\to+\infty} b_n^2 = 0 \end{cases}$$

Based on the results of exercise (1),

$$\lim_{n\to+\infty} a_n = 0, \text{ et } \lim_{n\to+\infty} b_n = 0$$

Exercise 3

Let (u_n) be a sequence of real numbers

1. If $\lim u_n = +\infty$, show that $\lim E(u_n) = +\infty$.
2. If the sequence (u_n) converges, can we say that $E(u_n)$ converges?

Correction 3

1. We know that $\forall n \in \mathbb{N} : \; E(u_n) \leq u_n < E(u_n) + 1$. Using the squeeze theorem:

$$\lim_{n\to+\infty} E(u_n) \leq \lim_{n\to+\infty} u_n \leq 1 + \lim_{n\to+\infty} E(u_n)$$

We see that the sequence u_n converges to $+\infty$

2. For the second question, we use the following counter-example:
Let's take the sequence $u_n = \dfrac{\cos(n\pi)}{n}$; $n \in \mathbb{N}^*$. We know that, $\forall n \in \mathbb{N}^* : \dfrac{-1}{n} \leq \dfrac{\cos(n\pi)}{n} \leq \dfrac{1}{n}$, so u_n converges to 0. On the other hand, the sequence $E(u_n)$ diverges because it has two extracted sequences ($E(u_{2n}) = E(\dfrac{1}{2n}) = 0$ and $E(u_{2n+1}) = E(\dfrac{-1}{2n+1}) = -1$) with different limits.

Exercise 4

Study the nature of the following sequences and determine their possible limits:

1. $\sqrt{n^2+n+1} - \sqrt{n}$
2. $\dfrac{n\cos n}{n^3+1}$
3. $\dfrac{\sin n^2 + 2\cos n}{n^2}$
4. $\dfrac{a^n - b^n}{a^n + b^n}$; $a,\ b \in]0, +\infty[$
5. $(1+\dfrac{2}{n})^n$
6. $n^3(\tan\dfrac{3}{n} - \sin\dfrac{3}{n})$

Correction 4

1.
$$\begin{aligned}\sqrt{n^2+n+1} - \sqrt{n} &= n(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}) + \frac{n}{\sqrt{n}} \\ &= n[\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} + \frac{1}{\sqrt{n}}]\end{aligned}$$

Since $\lim\limits_{n\to+\infty} \sqrt{1+\frac{1}{n}+\frac{1}{n^2}} + \frac{1}{\sqrt{n}} = 1$, so
$$\lim_{n\to+\infty} \sqrt{n^2+n+1} - \sqrt{n} = 1 \times \lim_{n\to+\infty} n = +\infty$$

2. We have for all $n \in \mathbb{N}$:
$$|\cos(n)| \leq 1 \implies \left|\frac{n\cos(n)}{n^2+1}\right| \leq \frac{n}{n^2+1} \implies \frac{-n}{n^2+1} \leq \frac{n\cos(n)}{n^2+1} \leq \frac{n}{n^2+1}$$
And since
$$\lim_{n\to+\infty} \frac{n}{n^2+1} = \lim_{n\to+\infty} \frac{-n}{n^2+1} = 0$$
According to Squeeze theorem, we find :
$$0 \leq \lim_{n\to+\infty} \frac{n\cos(n)}{n^2+1} \leq 0 \implies \lim_{n\to+\infty} \frac{n\cos(n)}{n^2+1} = 0$$

3. We have for all $n \in \mathbb{N}$:
$$|\sin(n^2) + 2\cos(n)| \leq 3 \implies \left|\frac{\sin(n^2)+2\cos(n)}{n^2}\right| \leq \frac{3}{n^2} \implies \frac{-3}{n^2} \leq \frac{\sin(n^2)+2\cos(n)}{n^2} \leq \frac{3}{n^2}$$
And since
$$\lim_{n\to+\infty} \frac{-3}{n^2} = \lim_{n\to+\infty} \frac{3}{n^2} = 0$$
According to Squeeze theorem, we find :
$$0 \leq \lim_{n\to+\infty} \frac{\sin(n^2)+2\cos(n)}{n^2} \leq 0 \implies \lim_{n\to+\infty} \frac{\sin(n^2)+2\cos(n)}{n^2} = 0$$

4. since $a > 0$, and $b > 0$ we can write

$$\frac{a^n - b^n}{a^n + b^n} = \frac{W^n - 1}{W^n + 1}, \text{ with } W = \frac{a}{b}$$

We have three cases to look at:

First case: $0 < W < 1 \Leftrightarrow a < b$ We can write:

$W^n = e^{\ln w^n} = e^{n \ln W}$, So $\lim_{n \to +\infty} W^n = 0$, $[\ln W < 0]$ which implies that

$$\lim_{n \to +\infty} \frac{a^n - b^n}{a^n + b^n} = \frac{0 - 1}{1 + 0} = -1$$

Second case: $W = 1 \Leftrightarrow a = b$

We have $\lim_{n \to +\infty} w^n = 1$ which implies that

$$\lim_{n \to +\infty} \frac{a^n - b^n}{a^n + b^n} = \frac{1 - 1}{1 + 1} = 0$$

Third case: $W > 1 \Leftrightarrow a > b$ we can write:

$$\frac{a^n - b^n}{a^n + b^n} = \frac{W^n - 1}{W^n + 1} = \frac{1 - \frac{1}{W^n}}{1 + \frac{1}{W^n}}$$

as, $W^n = e^{\ln w^n} = e^{n \ln W}$, So $\lim_{n \to +\infty} W^n = +\infty$, $[\ln W > 0]$ which implies that

$$\lim_{n \to +\infty} \frac{a^n - b^n}{a^n + b^n} = \lim_{n \to +\infty} \frac{1 - \frac{1}{W^n}}{1 + \frac{1}{W^n}} = 1$$

5. We have:

$$\lim_{n \to +\infty} (1 + \frac{2}{n})^n = \lim_{n \to +\infty} e^{\ln(1+2/n)^n} = \lim_{n \to +\infty} e^{n \ln(1+2/n)} = \lim_{n \to +\infty} e^{\frac{2\ln(1+2/n)}{2/n}}$$

Let $x = \frac{2}{n}$ which implies that if $n \longrightarrow +\infty \Longleftrightarrow x \longrightarrow 0^+$, and according to the known limit $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$, So:

$$\lim_{n \to +\infty} (1 + \frac{2}{n})^n = \lim_{n \to +\infty} e^{\frac{2\ln(1+2/n)}{2/n}} = e^2$$

6. We have:

$$n^3(\tan\frac{3}{n} - \sin\frac{3}{n}) = n^2\left(\frac{3\sin\frac{3}{n}}{\frac{3}{n}\cos\frac{3}{n}} - \frac{3\sin\frac{3}{n}}{\frac{3}{n}}\right)$$
$$= \frac{3\sin\frac{3}{n}}{\frac{3}{n}}\left(\frac{1}{\cos\frac{3}{n}} - 1\right)n^2$$
$$= 3\times 9\left(\frac{\sin\frac{3}{n}}{\frac{3}{n}}\right)\left(\frac{1-\cos\frac{3}{n}}{\left(\frac{3}{n}\right)^2\cos\frac{3}{n}}\right)$$
$$= 27\left(\frac{\sin\frac{3}{n}}{\frac{3}{n}}\right)\left(\frac{1-\cos\frac{3}{n}}{\left(\frac{3}{n}\right)^2}\right)\frac{1}{\cos\frac{3}{n}}$$

Let $x = \frac{3}{n}$ which implies that $n \longrightarrow +\infty \Longleftrightarrow x \longrightarrow 0^+$, and according to the two usual limits

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1 \text{ and } \lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \frac{1}{2}$$

Therefore;

$$\lim_{n\to+\infty} n^3(\tan\frac{3}{n} - \sin\frac{3}{n}) = 27\times 1\times\frac{1}{2}\times\frac{1}{1} = \frac{27}{2}$$

Exercise 5

Let (u_n) be a sequence such that $u_0 = 4$ and $\forall n \in \mathbb{N}: \ u_{n+1} = 3 - \dfrac{4}{2+u_n}$.

1. Show that $u_n \geq 2$, for all $n \in \mathbb{N}$.
2. Prove that (u_n) is a monotonic sequence.
3. Study its convergence. If it converges, compute its limit.

Correction 5

1. We use recurrence reasoning (Proof by Induction) to prove that $\forall n \in \mathbb{N}^*: \ u_n \geq 2$.
   (a) **Statement**. $P_n: \forall n \in \mathbb{N}^*: \ u_n \geq 2$.
   (b) **Base Case**. $u_0 = 4 > 2$, so the property P_0 holds.
   (c) **Induction hypothesis**. Let $n \in \mathbb{N}$ such that P_n is true, i.e. $u_n \geq 2$, so we have:

$$u_n \geq 2 \implies u_n + 2 \geq 4 \implies \frac{4}{U_n+2} \leq 1 \implies \frac{-4}{U_n+2} \geq -1$$
$$\implies 3 + \frac{-4}{U_n+2} \geq -1+3 = 2 \implies u_{n+1} \geq 2$$

(d) **Conclusion**. the property P_n is true for all $n \in \mathbb{N}$.

2. We have

$$\begin{aligned} u_{n+1} - u_n &= 3 - \frac{4}{u_n + 2} - u_n \\ &= \frac{-u_n^2 + u_n + 2}{u_n + 2} \\ &= -\frac{(u_n - 2)(u_n + 1)}{u_n + 2} \\ &\leq 0 \qquad\qquad \text{because } u_n \geq 2 \end{aligned}$$

Thus the sequence u_n is decreasing.
Conclution: Since the sequence u_n is decreasing and bounded from below by 2 . So it converges to a limit $l \in \mathbb{R}$. On the other hand u_{n+1} is a sub-sequence of u_n so it also converges to l. According to the recurrence relation for u_n, we obtain:

$$\lim_{n\to+\infty} u_{n+1} = 3 - \frac{4}{2 + \lim\limits_{n\to+\infty} u_n} \iff l = 3 - \frac{4}{l+2} \iff \frac{(l-2)(l+1)}{l+2} = 0$$

And since $u_n \geq 2$, this means that $\lim\limits_{n\to+\infty} u_n = l \geq 2$. As a result $l = 2$

Exercise 6

Let $(u_n)_{n\in\mathbb{N}^*}$ be the sequence defined by :

$$u_n = \frac{n+2}{n+1}$$

1. Show that $(u_n)_{n\in\mathbb{N}^*}$ converges to 1.
2. Find an integer $n_0 \in \mathbb{N}$ such that all terms u_n of index $n \geq n_0$ are in the interval $I =]0.98; 1.2[$.

Correction 6

1. To show that $\lim\limits_{n\to+\infty} u_n = 1$, we can use this rough draft to compute n_0.

rough draft

$$\left|\frac{n+1}{n+2} - 1\right| \leq \epsilon \implies \left|\frac{-1}{n+2}\right| \leq \epsilon \implies \frac{1}{\epsilon} \leq n+2 \implies \frac{1}{\epsilon} - 2 \leq n$$

$$n_0 = E(\frac{1}{\epsilon}) + 1$$

Proof: Let $\epsilon > 0, \exists n_0 = \left(E(\frac{1}{\epsilon}) + 1\right) \in \mathbb{N}; \forall n \in \mathbb{N}$

$$\begin{aligned} n \geq n_0 &\Longrightarrow n \geq E(\frac{1}{\epsilon}) + 1 \\ &\Longrightarrow n \geq E(\frac{1}{\epsilon}) + 1 - 2 \\ &\Longrightarrow n \geq \frac{1}{\epsilon} - 2 \\ &\Longrightarrow n + 2 \geq \frac{1}{\epsilon} \\ &\Longrightarrow \frac{1}{n+2} \leq \epsilon \\ &\Longrightarrow \left|\frac{-1}{n+2}\right| \leq \epsilon \\ &\Longrightarrow \left|\frac{n+1}{n+2} - 1\right| \leq \epsilon \\ &\Longrightarrow |u_n - 1| \leq \epsilon \end{aligned}$$

2.

$$\begin{aligned} 0.98 \leq \frac{n+1}{n+2} \leq 1.2 &\Longleftrightarrow 0.98 \leq 1 - \frac{1}{n+2} \leq 1.2 \\ &\Longleftrightarrow 0.98 - 1 \leq -\frac{1}{n+2} \leq 1.2 - 1 \\ &\Longleftrightarrow -0.02 \leq -\frac{1}{n+2} \leq 0.2 \\ &\Longleftrightarrow \frac{1}{n+2} \leq 0.02 \\ &\Longleftrightarrow n + 2 \geq 50 \\ &\Longleftrightarrow n \geq 48 \\ &\Longleftrightarrow n \geq 48 = n_0 \end{aligned}$$

Exercise 7

For any $n \in \mathbb{N}^*$, consider the sequence defined by: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$

1. Compute $H_{2n} - H_n$
2. Show that H_n is divergent.

Correction 7

1. We have

$$\begin{aligned} H_{2n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+....+\frac{1}{n} \\ &+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+...+\frac{1}{n+(n-2}+\frac{1}{n+(n-1)}+\frac{1}{n+n} \\ &= H_n+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+...+\frac{1}{n+(n-2}+\frac{1}{n+(n-1)}+\frac{1}{2n} \end{aligned}$$

Which implies that

$$\begin{aligned} H_{2n}-H_n &= \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+...+\frac{1}{n+(n-2}+\frac{1}{n+(n-1)}+\frac{1}{2n} \\ &= \sum_{k=1}^{k=n}\frac{1}{n+k} \end{aligned}$$

On the other hand, for any $1 \leq k \leq n$:

$$\begin{aligned} n+k \leq 2n &\implies \frac{1}{n+k} \geq \frac{1}{2n} \\ &\implies \sum_{k=1}^{k=n}\frac{1}{n+k} \geq \sum_{k=1}^{k=n}\frac{1}{2n} \\ &\implies \sum_{k=1}^{k=n}\frac{1}{n+k} \geq \frac{1}{2n}\sum_{k=1}^{k=n}U_k \qquad \text{(with } u_k=1) \\ &\implies \sum_{k=1}^{k=n}\frac{1}{n+k} \geq \frac{1}{2n}\times n \\ &\implies \sum_{k=1}^{k=n}\frac{1}{n+k} \geq \frac{1}{2} \end{aligned}$$

Consequently

$$H_{2n}-H_n \geq \frac{1}{2} \tag{1}$$

2. Assuming that there exists $l \in \mathbb{R}$ such that: $\lim\limits_{n\to+\infty} H_n = l$, it is clear that H_{2n} is a sub-sequence of H_n. From the inequality (1), and the Squeeze theorem:

$$\lim_{n\to+\infty}(H_{2n}-H_n) \geq \frac{1}{2} \implies \lim_{n\to+\infty}H_{2n}-\lim_{n\to+\infty}H_n \geq \frac{1}{2} \implies l-l=0 \geq \frac{1}{2}$$

Contradiction, so the harmonic sequence H_n diverges.

Chapter 4

Limits and continuous functions

4.1 Overview concepts:

In this chapter we are going to study real functions of one real variables, or simply functions which are defined on a non-empty part $\mathbb{E}$ of $\mathbb{R}$ to $\mathbb{R}$ with ($\mathbb{E} \subset \mathbb{R}$; or $\mathbb{E} = \mathbb{R}$).

4.1.1 Real function of one real variable

Definition 4.1

Any application from $\mathbb{E}$ to $\mathbb{R}$ is called a numerical function.
If $\mathbb{E} \subset \mathbb{R}$, we say that f is a numerical function of a real variable, or a real function of a real variable.
We write;

$$f: \begin{array}{ccc} \mathbb{E} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) \end{array}$$

$\mathbb{E}$ is called the domain of definition of f and is denoted by D_f.

Example 4.1

For example, the function defined by:

$$f: \begin{array}{ccc} \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{x} \end{array}$$

is a numerical function of one real variable. In this case the domain of definition of f is $D_f = \mathbb{R}^*$.

4.1.2 The Graph of a function

Definition 4.2

Let $f : D_f \longrightarrow \mathbb{R}$ be a numerical function of a real variable, the Graph of f is a set of ordered pairs of the form $(x, f(x))$. And denote it by Γ_f i.e:

$$\Gamma_f = \{(x, f(x))/x \in D_f\} \subset \mathbb{R}^2$$

Remark 4.1 *Γ_f is a subset of $\mathbb{R}^2$, i.e $\Gamma_f \subset \mathbb{R}^2$*

Example 4.2

The graph of $f(x) = \dfrac{1}{x}$ is shown below

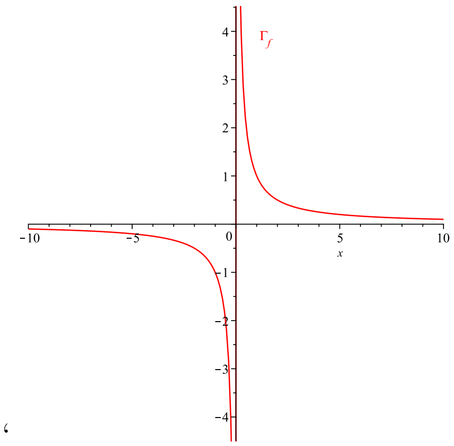


Figure 4.1: Graph of $f(x) = \dfrac{1}{x}$

4.1.3 Operations on Functions

Definition 4.3: (The sum and product of two functions)

Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ two functions defined on D to $\mathbb{R}$

- The sum of f and g is the function defined by $f + g$:

$$\begin{array}{rccl} f+g: & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & (f+g)(x) = f(x) + g(x) \end{array}$$

- The product of f and g is the function defined by $f.g$:

$$\begin{array}{rccl} f.g: & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & (f.g)(x) = f(x).g(x) \end{array}$$

- Let $\lambda \in \mathbb{R}$, the function $\lambda.f$ is defined by:

$$\begin{array}{rccl} \lambda.f: & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & (\lambda.f)(x) = \lambda.f(x) \end{array}$$

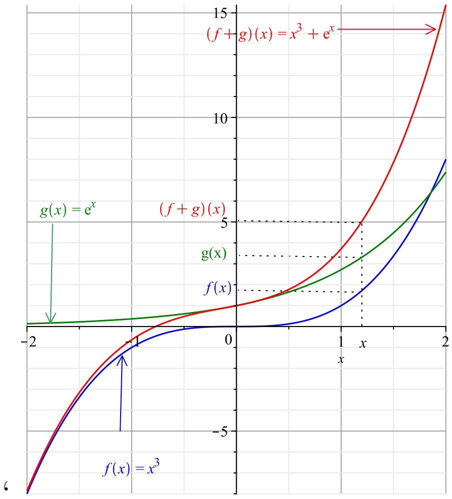


Figure 4.2: Graph of the sum of two functions $f + g$

4.1.4 Monotonicity, parity and periodicity

Definition 4.4

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function.

- The function f is said to be increasing on D_f iff:

$$\forall x, y \in D_f;\ x \leq y \implies f(x) \leq f(y)$$

- The function f is said to be strictly increasing on D_f iff:

$$\forall x, y \in D_f;\ x < y \implies f(x) < f(y)$$

- The function f is said to be decreasing on D_f iff:

$$\forall x, y \in D_f;\ x \leq y \implies f(x) \geq f(y)$$

- The function f is said to be strictly decreasing on D_f iff:

$$\forall x, y \in D_f;\ x < y \implies f(x) > f(y)$$

- The function f is said to be a constant function on D_f iff:

$$\exists a \in \mathbb{R}, \forall x, y \in D_f; f(x) = f(y) = a$$

- The function f is said to be monotonic on D_f if it is either increasing or decreasing on D_f
- The function f is said to be strictly monotonic on D_f if it is either strictly increasing or strictly decreasing on D_f

Example 4.3

1. The $\sqrt{x}$ function is strictly increasing on $[0, +\infty[$.
2. The function $\exp(x)$ is strictly increasing on $\mathbb{R}$ and $\ln(x)$ is strictly increasing on $]0, +\infty[$.
3. The function $\lfloor x \rfloor$ is increasing on $\mathbb{R}$.
4. The function $|x|$ is neither increasing nor decreasing on $\mathbb{R}$.

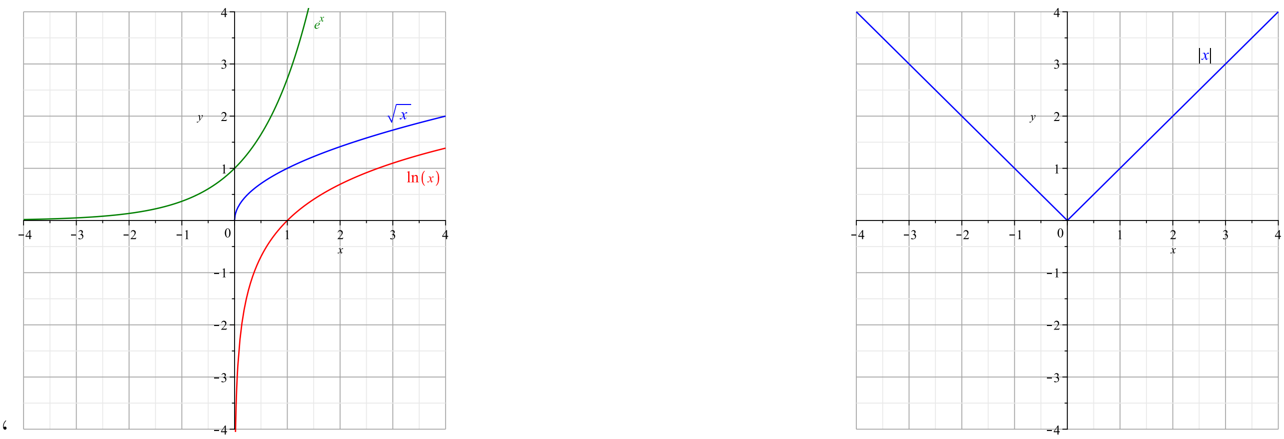


Figure 4.3: The functions $\exp(x)$,$\sqrt{x}$ and $\ln(x)$ (The function $|x|$ on the right)

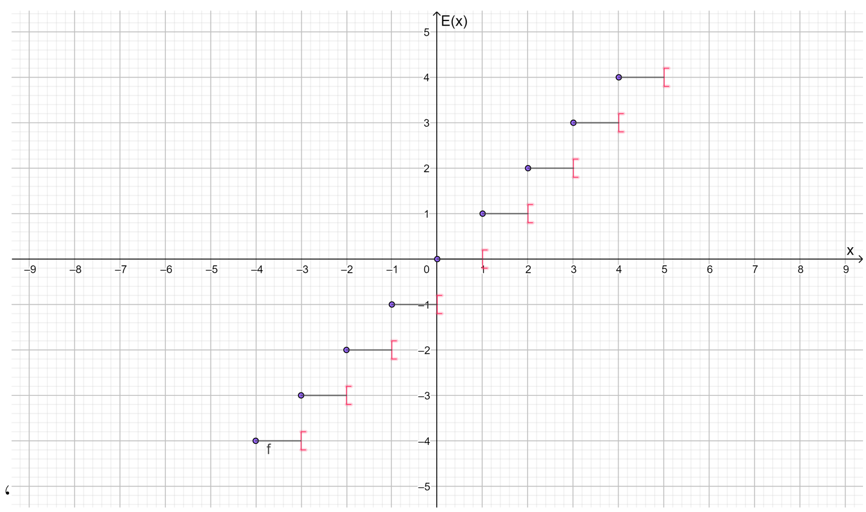


Figure 4.4: The integer part function

Definition 4.5

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function.

- We say that f is even iff: $\begin{cases} \forall x \in \mathbb{R};\ x \in D_f \implies -x \in D_f \\ \forall x \in D_f :\ f(-x) = f(x) \end{cases}$

- We say that f is odd iff: $\begin{cases} \forall x \in \mathbb{R};\ x \in D_f \implies -x \in D_f \\ \forall x \in D_f :\ f(-x) = -f(x) \end{cases}$

Graphical interpretation:

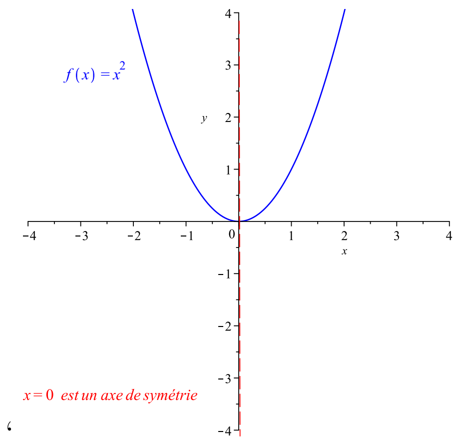
- The graphical representation of an even function has the y-axis as the axis of symmetry.

- The graphical representation of an odd function has the origin of the coordinate system as the centre of symmetry.

Example 4.4

1. Since:
$$\begin{cases} \forall x \in D_f = \mathbb{R} \implies -x \in D_f \\ \forall x \in D_f; f(-x) = (-x)^2 = x^2 = f(x), \end{cases}$$
then the function $f(x) = x^2$ is even.

2. Since:
$$\begin{cases} D_f = \mathbb{R} \\ \forall x \in D_f; f(-x) = -x^3 = -f(x), \end{cases}$$
then the function $f(x) = x^3$ is odd.



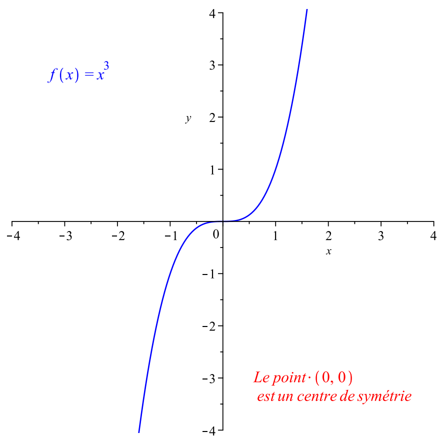


Figure 4.5: The function x^2 and x^3

Definition 4.6

Let $f : D_f \longleftarrow \mathbb{R}$ be a real function.
We say that a function f is periodic, with period $p \in \mathbb{R}_+^*$, if

$$\begin{cases} \forall x \in \mathbb{R};\ x \in D_f \implies x + p \in D_f \\ \forall x \in D_f;\ f(x+p) = f(x) \end{cases}$$

Graphical interpretation:

- If f is a periodic function with period p, then the graph of f is invariant by the translation of vector $p\,\overrightarrow{i}$.

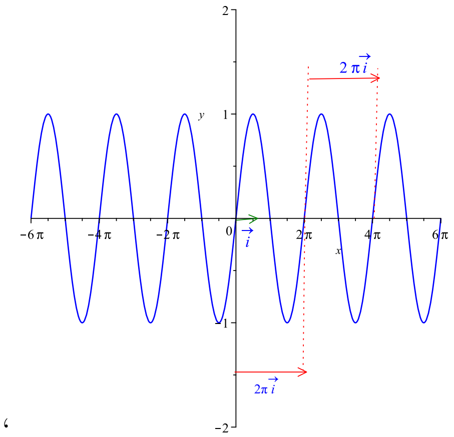


Figure 4.6: $\sin(x)$ is 2π-periodic

4.1.5 Bounded functions

Definition 4.7

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function.

- If there exists $m \in \mathbb{R}$ such that: $m \le f(x)$ for all $x \in D_f$, then the function f is said to be bounded from below by m. i.e

$$\exists m \in \mathbb{R}, \forall x \in D_f;\ m \le f(x) \Leftrightarrow \text{the function f is bounded from below}$$

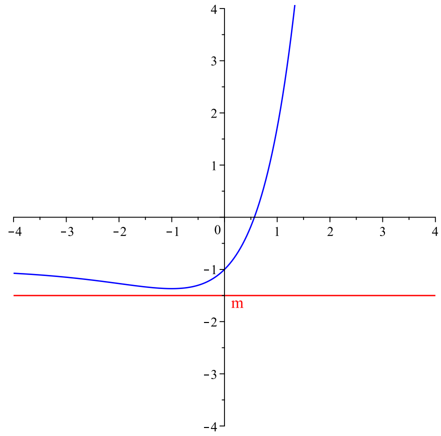
- If there exists $M \in \mathbb{R}$ such that: $f(x) \le M$ for all $x \in D_f$, then the function f is said to be bounded from above by M. i.e

$$\exists M \in \mathbb{R}, \forall x \in D_f;\ f(x) \le M \Leftrightarrow \text{the function f is bounded from above}$$

- If there exists $M, m \in \mathbb{R}$ such that: $m \le f(x) \le M$ for all $x \in D_f$, then the function f is said to be bounded. i.e

$$\exists M, m \in \mathbb{R}, \forall x \in D_f;\ m \le f(x) \le M \Leftrightarrow \text{the function f is bounded}$$

Remark 4.2 *Also, we can say that f is bounded on D_f iff:* $\exists M \in \mathbb{R}_+, \forall x \in D_f;\ |f(x)| \le M$.



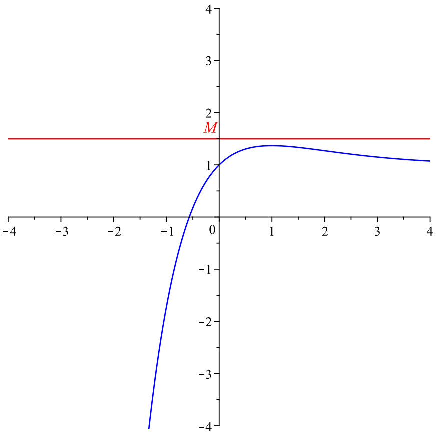


Figure 4.7: The bounded fom below function (in the left) and The bounded from above function (in the right)

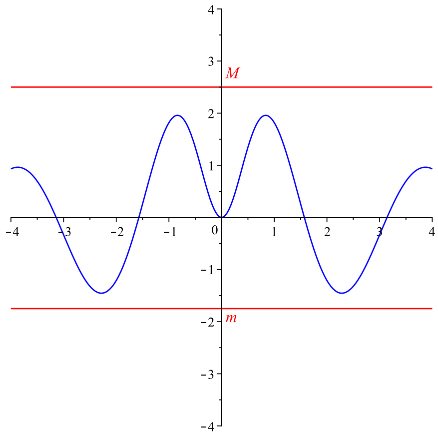


Figure 4.8: bounded function

Definition 4.8

Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ tow functions. We can write:

- $f \leq g$ iff: $\forall x \in D;\ f(x) \leq g(x)$
- $f < g$ iff: $\forall x \in D;\ f(x) < g(x)$
- $f = g$ iff: $\forall x \in D;\ f(x) = g(x)$

Rappel:-

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function. Recall that $f(D_f)$ is the set of all values of f denoted by:

$$f(D_f) = \{f(x)/x \in D_f\}$$

Let's put:

$$\begin{cases} \sup\limits_{x \in D_f} (f(x)) = \sup(f(D_f)) \\ \inf\limits_{x \in D_f} (f(x)) = \inf(f(D_f)) \end{cases}$$

Definition 4.9

- The smallest upper bound of f on D_f is called $\sup\limits_{x\in D_f}(f(x))$ and is denoted by :

$$\sup_{x\in D_f} f = \sup_{x\in D_f}(f(x))$$

- The greatest lower bound of f on D_f is called $\inf\limits_{x\in D_f}(f(x))$ and is denoted by :

$$\inf_{x\in D_f} f = \inf_{x\in D_f}(f(x))$$

Proposition 4.1

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function, then we have the following equivalences:

- f is bounded from above on D_f. $\Leftrightarrow \sup\limits_{x\in D_f} f \in \mathbb{R}$ and we write : $\sup\limits_{x\in D_f} f < +\infty$.
- f is bounded from below on D_f. $\Leftrightarrow \inf\limits_{x\in D_f} f \in \mathbb{R}$ and we write : $\inf\limits_{x\in D_f} f > -\infty$.
- f is bounded on D_f. $\Leftrightarrow \sup\limits_{x\in D_f} f, \inf\limits_{x\in D_f} f \in \mathbb{R}$ and we write : $\sup\limits_{x\in D_f} f < +\infty$ and $\inf\limits_{x\in D_f} f > -\infty$.
- $M = \sup\limits_{x\in D_f}(f(x)) \Leftrightarrow \begin{cases} \forall x \in D_f;\ f(x) \leq M \\ \forall \varepsilon > 0, \exists x_0 \in D_f;\ M - \varepsilon < f(x_0) \end{cases}$
- $m = \inf\limits_{x\in D_f}(f(x)) \Leftrightarrow \begin{cases} \forall x \in D_f;\ m \leq f(x) \\ \forall \varepsilon > 0, \exists x_0 \in D_f;\ f(x_0) < m + \varepsilon \end{cases}$

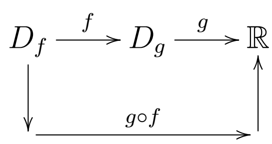
4.1.6 The composition of two functions

Definition 4.10

Consider $f : D_f \longrightarrow \mathbb{R}$ and $g : D_g \longrightarrow \mathbb{R}$ be two functions such that: $f(D_f) \subset D_g$. Then the composition of f and g, denoted by $g \circ f$ is defined as the function:

$$\forall x \in D_f;\ (g \circ f)(x) = g(f(x))$$

The below figure shows the representation of composite functions:



Example 4.5

Let f and g be two functions defined by:

$$f: \begin{array}{ll} \mathbb{R} & \longrightarrow \mathbb{R} \\ x & \longmapsto x^2+1 \end{array} \qquad g: \begin{array}{ll} [-1,+\infty[& \longrightarrow \mathbb{R} \\ x & \longmapsto \sqrt{x+1} \end{array}$$

We have $f(\mathbb{R}) = [1, +\infty[\implies f(D_f) \subset D_g$
So $g \circ f$ defined as follows:

$$\forall x \in \mathbb{R};\ (g \circ f)(x) = g(f(x)) = \sqrt{x^2+2}$$

4.2 Limits of Functions

4.2.1 Limite finie en un point x_0

Definition 4.11

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function, x_0 and l two numbers (with $x_0 \in D_f$ or $x_0 \notin D_f$).
We say that $f(x)$ tends to l when x tends to x_0 iff:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ |x - x_0| < \delta \implies |f(x) - l| < \varepsilon$$

and we write : $\lim\limits_{x \to x_0} f(x) = l$

Remark 4.3

1. *Inequality* $|x - x_0| < \delta \iff x \in]x_0 - \delta, x_0 + \delta[$.
2. *Inequality* $|f(x) - l| < \varepsilon \iff f(x) \in]l - \varepsilon, l + \varepsilon[$.
3. *We can replace inequality "$<$" by "$\leq$" in the definition.*

Graphical interpretation:
For any interval of type $J =]l - \varepsilon, l + \varepsilon[$ with $\varepsilon > 0$, we can find an interval of type $I =]x_0 - \delta, x_0 + \delta[$, such that the graphical representation of f restricted to I is included in J.

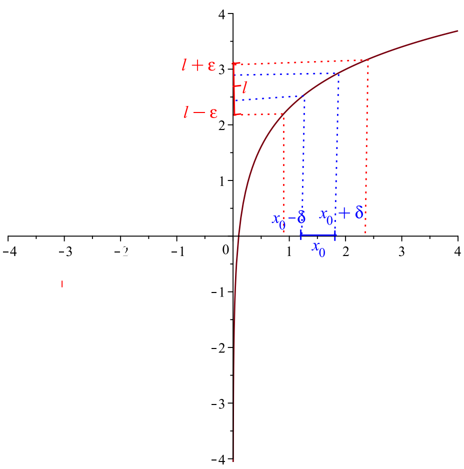


Figure 4.9:

Example 4.6

Show that $\lim_{x\to 2}(4x+1)=9$.
Let $\varepsilon>0$, we have:

$$|(4x+1)-9|\le\varepsilon \Leftrightarrow |4x-8|\le\varepsilon \Leftrightarrow 4|x-2|\le\varepsilon \Leftrightarrow |x-2|\le\frac{\varepsilon}{4}$$

Let's put $\delta=\frac{\varepsilon}{4}$ we obtain:

$$\forall\varepsilon>0,\exists\delta>0(\delta=\frac{\varepsilon}{4}),\forall x\in\mathbb{R};|x-2|\le\delta \implies |(4x+1)-9|\le\varepsilon$$

$$\implies \lim_{x\to 2}(4x+1)=9$$

Proposition 4.2

If a function f has a limit at x_0, then this limit is unique.

Proof 1

By contradiction, suppose that, f has two distinct limits l_1 and l_2. $(l_1\neq l_2)$ en x_0.
By setting: $\varepsilon=\frac{1}{3}|l_1-l_2|>0$ because $l_1\neq l_2$.
We have:

$$\begin{cases}\lim_{x\to x_0} f(x)=l_1\\ \quad et\\ \lim_{x\to x_0} f(x)=l_2\end{cases}$$

$$\implies \begin{cases}\exists\delta_1(\varepsilon)>0,\forall x\in D_f;\ |x-x_0|\le\delta_1 \implies |f(x)-l_1|\le\varepsilon\\ \quad et\\ \exists\delta_2(\varepsilon)>0,\forall x\in D_f;\ |x-x_0|\le\delta_2 \implies |f(x)-l_2|\le\varepsilon\end{cases}$$

By choosing: $\delta=\min(\delta_1,\delta_2)$ we get:

$$\begin{cases}\forall x\in D_f;\ |x-x_0|\le\delta \implies |f(x)-l_1|\le\varepsilon\\ \quad and\\ \forall x\in D_f;\ |x-x_0|\le\delta \implies |f(x)-l_2|\le\varepsilon\end{cases}$$

$$\implies \forall x\in D_f;\ |x-x_0|\le\delta \implies |f(x)-l_1|+|f(x)-l_2|\le 2\varepsilon \tag{4.1}$$

According to the triangle inequality we have:

$$|l_1-l_2|=|l_1-f(x)+f(x)-l_2|\le|f(x)-l_1|+|f(x)-l_2| \tag{4.2}$$

$$(3.1)\text{ and }(3.2) \implies |l_1-l_2|\le 2\varepsilon \implies |l_1-l_2|\le\frac{2}{3}|l_1-l_2| \implies 1\le\frac{2}{3}$$

So we end up with a contradiction, this means that f admits a unique limit at point x_0.

4.2.2 Left and Right-Hand Limits

Definition 4.12

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function, x_0 and l be two real numbers. (with $x_0 \in D_f$ or $x_0 \notin D_f$).

- We say that l is the left limit of the function f at a point x_0 iff:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ x_0 - \delta \leq x < x_0 \implies |f(x) - l| \leq \varepsilon$$

  and we write: $\lim\limits_{x \overset{<}{\longrightarrow} x_0} f(x) = l$ or $\lim\limits_{x \to x_0^-} f(x) = l$

- We say that l is the right limit of the function f at a point x_0 iff:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ x_0 < x \leq x_0 + \delta \implies |f(x) - l| \leq \varepsilon$$

  and we write: $\lim\limits_{x \overset{>}{\longrightarrow} x_0} f(x) = l$ or $\lim\limits_{x \to x_0^+} f(x) = l$

Example 4.7

- prove that: $\lim\limits_{x \to 0^+} x\cos(\frac{1}{x}) = 0$
  We have:

$$|x\cos(\frac{1}{x})| \leq |x||\cos(\frac{1}{x})| \leq |x|$$
$$(\text{as } |\cos(\frac{1}{x})| \leq 1)$$
$$\implies |x\cos(\frac{1}{x})| \leq |x| \tag{4.3}$$

  Let $\varepsilon > 0$, with $\delta = \varepsilon$. If we have: $0 < x \leq \delta \Leftrightarrow 0 < x \leq \varepsilon \implies |x| \leq \varepsilon$

  $(3.3) \implies |x\cos(\frac{1}{x})| \leq \varepsilon.$

  So, $\forall \varepsilon > 0, \exists \delta > 0\,(\delta = \varepsilon), \forall x \in \mathbb{R}^*;\ 0 < x \leq \delta \implies |x\cos(\frac{1}{x})| \leq \varepsilon$

  $\implies \lim\limits_{x \to 0^+} x\cos(\frac{1}{x}) = 0$

- Show that $\lim\limits_{x \to 0^-} x\cos(\frac{1}{x}) = 0$ (Using the same technique as above)

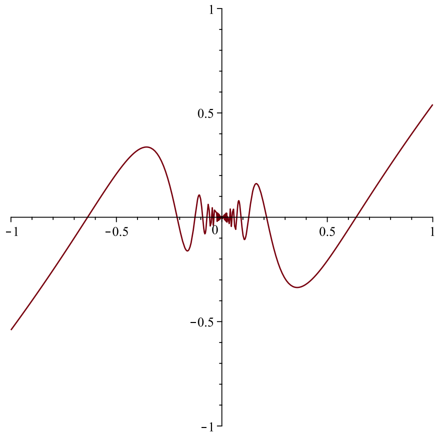


Figure 4.10: Graph of the function $x\cos(\frac{1}{x})$

Theorem 4.1

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function, $x_0, l \in \mathbb{R}$ (with $x_0 \in D_f$ or $x_0 \notin D_f$). The following propositions are equivalent

1. $\lim\limits_{x \to x_0} f(x) = l$
2. $\lim\limits_{x \overset{<}{\longrightarrow} x_0} f(x) = \lim\limits_{x \overset{>}{\longrightarrow} x_0} f(x) = l$

Result:

If we have: $\lim\limits_{x \overset{<}{\longrightarrow} x_0} f(x) \neq \lim\limits_{x \overset{>}{\longrightarrow} x_0} f(x)$ then $\lim\limits_{x \to x_0} f(x)$ doesn't exist.

Example 4.8

Let's consider the function $f(x) = \dfrac{|x|}{x}$. We have:

$$\begin{cases} \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} -\dfrac{x}{x} = -1 \\ \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{x}{x} = 1 \end{cases}$$

$\Longrightarrow \lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$ then $\lim\limits_{x \to 0} f(x)$ doesn't exist.

4.2.3 Infinite limit of a function at x_0.

Definition 4.13

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function and $x_0 \in \mathbb{R}$ (with $x_0 \in D_f$ or $x_0 \notin D_f$)

- It is said that f tends to $+\infty$ when x tends to x_0 iff:

$$\forall A > 0, \exists \delta > 0, \forall x \in D_f;\ |x - x_0| \leq \delta \implies f(x) \geq A$$

  and we write $\lim_{x \to x_0} f(x) = +\infty$

- It is said that f tends to $-\infty$ when x tends to x_0 iff:

$$\forall A > 0, \exists \delta > 0, \forall x \in D_f;\ |x - x_0| \leq \delta \implies f(x) \leq -A$$

  and we write $\lim_{x \to x_0} f(x) = -\infty$

Example 4.9

Show that $\lim_{x \to 0} \frac{1}{x^2} = +\infty$
Let $A > 0$ we have:

$$\frac{1}{x^2} \geq A \Leftrightarrow x^2 \leq \frac{1}{A} \Leftrightarrow x^2 - \frac{1}{A} \leq 0 \Leftrightarrow x \in \left[-\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{A}}\right] \Leftrightarrow |x| \leq \frac{1}{\sqrt{A}}$$

Putting $\delta = \frac{1}{\sqrt{A}}$ then $\forall x \in D_f;\ |x| \leq \delta \implies \frac{1}{x^2} \geq A$

$\implies \forall A > 0, \exists \delta > 0\,(\delta = \frac{1}{\sqrt{A}}), \forall x \in D_f;\ |x| \leq \delta \implies \frac{1}{x^2} \geq A$ therfore $\lim_{x \to 0} \frac{1}{x^2} = +\infty$

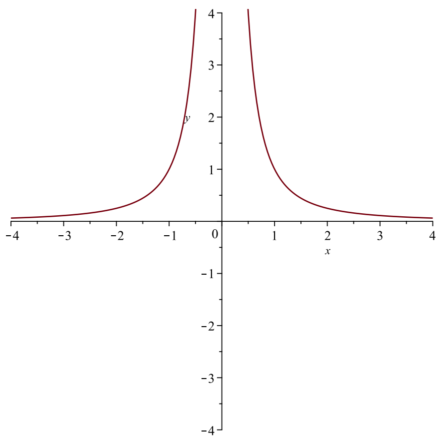


Figure 4.11: The graph of a function $\frac{1}{x^2}$

4.2.4 Finite limit of a function at $-\infty$ and $+\infty$

Definition 4.14

- Let f be a function defined on an interval of type $]-\infty, a]$ (i.e $]-\infty, a] \subset D_f$)
We say that f tends to l ($l \in \mathbb{R}$) when x tends to $-\infty$ iff:

$$\forall \varepsilon > 0, \exists B > 0, \forall x \in D_f;\ x \leq -B \implies |f(x) - l| \leq \varepsilon$$

and we write $\lim\limits_{x \to -\infty} f(x) = l$

- Let f be a function defined on an interval of type $[a, +\infty[$ (i.e. $[a, +\infty[\subset D_f$).
We say that f tends to l ($l \in \mathbb{R}$) when x tends to $+\infty$ iff:

$$\forall \varepsilon > 0, \exists B > 0, \forall x \in D_f;\ x \geq B \implies |f(x) - l| \leq \varepsilon$$

and we write $\lim\limits_{x \to +\infty} f(x) = l$

Example 4.10

prove that $\lim\limits_{x \to +\infty} \frac{x}{x+1} = 1$ Let $\varepsilon > 0$, we have:

$$\left|\frac{x}{x+1} - 1\right| \leq \varepsilon \Leftrightarrow \left|\frac{1}{x+1}\right| \leq \varepsilon \Leftrightarrow |x+1| \geq \frac{1}{\varepsilon}$$

$$\Leftrightarrow \begin{cases} x + 1 \geq \frac{1}{\varepsilon} \\ or \\ x + 1 \leq -\frac{1}{\varepsilon} \end{cases} \Leftrightarrow \begin{cases} x \geq \frac{1}{\varepsilon} - 1 \\ or \\ x \leq -1 - \frac{1}{\varepsilon} \end{cases}$$

We set $B = \frac{1}{\varepsilon} - 1$, if $x \geq B \implies \left|\frac{x}{x+1} - 1\right| \leq \varepsilon$

So, $\forall \varepsilon > 0, \exists B > 0\,(B = \frac{1}{\varepsilon} - 1), \forall x \in D_f;\ x \geq B \implies \left|\frac{x}{x+1} - 1\right| \leq \varepsilon$

$\implies \lim\limits_{x \to +\infty} \frac{x}{x+1} = 1.$

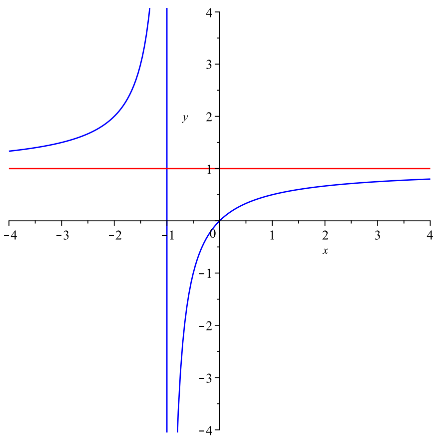


Figure 4.12: The graph of a function $\frac{x}{x+1}$

4.2.5 Infinite limit of a function at $+\infty$ and $-\infty$

Definition 4.15

- Let f be a function defined on an interval of type $[a,+\infty[$ (i.e. $[a,+\infty[\subset D_f$). We say that f tends to $+\infty$ when x tends to $+\infty$ if:

$$\forall A>0, \exists B>0, \forall x\in D_f;\ x\geq B \implies f(x)\geq A$$

and we write: $\lim_{x\to+\infty} f(x)=+\infty$

- Let f be a function defined on an interval of type $]-\infty,a]$ (i.e. $]-\infty,a]\subset D_f$) . We say that f tends to $+\infty$ when x tends to $-\infty$ if:

$$\forall A>0, \exists B>0, \forall x\in D_f;\ x\leq -B \implies f(x)\geq A$$

and we write: $\lim_{x\to-\infty} f(x)=+\infty$

- Let f be a function defined on an interval of type $[a,+\infty[$ (i.e. $[a,+\infty[\subset D_f$) . We say that f tends to $-\infty$ when x tends to $+\infty$ if:

$$\forall A>0, \exists B>0, \forall x\in D_f;\ x\geq B \implies f(x)\leq -A$$

and we write: $\lim_{x\to+\infty} f(x)=-\infty$

- Let f be a function defined on an interval of type $]-\infty,a]$ (i.e. $]-\infty,a]\subset D_f$) . We say that f tends to $-\infty$ when x tends to $-\infty$ if:

$$\forall A>0, \exists B>0, \forall x\in D_f;\ x\leq -B \implies f(x)\leq -A$$

and we write: $\lim_{x\to-\infty} f(x)=-\infty$

Notation: Let $\overline{\mathbb{R}}$ denote the set defined by:

$$\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty,-\infty\}$$

$\overline{\mathbb{R}}$ is called the extended real line.

4.2.6 Relationship between limits and sequences

Theorem 4.2

Let $f: D_f \longrightarrow \mathbb{R}$ be a real function, $x_0\in\mathbb{R}$ (with $x_0\in D_f$ or $x_0\notin D_f$) and $l\in\overline{\mathbb{R}}$. The following properties are equivalent:

1. $\lim_{x\to x_0} f(x)=l$
2. For any sequence $(x_n)_{n\in\mathbb{N}}$ in D_f such that: $\forall n\in\mathbb{N}; x_n\neq x_0$ and $\lim_{n\to+\infty} x_n=x_0$, then we have $\lim_{n\to+\infty} f(x_n)=l$

Proof 2

- First, we prove implication $(1 \implies 2)$.
  Let $\varepsilon > 0$,
$$\exists\delta_\varepsilon > 0, \forall x \in D_f;\ |x - x_0| \leq \delta_\varepsilon \implies |f(x) - l| \leq \varepsilon \tag{4.4}$$
$$(\text{As} \lim_{x \to x_0} f(x) = l)$$
$$\delta_\varepsilon > 0 \implies \exists n_0(\delta_\varepsilon) \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_0 \implies |x_n - x_0| \leq \delta_\varepsilon \tag{4.5}$$
$$(\text{Because} \lim_{n \to +\infty} x_n = x_0)$$
$$(3.4) \text{ et } (3.5) \implies |f(x_n) - l| \leq \varepsilon$$
$$\implies \forall\varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_0 \implies |f(x_n) - l| \leq \varepsilon$$
$$\implies \lim_{n \to +\infty} f(x_n) = l$$

- Next, we prove the implication $(2 \implies 1)$ by contradiction proof,
  we assume that for any sequence $(x_n)_{n\in\mathbb{N}} \subset D_f$ that converges to x_0 we have $f(x_n)$ converges to l and $\lim_{x \to x_0} f(x) \neq l$.
$$\lim_{x \to x_0} f(x) \neq l \Leftrightarrow \exists\varepsilon > 0, \forall\delta > 0, \exists x^* \in D_f;\ |x^* - x| \leq \delta \wedge |f(x^*) - l| \geq \varepsilon \tag{4.6}$$
  We set: $\delta = \frac{1}{n}/n \in \mathbb{N}^*$
$$(3.6) \implies \forall n \in \mathbb{N}^*, \exists x_n \in D_f; (|x_n - x_0| \leq \frac{1}{n}) \wedge (|f(x_n) - l| > \varepsilon)$$
  So we have found a sequence $(x_n)_{n\in\mathbb{N}^*} \subset D_f$ that converges to x_0.

  (since $\forall n \in \mathbb{N}^*; |x_n - x_0| \leq \frac{1}{n}$) et $f(x_n)$ doesn't converge to l (as $\forall n \in \mathbb{N}^*; |f(x_n) - l| > \varepsilon$), which contradicts our hypothesis.

Remark 4.4 *If there are two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ of D_f such that:*
$$\begin{cases} \lim_{n \to +\infty} x_n = x_0 \\ \quad\textit{and} \\ \lim_{n \to +\infty} y_n = x_0 \end{cases} \wedge \lim_{n \to +\infty} f(x_n) \neq \lim_{n \to +\infty} f(y_n)$$
Then $\lim_{x \to x_0} f(x)$ doesn't exist.

Example 4.11

Let $\begin{array}{llll} f: & \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \sin(\frac{1}{x}) \end{array}$

We have $\lim\limits_{x\to 0}\sin(\frac{1}{x})$ doesn't exist because: If we set $x_n = \dfrac{1}{n\pi}$ et $y_n = \dfrac{1}{\frac{\pi}{2}+2n\pi}$

The sequences $(x_n)_{n\in\mathbb{N}^*}$, $(y_n)_{n\in\mathbb{N}^*}$ in $\mathbb{R}^*$ and $\lim\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} y_n = 0$

On the other hand, we have: $\begin{cases} \lim\limits_{n\to+\infty} f(x_n) = \lim\limits_{n\to+\infty} \sin(n\pi) = 0 \\ \quad et \\ \lim\limits_{n\to+\infty} f(y_n) = \lim\limits_{n\to+\infty} \sin(\frac{\pi}{2}+2n\pi) = 1 \end{cases}$

$$\Longrightarrow \lim_{n\to+\infty} f(x_n) \neq \lim_{n\to+\infty} f(y_n)$$

$$\Longrightarrow \lim_{x\to x_0} f(x) \text{ doesn't exist.}$$

4.2.7 Limits operations

Proposition 4.3: (The limit of sum of two or more functions)

Let f, g be two functions and $x_0 \in \overline{\mathbb{R}}$. Then we have:

$\lim\limits_{x\to x_0} f(x)$	$\lim\limits_{x\to x_0} g(x)$	$\lim\limits_{x\to x_0} (f(x)+g(x))$
$l_1 \in \mathbb{R}$	$l_2 \in \mathbb{R}$	$l_1 + l_2$
$l_1 \in \mathbb{R}$	$\pm\infty$	$\pm\infty$
$+\infty$	$+\infty$	$+\infty$
$-\infty$	$-\infty$	$-\infty$
$+\infty$	$-\infty$	Indeterminate form
$-\infty$	$+\infty$	Indeterminate form

Proposition 4.4: (The limit of product of two or more functions)

Let f, g be two functions and $x_0 \in \overline{\mathbb{R}}$. Then we have:

$\lim\limits_{x\to x_0} f(x)$ / $\lim\limits_{x\to x_0} g(x)$	$l_2 > 0$	$l_2 < 0$	0	$+\infty$	$-\infty$
$l_1 > 0$	$\lim\limits_{x\to x_0} fx)g(x) = l_1 l_2$	$l_1 l_2$	0	$+\infty$	$-\infty$
$l_1 < 0$	$l_1 l_2$	$l_1 l_2$	0	$-\infty$	$+\infty$
0	0	0	0	Indeterminate form	Indeterminate form
$+\infty$	$+\infty$	$-\infty$	Indeterminate form	$+\infty$	$-\infty$
$-\infty$	$-\infty$	$+\infty$	Indeterminate form	$-\infty$	$+\infty$

Proposition 4.5: (The limit of quotient of two functions)

Let f, g be two functions defined on D with $g(x) \neq 0$ on D and $x_0 \in \overline{\mathbb{R}}$. Then we have:

$\lim_{x\to x_0} f(x)$ \ $\lim_{x\to x_0} g(x)$	$l_2 > 0$	$l_2 < 0$	0^+	0^-	$+\infty$	$-\infty$
$l_1 > 0$	$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{l_1}{l_2}$	$\frac{l_1}{l_2}$	$+\infty$	$-\infty$	0	0
$l_1 < 0$	$\frac{l_1}{l_2}$	$\frac{l_1}{l_2}$	$-\infty$	$+\infty$	0	0
0^+	0	0	Indeterminate form	Indeterminate form	0	0
0^-	0	0	Indeterminate form	Indeterminate form	0	0
$+\infty$	$+\infty$	$-\infty$	$+\infty$	$-\infty$	IF	IF
$-\infty$	$-\infty$	$+\infty$	$-\infty$	$+\infty$	IF	IF

Remark 4.5 *According to the previous propositions, the indeterminate forms are:* $+\infty - \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$. *Also we can deduce the other forms which are:* 0^0, ∞^0, 1^∞

4.2.8 Limit of Composite Functions

Proposition 4.6

Let $f : D_f \longrightarrow \mathbb{R}$, $g : D_g \longrightarrow \mathbb{R}$ and $x_0, y_0, l \in \overline{\mathbb{R}}$.

If we have: $\begin{cases} \lim_{x\to x_0} f(x) = y_0 \\ \quad et \\ \lim_{x\to y_0} g(x) = l \end{cases}$ then $\lim_{x\to x_0} (g \circ f)(x) = l$

Example 4.12

Let $\begin{array}{rccc} f: & \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \dfrac{e^x - 1}{x} \end{array}$ and $\begin{array}{rccc} g: & \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \ln(x) \end{array}$

We have: $\begin{cases} \lim_{x\to 0} f(x) = \lim_{x\to 0} \left(\dfrac{e^x - 1}{x}\right) = 1 \\ \quad and \\ \lim_{x\to 1} g(x) = \lim_{x\to 1} \ln(x) = 0 \end{cases}$

Then $\lim_{x\to 0} (g \circ f)(x) = \lim_{x\to 0} \ln\left(\dfrac{e^x - 1}{x}\right) = 0$

4.2.9 Finding Limits: Properties of Limits

Proposition 4.7

1. If we have: $\lim_{x \to x_0} f(x) = l$ then there exists $\alpha > 0$ such that the function f is bounded on $]x_0 - \alpha, x_0 + \alpha[$.
2. If we have: $f(x) \leq g(x)$ in the neighbourhood of x_0 and $\lim_{x \to x_0} f(x) = l_1$, $\lim_{x \to x_0} g(x) = l_2$ then $l_1 \leq l_2$.
3. The Squeeze Theorem: Let f, g, h be three functions with the following property $f(x) \leq g(x) \leq h(x)$ in the neighbourhood of x_0.

$$\text{If we have: } \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l \text{ then } \lim_{x \to x_0} g(x) = l$$

4. Let f, g two functions which verify $f(x) \leq g(x)$ in the neighbourhood of x_0

$$\text{If we have: } \begin{cases} \lim_{x \to x_0} f(x) = +\infty \text{ then } \lim_{x \to x_0} g(x) = +\infty \\ \lim_{x \to x_0} g(x) = -\infty \text{ then } \lim_{x \to x_0} f(x) = -\infty \end{cases}$$

5. Let f be a bounded function in the neighborhood of x_0 and g a function verifying $\lim_{x \to x_0} g(x) = 0$ then $\lim_{x \to x_0} f(x)g(x) = 0$.

Definition 4.16: (important definition)

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function. We say that f is defined in the neighborhood of x_0 iff: there exists an interval of the following type $I =]x_0 - \varepsilon, x_0 + \varepsilon[$ such that: $I \subset D_f$. (I is an interval with center x_0 and radius $\varepsilon > 0$).

4.3 Continuous Functions

4.3.1 Continuity at a point x_0

Definition 4.17

Let f be a function defined in the neighborhood of x_0. We say that f is continuous at the point x_0 iff:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

$$\text{i.e} \qquad \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ |x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \varepsilon$$

Example 4.13

Let f be a function defined by:

$$f(x) = \begin{cases} x \sin(\frac{1}{x^2}), & \text{si } x \neq 0 \\ 0 & \text{si } x = 0 \end{cases}$$

Show that f is continuous at $x_0 = 0$.

1. $D_f = \mathbb{R} \implies f$ is defined in the neighbourhood of $x_0 = 0$.
2. We'll show that $\lim_{x \to 0} f(x) = 0$.
   We have:
   $$\forall x \in \mathbb{R}^*;\ |x \sin(\frac{1}{x^2}| \leq |x|$$
   If we choose $\delta = \varepsilon$ (with $\varepsilon > 0$), we find:
   $$\forall \varepsilon > 0, \exists \delta > 0\ (\delta = \varepsilon), \forall x \in \mathbb{R};\ |x| \leq \delta \implies |x \sin(\frac{1}{x^2}| \leq \varepsilon$$
   $$\implies \lim_{x \to 0} f(x) = 0 \implies f \text{ is continuous at } x_0$$

4.3.2 Left and right continuity at a point x_0

Definition 4.18

- Let f be a function defined on an interval of kind $[x_0, x_0 + h[$ with $h > 0$ (i.e.;$[x_0, x_0 + h[\subset D_f$). A function f is right continuous at a point x_0 iff:
  $$\lim_{x \overset{>}{\longrightarrow} x_0} f(x) = f(x_0)$$
  $$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ x_0 \leq x < x_0 + \delta \implies |f(x) - f(x_0)| \leq \varepsilon$$
- Let f be a function defined on an interval of kind $[x_0 - h, x_0[$ with $h > 0$ (i.e.;$[x_0 - h, x_0[\subset D_f$). A function f is left continuous at a point x_0 iff:
  $$\lim_{x \overset{<}{\longrightarrow} x_0} f(x) = f(x_0)$$
  $$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ x_0 - \delta < x \leq x_0 \implies |f(x) - f(x_0)| \leq \varepsilon$$

Example 4.14

Let f be a function defined by:

$$f(x) = \begin{cases} \dfrac{\sin(x)}{|x|}, & \text{si } x \neq 0 \\ 1 & \text{si } x = 0 \end{cases}$$

1. We'll study the right continuity of f at $x_0 = 0$
   - We have $D_f = \mathbb{R} \implies f$ is defined in the right of $x_0 = 0$
   - $\lim\limits_{x \overset{>}{\longrightarrow} 0} \dfrac{\sin(x)}{|x|} = \lim\limits_{x \overset{>}{\longrightarrow} 0} \dfrac{\sin(x)}{x} = 1 = f(1)$

   So f is right continuous at x_0.
2. The continuity of f at the left of $x_0 = 0$.
   - We have $D_f = \mathbb{R} \implies f$ is defined in the left of $x_0 = 0$.
   - $\lim\limits_{x \overset{<}{\longrightarrow} 0} \dfrac{\sin(x)}{|x|} = \lim\limits_{x \overset{<}{\longrightarrow} 0} -\dfrac{\sin(x)}{x} = -1 \neq f(1)$

   So f is not left continuous at x_0.

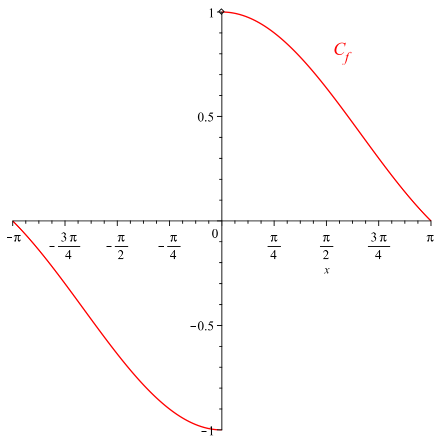


Figure 4.13: Graph of the function f

Theorem 4.3

Let f be a function defined in the neighborhood of x_0. The following two propositions are equivalent:

1. f is left and right continuous at x_0.
2. f is continuous at x_0.

Remark 4.6 *Our example (4.14) shows that f is right continuous at $x_0 = 0$ and is not left continuous at $x_0 = 0$. which implies that f is not continuous at $x_0 = 0$.*

4.3.3 Continuous extension to a point

Definition 4.19

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function and $x_0 \in \mathbb{R}$ Such that: $x_0 \notin D_f$. If $\lim\limits_{x \to x_0} f(x) = l \in \mathbb{R}$.exists, but $f(x_0)$ is not defined, we define a new function:

$$\begin{aligned} \tilde{f} :\ D_f \cup \{x_0\} &\longrightarrow \mathbb{R} \\ x &\longmapsto \tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ l & \text{if } x = x_0 \end{cases} \end{aligned}$$

which is continuous at x_0. It is called the continuous extension of f to x_0.

Example 4.15

Let

$$\begin{aligned} f :\ \mathbb{R}^* &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{\sin(x)}{x} \end{aligned}$$

Can we extend the function f to be continuous at $x_0 = 0$.

We have: $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1 \implies f$ has a finite limit at $x_0 = 0$
So f is extendable by continuity at $x_0 = 0$ and the extension by continuity of f at $x_0 = 0$ is defined by:

$$\begin{aligned} \tilde{f} :\ \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \tilde{f}(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \end{aligned}$$

4.3.4 Operations on continuous functions at x_0

Theorem 4.4

Let f, g be two continuous functions at a point $x_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Then:

1. The function $|f|$ is continuous at x_0.
2. The functions λf, $f + g$ and fg are continuous at x_0.
3. If $g(x_0) \neq 0$ then $\dfrac{f}{g}$ is continuous at x_0.

Proposition 4.8

Let $f : D_f \longrightarrow \mathbb{R}$ and $g : D_g \longrightarrow \mathbb{R}$ be two functions such that: $f(D_f) \subset D_g$.

If we have:

$$\begin{cases} f \text{ is defined in a neighborhood of } x_0 \text{ and continuous at } x_0 \\ \qquad \text{et} \\ g \text{ is defined in a neighborhood of } y_0 = f(x_0) \text{ and continuous at } y_0 \end{cases}$$

Then $(g \circ f)(x)$ is continuous at x_0.

4.3.5 The sequential continuity theorem

Theorem 4.5

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function. The following two statements are equivalent:

1. f is continuous at x_0.
2. for each sequence $(x_n)_{n\in\mathbb{N}} \subset D_f$ such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Proof 3

The proof of this theorem follows from theorem (4.2)

Proposition 4.9

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function and $x_0 \in D_f$.
If f is continuous at x_0 and $f(x_0) \neq 0$ then there exists a neighborhood $(\mathcal{V})$ of x_0 such that:

$$\forall x \in \mathcal{V};\ f(x) \neq 0$$

Proof 4

We have f is continuous at x_0 so,

1. f is defined in a neighborhood of x_0

$$\Leftrightarrow \exists \eta > 0 \text{ such that: } I =]x_0 - \eta, x_0 + \eta[\subset D_f \tag{4.7}$$

2. $\lim_{x \to x_0} f(x) = f(x_0)$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ |x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \varepsilon \tag{4.8}$$

If we put: $\varepsilon = \frac{1}{2}|f(x_0)|$ then:

$$\exists \delta > 0, \forall x \in D_f;\ |x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \frac{1}{2}|f(x_0)|$$

$$(3.7) \implies \exists \delta > 0, \forall x \in I;\ |x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \frac{1}{2}|f(x_0)|$$

$$\implies \forall x \in I \cap]x_0 - \eta, x_0 + \eta[;\ |f(x) - f(x_0)| \leq \frac{1}{2}|f(x_0)| \tag{4.9}$$

Let's put : $\mathcal{V} = I \cap]x_0 - \delta, x_0 + \delta[$, then $\mathcal{V}$ is a neighborhood of x_0.
On the other hand, according to the triangular inequality, we have:

$$|f(x_0)| - |f(x)| \leq ||f(x_0)| - |f(x)|| \leq |f(x_0) - f(x)| \leq \frac{1}{2}|f(x_0)|$$

$$(4.9) \implies \forall x \in \mathcal{V}; |f(x_0)| - |f(x)| \leq \frac{1}{2}|f(x_0)|$$

$$\implies \forall x \in \mathcal{V};\ |f(x)| \geq \frac{1}{2}|f(x_0)| \neq 0$$

So there is a neighbourhood $\mathcal{V}$ of x_0 such that: $\forall x \in \mathcal{V}; f(x) \neq 0$.

4.3.6 Continuity over an interval

Definition 4.20

1. f is said to be continuous on an open interval of type $]a, b[$ iff: it is continuous at any point on the interval $]a, b[$.
2. f is said to be continuous on an interval of type $[a, b]$ iff: it is continuous on $]a, b[$ and continuous to the right of a and to the left of b.
3. f is said to be continuous on an interval of type $]a, b]$ iff: it is continuous on $]a, b[$ and continuous to the left of b.
4. f is said to be continuous on an interval of type $[a, b[$ iff: it is continuous on $]a, b[$ and continuous to the right of a.

4.3.7 Uniform continuity

Definition 4.21

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function. We say that f is uniformly continuous on D_f iff:

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0, \forall x, y \in D_f;\ |x - y| \leq \delta(\varepsilon) \implies |f(x) - f(y)| \leq \varepsilon$$

Remark 4.7 *Note that uniform continuity is a property of the function over the set D_f, while continuity can be defined at a point $x_0 \in D_f$. The number δ depends only on ε in the case of uniform continuity, but in the case of continuity at a point x_0, δ depends on ε and x_0.*

Example 4.16

Show that the function $f(x) = \sqrt{x}$ is uniformly continuous on $\mathbb{R}_+$.
Solution: Step 01: In this step, we'll show that:
$\forall x, y \in \mathbb{R}_+ : \sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$ and $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$

1. Let $x, y \in \mathbb{R}_+$ we have: $0 \leq 2\sqrt{x}\sqrt{y} \Leftrightarrow x + y \leq x + 2\sqrt{x}\sqrt{y} + y \Leftrightarrow x + y \leq \left(\sqrt{x} + \sqrt{y}\right)^2 \Leftrightarrow \sqrt{x + y} \leq |\sqrt{x} + \sqrt{y}| = \sqrt{x} + \sqrt{y}$
   So $\forall x, y \in \mathbb{R}_+;\ \sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$.

2. Let $x, y \in \mathbb{R}_+$ we have:

$$\begin{aligned} (x - y) \leq |x - y| &\implies y + (x - y) \leq y + |x - y| \\ &\implies \sqrt{x} \leq \sqrt{y + |x - y|} \leq \sqrt{y} + \sqrt{|x - y|} \\ &\implies \sqrt{x} - \sqrt{y} \leq \sqrt{|x - y|} \qquad (4.10) \end{aligned}$$

   on the other hand, we have:

$$\begin{aligned} (y - x) \leq |y - x| &\implies x + (y - x) \leq x + |x - y| \\ &\implies \sqrt{y} \leq \sqrt{x + |x - y|} \leq \sqrt{x} + \sqrt{|x - y|} \\ &\implies -\sqrt{|x - y|} \leq \sqrt{x} - \sqrt{y} \qquad (4.11) \end{aligned}$$

   (4.10) and (4.11) $\implies \forall x, y \in \mathbb{R}_+;\ |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$

Step 02:
In this step we will show the uniform continuity of the function $f(x) = \sqrt{x}$
Let $\varepsilon > 0$ and $x, y \in \mathbb{R}_+$ we have:

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$$

Let's put $\delta = \varepsilon^2$ then:

$$|x - y| \leq \delta \implies \sqrt{|x - y|} \leq \varepsilon \implies |\sqrt{x} - \sqrt{y}| \leq \varepsilon$$

$$\implies \forall \varepsilon > 0, \exists \delta > 0\ (\delta = \varepsilon^2), \forall x, y \in \mathbb{R}_+;\ |x - y| \leq \delta \implies |\sqrt{x} - \sqrt{y}| \leq \varepsilon$$

therefore, $f(x) = \sqrt{x}$ is uniformly continuous on $\mathbb{R}_+$.

Example 4.17

Show that the function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.
Solution:
$f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$

$$\Leftrightarrow \exists \varepsilon > 0, \forall \delta > 0, \exists x, y \in \mathbb{R};\ |x-y| \leq \delta \wedge |x^2 - y^2| > \varepsilon$$

Let's put: $\varepsilon = 1$
Let $\delta > 0$, we will confirm the existence of $x, y \in \mathbb{R}$ such that:$|x-y| \leq \delta \wedge |x^2 - y^2| > \varepsilon$
Let's take : $y = x + \frac{1}{2}\delta \implies x - y = \frac{1}{2}\delta \implies |x-y| = \frac{1}{2}\delta \leq \delta$
$|x^2 - y^2| > 1 \Leftrightarrow |x^2 - x^2 - x\delta - \frac{1}{4}\delta^2| > 1 \Leftrightarrow |\frac{1}{4}\delta^2 + x\delta| > 1$
If we choose $x = \frac{1}{\delta} + \frac{3}{4}\delta$, then $y = \frac{1}{\delta} + \frac{3}{4}\delta + \frac{1}{2}\delta = \frac{1}{\delta} + \frac{5}{4}\delta$

$$\implies \begin{cases} |x-y| = \frac{1}{2}\delta \leq \delta \\ \wedge \\ |x^2 - y^2| = |1 + \delta^2| > 1 \end{cases}$$

$$\implies \exists \varepsilon > 0\ (\varepsilon = 1), \forall \delta > 0, \exists x, y \in \mathbb{R}\ (x = \frac{1}{\delta} + \frac{3}{4}\delta, y = \frac{1}{\delta} + \frac{5}{4}\delta); \begin{cases} |x-y| \leq \delta \\ \wedge \\ |x^2 - y^2| > 1 \end{cases}$$

$\implies f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proposition 4.10

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function, then we have the following implication:

$$f \text{ is uniformly continuous on } D_f \implies f \text{ is continuous on } D_f$$

Remark 4.8 *The converse is false: a function can be continuous on D_f without being uniformly continuous on D_f. From example (4.16) we have: $f(x) = x^2$ is continuous on $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$.*

4.3.8 Theorems about continuous functions

Theorem 4.6: (Heine's theoem)

Every continuous function on an interval of type $[a, b]$ is uniformly continuous on this interval.

Proof 5

In this theorem, we'll show the following implication:
f is continuous on $[a, b] \implies f$ is uniformly continuous on $[a, b]$. By contradiction, we assume that f is continuous on $[a, b]$ and not uniformly continuous on $[a, b]$.
f is not uniformly continuous on $[a, b] \Leftrightarrow$

$$\exists \varepsilon_0, \forall \delta > 0, \exists x, y \in [a, b]; (|x - y| \leq \delta) \wedge (|f(x) - f(y)| > \varepsilon_0)$$

Let's put: $\delta = \dfrac{1}{n}$ tq: $n \in \mathbb{N}^*$

$$\implies \forall n \in \mathbb{N}^*, \exists x_n, y_n \in [a, b];\ |x_n - y_n| \leq \frac{1}{n} \wedge |f(x_n) - f(y_n)| > \varepsilon_0 \tag{4.12}$$

So we have constructed two sequences $(x_n)_{n\in\mathbb{N}^*}$ and $(y_n)_{n\in\mathbb{N}^*}$ which are included in $[a, b]$.
$(x_n)_{n\in\mathbb{N}^*} \subset [a, b] \implies (x_n)_{n\in\mathbb{N}^*}$ is a bounded sequence.
According to bolzano weierstrass's theorem, there exists a sub-sequence $(x_{\phi(n)})_{n\in\mathbb{N}^*}$ such that: $\lim\limits_{n\to+\infty} x_{\phi(n)} = l$ with $l \in [a, b]$.

On the one hand we have: $|x_{\phi(n)} - y_{\phi(n)}| \leq \dfrac{1}{\phi(n)} \implies \lim\limits_{n\to+\infty}(x_{\phi(n)} - y_{\phi(n)}) = 0$

So $\begin{cases} \lim\limits_{n\to+\infty}(x_{\phi(n)} - y_{\phi(n)}) = 0 \\ \quad\text{and} \\ \lim\limits_{n\to+\infty} x_\phi(n) = l \end{cases} \implies \lim\limits_{n\to+\infty} y_\phi(n) = l$

f is continuous at $l \implies \exists \eta > 0, \forall x, y \in [a, b];\ |x - y| \leq \eta \implies |f(x) - f(l)| \leq \dfrac{\varepsilon_0}{3}$.
The sequences $(x_\phi(n))_{n\in\mathbb{N}^*}$ and $(y_\phi(n))_{n\in\mathbb{N}^*}$ converges to l

$$\implies \begin{cases} \exists n_0 \in \mathbb{N}^*, \forall n \in \mathbb{N}^*;\ n \geq n_0 \implies |x_{\phi(n)} - l| \leq \eta \implies |f(x_{\phi(n)}) - f(l)| \leq \dfrac{\varepsilon_0}{3} \\ \quad\text{and} \\ \exists n_1 \in \mathbb{N}^*, \forall n \in \mathbb{N}^*;\ n \geq n_1 \implies |y_{\phi(n)} - l| \leq \eta \implies |f(y_{\phi(n)}) - f(l)| \leq \dfrac{\varepsilon_0}{3} \end{cases}$$

If we put: $n^* = \max(n_0, n_1)$ we get:

$$\exists n^* \in \mathbb{N}^*, \forall n \in \mathbb{N}^*;\ n \geq n^* \implies |f(x_{\phi(n)}) - f(l)| + |f(y_{\phi(n)}) - f(l)| \leq \frac{2\varepsilon_0}{3}$$

According to the triangular inequality we have:

$$|f(x_{\phi(n)}) - f(y_{\phi(n)})| \leq |f(x_{\phi(n)}) - f(l)| + |f(y_{\phi(n)}) - f(l)| \leq \frac{2\varepsilon_0}{2}$$

.

$$\implies \forall n \geq n^* \text{ we have: } |f(x_{\phi(n)}) - f(y_{\phi(n)})| \leq \frac{2\varepsilon_0}{2} \tag{4.13}$$

$$(3.12) \text{ and } (3.13) \implies \forall n \geq n^*; \varepsilon_0 < |f(x_{\phi(n)}) - f(y_{\phi(n)})| \leq \frac{2\varepsilon_0}{2}$$

$$\implies \varepsilon_0 < \frac{2\varepsilon_0}{2} \quad \text{is a contradiction}$$

so the multiplication (f is continuous on $[a, b] \implies f$ is uniformly continuous on $[a, b]$) is true.

Theorem 4.7: (Weirstrass's theorem)

Let f be a continuous function on $[a,b]$, then:

$$\begin{cases} f \text{ is bounded on } [a,b] \\ \quad\text{and} \\ \exists x_1, x_2 \in [a,b] \text{ tq:} f(x_1) = \min\limits_{x\in[a,b]}(f(x)) \text{ and } f(x_2) = \max\limits_{x\in[a,b]}(f(x)) \end{cases}$$

(i.e. f is bounded and reaches its bounds on $[a,b]$.)

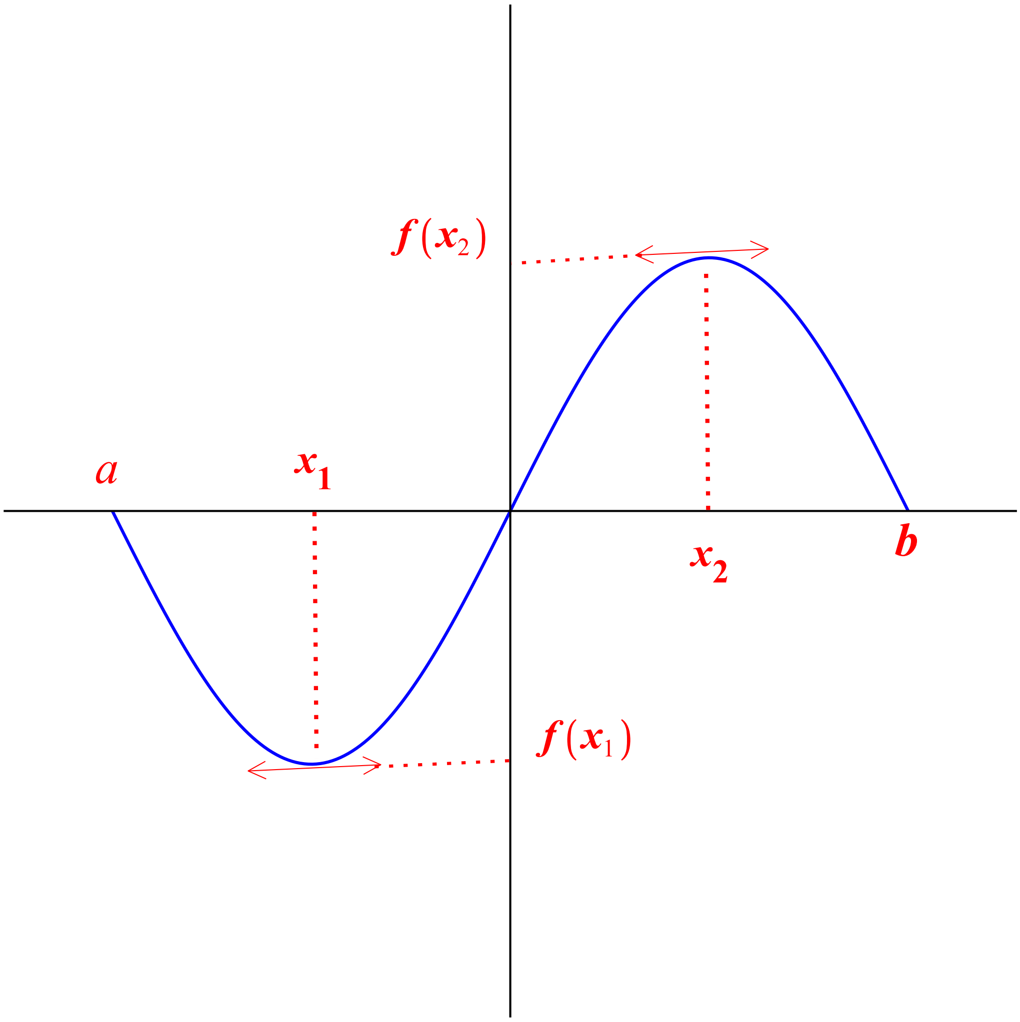


Figure 4.14: A continuous function on $[a,b]$

Proof 6

1. Let's assume that f is not bounded on $[a, b] \Leftrightarrow$

$$\forall n \in \mathbb{N}^*, \exists x_n \in [a, b] \text{ tq: } |f(x_n)| > n \tag{4.14}$$

So we constructed a sequence $(x_n)_{n\in\mathbb{N}^*} \subset [a, b] \implies (x_n)_{n\in\mathbb{N}^*}$ is bounded.
According to B.W's theorem there exists a sub-sequence $(x_{\phi(n)})_{n\in\mathbb{N}^*}$ of $(x_n)_{n\in\mathbb{N}^*}$ such that:

$$\lim_{n\to+\infty} x_{\phi(n)} = l \text{ avec } l \in [a, b]$$

$l \in [a, b] \implies f$ is continuous at $l \implies \lim_{n\to+\infty} f(x_{\phi(n)}) = f(l) \in \mathbb{R}$
(3.14) $\implies \forall n \in \mathbb{N}^*;\ |f(x_{\phi(n)}| > \phi(n) \implies \lim_{n\to+\infty} f(x_{\phi(n)}) = +\infty$ is a contradiction
$\implies f$ is bounded.

2. We put $\begin{cases} m = \inf\limits_{x\in[a,b]}(f(x)) = \inf(f([a, b])) \\ \quad\text{and} \\ M = \sup\limits_{x\in[a,b]}(f(x)) = \sup(f([a, b])) \end{cases}$
From the definition of sup and inf we have:

$$\forall \varepsilon > 0, \begin{cases} \exists x^* \in [a, b];\ f(x^*) < m + \varepsilon \\ \quad\text{and} \\ \exists y^* \in [a, b];\ M - \varepsilon < f(y^*) \end{cases}$$

Let's put: $\varepsilon = \frac{1}{n}/\ n \in \mathbb{N}^*$, we get:

$$\forall n \in \mathbb{N}^*; \begin{cases} \exists x_n \in [a, b];\ f(x_n) < m + \frac{1}{n} \\ \quad\text{and} \\ \exists y_n \in [a, b];\ M < f(y_n) + \frac{1}{n} \end{cases}$$

So we constructed two sequences $(x_n)_{n\in\mathbb{N}^*}$ and $(y_n)_{n\in\mathbb{N}^*}$ which are included in $[a, b] \implies (x_n)_{n\in\mathbb{N}^*}$ and $(y_n)_{n\in\mathbb{N}^*}$ are bounded. According to B.W's theorem we have:

$$\begin{cases} \exists (x_{\phi(n)})_{n\in\mathbb{N}^*} \text{ such that: } \lim\limits_{n\to+\infty} x_{\phi(n)} = \alpha /\ \alpha \in [a, b] \\ \quad\text{and} \\ \exists (y_{\sigma(n)})_{n\in\mathbb{N}^*} \text{ such tnat: } \lim\limits_{n\to+\infty} y_{\sigma(n)} = \beta /\ \beta \in [a, b] \end{cases}$$

$$\alpha, \beta \in [a, b] \implies \begin{cases} f \text{ is continuous at } \alpha \implies \lim\limits_{n\to+\infty} f(x_{\phi(n)}) = f(\alpha) \\ \quad\text{and} \\ f \text{ is continuous at } \beta \implies \lim\limits_{n\to+\infty} f(y_{\sigma(n)}) = f(\beta) \end{cases}$$

$\implies \forall n \in \mathbb{N}^*; \begin{cases} f(x_{\phi(n)}) - \frac{1}{n} < m \leq f(x_{\phi(n)}) \\ \quad\text{and} \\ f(y_{\sigma(n)}) \leq M < f(y_{\sigma(n)}) + \frac{1}{n} \end{cases}$ Passing to the limits we obtain: $m = f(\alpha) = \inf\limits_{x\in[a,b]}(f(x)) = \min\limits_{x\in[a,b]}(f(x))$ with $\alpha \in [a, b]$.
and $M = f(\beta) = \sup\limits_{x\in[a,b]}(f(x)) = \max\limits_{x\in[a,b]}(f(x))$ with $\beta \in [a, b]$.

Theorem 4.8: (Bolzano-Cauchy)

Let f be a continuous function on the interval $[a, b]$ such that: $f(a).f(b) \leq 0$, then there exists at least $c \in [a, b]$ verifying $f(c) = 0$.

Proof 7

Assume that $f(a) < 0$ et $f(b) > 0$. Let's put: $F = \{x \in [a, b]/f(x) \leq 0\}$.
Since $(F \subset [a, b])$, the set F is bounded above.
According to the completeness axiom for the real numbers, we have: $\exists c \in \mathbb{R};\ \sup(F) = c$ with $a \leq c \leq b$ (since $b \in \text{Upper}(F)$ and $a \in F$).

1. $c = \sup(F) \implies \forall \varepsilon > 0, \exists x^* \in F;\ c - \varepsilon < x^* \leq c$
   Let's take $\varepsilon = \dfrac{1}{n}$
   $$\implies \forall n \in \mathbb{N}^*, \exists x_n \in F;\ c - \frac{1}{n} < x_n \leq c \tag{4.15}$$
   So we constructed a sequence $(x_n)_{n\in\mathbb{N}^*} \subset F$
   According to (3.15) $\lim\limits_{n\to+\infty} x_n = c$ (Squeeze theorem).
   f is continuous at $c \implies \lim\limits_{n\to+\infty} f(x_n) = f(c)$.
   On the other hand, we have: $(x_n)_{n\in\mathbb{N}^*} \subset F \implies \forall n \in \mathbb{N}^*;\ f(x_n) \leq 0 \implies f(c) \leq 0$

2. Let's consider the sequence $y_n = c + \dfrac{b-c}{n}/n \in \mathbb{N}^*$.
   We have: $y_{n+1} - y_n = -\dfrac{b-c}{n(n+1)} \leq 0 \implies (y_n)_{n\in\mathbb{N}^*}$ is decreasing, then:
   $$\forall n \in \mathbb{N}^*;\ c < y_n \leq y_1 = b$$
   $\implies (y_n)_{n\in\mathbb{N}^*}$ is a sequence in $[a, b]$ which converges to c.
   f is continuous at $c \implies \lim\limits_{n\to+\infty} f(y_n) = f(c)$.
   On the other hand, we have: $\forall n \in \mathbb{N}^*;\ c < y_n \implies f(y_n) > 0 \implies f(c) > 0$.

Finally, from (1) and (2) we get: $\exists c \in [a, b];\ f(c) = 0$

Example 4.18

Let
$$\begin{array}{rccl} f: & [0, 2\pi] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \sin(x) + (x-1)\cos(x) \end{array}$$

1. The function $f(x)$ is continuous on $[0, 2\pi]$ (since f is a sum of two continuous functions on $[0, 2\pi]$)

2. $f(0) = -1$ and $f(2\pi = 2\pi - 1 > 0 \implies f(0)f(2\pi) < 0$

According to B.C's theorem, there exists at least one real $c \in [0, 2\pi]$ such that: $f(c) = 0$

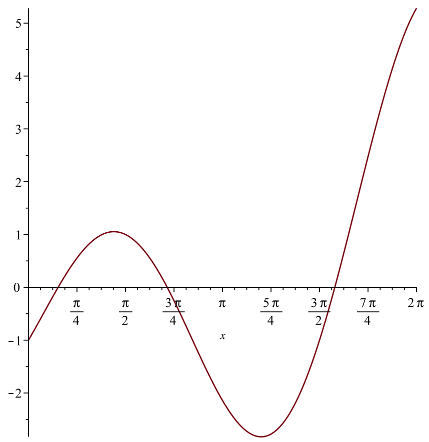


Figure 4.15: The graph of $f(x) = \sin(x) + (x-1)\cos(x)$ on the interval $[0, 2\pi]$.

Theorem 4.9: (The Intermediate Value Theorem)

Let f be a continuous function on $[a, b]$ we have:

1. If $f(a) < f(b)$ then $\forall \gamma \in [f(a), f(b)], \exists c \in [a, b]$ such that:$f(c) = \gamma$
2. If $f(b) < f(a)$ then $\forall \gamma \in [f(b), f(a)], \exists c \in [a, b]$ such that:$f(c) = \gamma$

Proposition 4.11

Let $f : I \longrightarrow \mathbb{R}$ be a continuous function on interval I (where I is an arbitrary interval). Then $f(I)$ is an interval.

Proof 8

Let $y_1, y_2 \in f(I)$ such that: $y_1 < y_2 \implies \exists x_1, x_2 \in I$ such that: $y_1 = f(x_1) \wedge y_2 = f(x_2)$.
Let's put: $a = \min(x_1, x_2)$ and $b = \max(x_1, x_2)$. We have: $a, b \in I$.
Let $y \in [y_1, y_2] \implies \exists c \in [a, b];\ f(c) = y$ (I.V.Th).
We have: $[a, b] \subset I$ (as I is an interval) $\Longrightarrow y = f(c) \in f(I)$.
$\forall y_1, y_2 \in f(I), \forall y \in \mathbb{R};\ y \in [y_1, y_2] \implies y \in f(I) \implies f(I)$ is an interval.

Remark 4.9 *If f is a continuous function on $[a, b]$ then, $f([a, b]) = [m, M]$ with $m = \min\limits_{x \in [a,b]}(f(x))$ and $M = \max\limits_{x \in [a,b]}(f(x))$*

4.3.9 Monotonic functions and continuity

Theorem 4.10

Let $f : I \longrightarrow \mathbb{R}$ be a function (I is an interval). If f is strictly monotone on the interval I, then f is injective on I.

Proof 9

Let's show that f is injective. consider $x_1, x_2 \in I;\ x_1 \neq x_2$

1. Si $x_1 < x_2$ et f is strictly increasing $\implies f(x_1) < f(x_2) \implies f(x_1) \neq f(x_2)$.
2. Si $x_1 < x_2$ et f is strictly decreasing
$$\implies f(x_1) > f(x_2) \implies f(x_1) \neq f(x_2)$$
.

The same technique is used for $x_1 > x_2$.
So $\forall x_1, x_2 \in I;\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \implies f$ is injective.

Theorem 4.11

Let $f : I \longrightarrow \mathbb{R}$ be a monotonic function defined on an interval I. Then the following two statements are equivalent

1. f is continuous on I.
2. $f(I)$ is an interval.

Theorem 4.12: (bijection theorem)

Let $f : I \longrightarrow \mathbb{R}$ be a function.
If f is strictly monotone and continuous on I, then

1. f is a bijection from I into $J = f(I)$.
2. The inverse function $f^{-1} : J = f(I) \longrightarrow I$ is strictly monotonic and continuous on J (and varies in the same direction as f).

Chapter's exercises with answers

Exercise 1

Find the domain of definition of the following functions:

1. $f_1(x) = \dfrac{1}{\sqrt{x}+\sqrt{1-x}}$
2. $f_2(x) = \ln(\ln x)$
3. $f_3(x) = \dfrac{1}{E(x)-2}$
4. $f_4(x) = \begin{cases} 1/(1-x) & \text{si } x \geq 0 \\ 1 & \text{si.} x < 0 \end{cases}$
5. $f_5(x) = \begin{cases} 1/(3-x) & \text{si } x \geq 0 \\ x^4 - x & \text{si.} x < -2 \end{cases}$
6. $f_6(x) = \begin{cases} 1/x(2-x) & \text{si } x \geq 3 \\ 1 & \text{si.} x < 0 \end{cases}$

Correction 1

1.

$$\begin{aligned} D_{f_1} &= \left\{x \in \mathbb{R} : x \geq 0, 1-x \geq 0, \sqrt{x}+\sqrt{1-x} \neq 0\right\} \\ &= \left\{x \in \mathbb{R} : 0 \leq x \leq 1 \wedge \sqrt{x}+\sqrt{1-x} \neq 0\right\} \end{aligned}$$

Let's assume that there exists $x \in [0,1]$, such that :$\sqrt{x}+\sqrt{1-x} = 0$. Since $\sqrt{x} \geq 0$ and $\sqrt{1-x} \geq 0$ hence the following implication:

$$\sqrt{x}+\sqrt{1-x} = 0 \implies \sqrt{x} = 0 \wedge \sqrt{1-x} = 0 \implies x = 0 \wedge x = 1.$$

Which is impossible, as a result $D_{f_1} = [0,1]$

2.

$$D_{f_2} = \{x \in \mathbb{R} : x > 0, \text{ and } \ln(x) > 0\}$$

According to the exp properties we have:

$$\ln(x) > 0 \iff e^{\ln(x)} > e^0 \iff x > 1$$

Consequently $D_{f_2} =]0,\infty[\cap]1,\infty[=]1,\infty[$

3.

$$D_{f_3} = \{x \in \mathbb{R} : E(x) \neq 2\}$$

On the other hand we have:

$$\begin{aligned} (E(x) = 2 \iff 2 \leq x < 2+1) &\iff (E(x) = 2 \iff 2 \leq x < 3) \\ &\iff (E(x) = 2 \iff x \in [2,3[) \\ &\iff (E(x) \neq 2 \iff x \in \mathbb{R} - [2,3[) \end{aligned}$$

As a result $D_{f_3} = \mathbb{R} - [2,3[$

4. $1 \in [0,+\infty[\implies D_{f_4} = \mathbb{R} - \{1\}$
5. $3 \in [0,+\infty[\implies D_{f_5} =]-\infty,-2[\cup\mathbb{R}^+ - \{3\}$
6. $0 \notin [3,+\infty[$, et $2 \notin [3,+\infty[\implies D_{f_6} =]-\infty,0[\cup[3,+\infty[$

Exercise 2

Solve the following equations in $\mathbb{R}$

1. $\ln(x-1)+\ln(2x-1)=0$
2. $2^{3x}-3^{x+2}=3^{x+1}-2^{3x+2}$
3. $(\sqrt{x})^x = x^{\sqrt{x}}$

Correction 2

1. The resolution domain of this equation is $D =]1,+\infty[\cap]\frac{1}{2},+\infty[=]1,+\infty[$

$$\ln(x-1)(2x-1)=0 \iff 2x^2-3x+1=1 \iff x(2x-3)=0 \iff x=0 \vee x=\frac{3}{2}$$

Since $0 \notin]1,+\infty[$, so the solution to the equation is $x=\frac{3}{2}$.

2. The resolution domain of this equation is $\mathbb{R}$

$$\begin{aligned} 2^{3x}-3^{x+2}=3^{x+1}-2^{3x+2}=0 &\iff 2^{3x}+2^{3x+2}=3^{x+1}+3^{x+2} \\ &\iff 5(2)^{3x}=12(3)^x \\ &\iff \ln(5)+3x\ln(2)=\ln(12)+x\ln(3) \\ &\iff \ln(12)-\ln(5)=x[3\ln(2)-\ln(3)] \\ &\iff \ln(12)-\ln(5)=x[\ln(8)-\ln(3)] \\ &\iff \ln\frac{12}{5}=x\ln\frac{8}{3} \\ &\iff x=\frac{\ln\frac{12}{5}}{\ln\frac{8}{3}}. \end{aligned}$$

3. The resolution domain of this equation is $]0,+\infty[$

$$\begin{aligned} (\sqrt{x})^x = x^{\sqrt{x}} &\iff x\ln\sqrt{x}=\sqrt{x}\ln x \\ &\iff \frac{1}{2}x\ln x=\sqrt{x}\ln x \\ &\iff \sqrt{x}\ln x[\frac{\sqrt{x}}{2}-1]=0 \\ &\iff \frac{1}{2}\sqrt{x}\ln x[\sqrt{x}-2]=0 \\ &\iff x=0 \vee x=1 \vee x=4 \end{aligned}$$

Since $0 \notin]0,+\infty[$, so the solution to the equation is $x=1 \vee x=4$.

Exercise 3

Prove that:

1. $\lim_{x\to 1} 3x+1=4$
2. $\lim_{x\to+\infty} x^2+x-2=+\infty$
3. $\lim_{x\to 0^-} \frac{1}{x}=-\infty$

Correction 3

1. $\lim_{x\to 1} 3x+1=4$. We're looking for $\delta>0$, such that:

$$\forall\epsilon>0,\exists\delta>0,\forall x\in\mathbb{R},(|x-1|<\delta \implies |3x+1-4|<\epsilon)$$

for this purpose, we use this draft:

Draft

$$\begin{aligned}|3x+1-4|<\epsilon=|3x-3|<\epsilon\\ \implies 3|x-1|<\epsilon\\ \implies |x-1|<\frac{\epsilon}{3}\end{aligned}$$

It is enough to take $\delta\leq\frac{\epsilon}{3}$

Hence

$$\begin{aligned}\forall\epsilon>0,\exists\delta=\frac{\epsilon}{3},\forall x\in\mathbb{R},|x-1|<\delta &\implies |x-1|<\frac{\epsilon}{3}\\ &\implies 3|x-1|<\epsilon\\ &\implies |3x-3|<\epsilon\\ &\implies |3x+1-4|<\epsilon\end{aligned}$$

2. $\lim_{x\to+\infty} x^2+x-2=+\infty$. We're looking for $B>0$, such that:

$$\forall A>0,\exists B>0,\forall x\in\mathbb{R},(x>B \implies x^2+x-2>A)$$

for this purpose, we use this draft:

Draft

$x^2+x-2>A \iff f(x)=x^2+x-(2+A)>0,\ \Delta=1+4(2+A)=4A+9>0$

$$x_1=\frac{-1-\sqrt{4A+9}}{2},x_2=\frac{-1+\sqrt{4A+9}}{2}$$

x	$-\infty$		x_1		x_2		$+\infty$
$f(x)$		>0	$=0$	<0	$=0$	>0	

It is enough to take $B=x_2=\frac{-1+\sqrt{4A+9}}{2}>\frac{-1+3}{2}=1>0$

So

$$\begin{aligned}\forall A>0,\exists B=\frac{-1+\sqrt{4A+9}}{2}>0,\forall x\in\mathbb{R},x>B &\implies x>\frac{-1+\sqrt{4A+9}}{2}\\ &\implies x^2+x-2-A>0\\ &\implies x^2+x-2>A\end{aligned}$$

3. $\lim_{x\to 0^-}\frac{1}{x}=-\infty$. We're looking for $\delta>0$, such that:

$$\forall A>0,\exists\delta>0,\forall x\in\mathbb{R},-\delta<x<0 \implies \frac{1}{x}<-A$$

for this purpose, we use this draft:

Draft

$$\frac{1}{x} < -A \iff \frac{-1}{A} < x < 0.$$

It is enough to take $\delta = \dfrac{-1}{A}$

So

$$\forall A > 0, \exists \delta = \frac{1}{A} > 0, \forall x \in \mathbb{R}, -\delta < x < 0 \implies \frac{-1}{A} < x < 0 \implies \frac{1}{x} < -A$$

Exercise 4

Determine the following limits when x converges to 0.

1. $\dfrac{\sqrt{1+x} - \left(1 + \frac{x}{2}\right)}{x^2}$
2. $\dfrac{\sqrt{2x^2+5x+9} - 3}{x}$
3. $x + 1 + \dfrac{|x|}{x}$
4. $\dfrac{\sqrt{1+x} - 1}{\sqrt[3]{1+x} - 1}$

Correction 4

1. We have

$$\begin{aligned}
\frac{\sqrt{1+x} - \left(1+\frac{x}{2}\right)}{x^2} &= \frac{[\sqrt{1+x} - \left(1+\frac{x}{2}\right)]}{x^2} \\
&= \frac{[\sqrt{1+x} - \left(1+\frac{x}{2}\right)][\sqrt{1+x} + \left(1+\frac{x}{2}\right)]}{x^2[\sqrt{1+x} + \left(1+\frac{x}{2}\right)]} \\
&= \frac{-x^2}{4x^2[\sqrt{1+x} + \left(1+\frac{x}{2}\right)]} \\
&= \frac{-1}{4[\sqrt{1+x} + \left(1+\frac{x}{2}\right)]}
\end{aligned}$$

Therefore

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \left(1+\frac{x}{2}\right)}{x^2} = \lim_{x \to 0} \frac{-1}{4[\sqrt{1+x} + \left(1+\frac{x}{2}\right)]} = \frac{-1}{8}$$

2. We have a

$$\begin{aligned}
\frac{\left(\sqrt{2x^2+5x+9} - 3\right)}{x} &= \frac{\left(\sqrt{2x^2+5x+9} - 3\right)\left(\sqrt{2x^2+5x+9} + 3\right)}{\left(\sqrt{2x^2+5x+9} + 3\right)} \\
&= \frac{2x^2+5x}{x\left(\sqrt{2x^2+5x+9} + 3\right)} \\
&= \frac{2x+5}{\left(\sqrt{2x^2+5x+9} + 3\right)}
\end{aligned}$$

Therefore

$$\lim_{x \to 0} \frac{\left(\sqrt{2x^2+5x+9} - 3\right)}{x} = \lim_{x \to 0} \frac{2x+5}{\left(\sqrt{2x^2+5x+9} + 3\right)} = \frac{5}{6}$$

3. We have,

$$\text{Si } x \longrightarrow 0^+ \quad \text{then} \quad x+1+\frac{|x|}{x} = x+1+1 = x+2 \longrightarrow 2$$
$$\text{Si } x \longrightarrow 0^- \quad \text{then} \quad x+1+\frac{|x|}{x} = x+1-1 = x \longrightarrow 0.$$

Consequently the limit does not exist

4. By the following change of variable:

$$y = \sqrt[6]{1+x} \Leftrightarrow y^6 = 1+x \Leftrightarrow \sqrt{1+x} = y^3 \wedge \sqrt[3]{1+x} = y^2 \wedge (\text{Si } x \longrightarrow 0 \text{ alors } y \longrightarrow 1)$$

we find

$$\frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1} = \frac{y^3-1}{y^2-1}$$

which implies that:

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1} = \lim_{y\to 1} \frac{y^3-1}{y^2-1} = \lim_{y\to 1} \frac{(y-1)(y^2+y+1)}{(y-1)(y+1)} = \lim_{y\to 1} \frac{(y^2+y+1)}{(y+1)} = \frac{3}{2}$$

Exercise 5

Determine the following limits when x tends to $+\infty$

1. $\dfrac{2x^2+3x-1}{3x^2+1}$
2. $x-\sqrt{x^2-x}$
3. $\dfrac{\sqrt{x}}{\sqrt{x+\sqrt{x}}}$
4. $\dfrac{2\ln(x)-\ln(3x^2-2)}{x\sin(1/x)}$

Correction 5

1.

$$\begin{aligned}
\lim_{x\to+\infty} \frac{2x^2+3x-1}{3x^2+1} &= \lim_{x\to+\infty} \frac{2x^2\left(1+\dfrac{3}{2x}-\dfrac{1}{x^2}\right)}{3x^2\left(1+\dfrac{1}{3x^2}\right)} \\
&= \lim_{x\to+\infty} \frac{2\left(1+\dfrac{3}{2x}-\dfrac{1}{x^2}\right)}{3\left(1+\dfrac{1}{3x^2}\right)} \\
&= \frac{2(1+0+0)}{3(1+0)} \\
&= \frac{2}{3}
\end{aligned}$$

2.

$$\begin{aligned}\lim_{x\to+\infty} x-\sqrt{x^2-x} &= \lim_{x\to+\infty}\frac{\left(x-\sqrt{x^2-x}\right)\left(x+\sqrt{x^2-x}\right)}{\left(x+\sqrt{x^2-x}\right)}\\ &= \lim_{x\to+\infty}\frac{x}{x+\sqrt{x^2-x}}\\ &= \lim_{x\to+\infty}\frac{x\times 1}{x\left(1+\sqrt{1-\frac{1}{x}}\right)}\\ &= \lim_{x\to+\infty}\frac{1}{\left(1+\sqrt{1-\frac{1}{x}}\right)}\\ &= \frac{1}{2}\end{aligned}$$

3.

$$\begin{aligned}\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x}}} &= \lim_{x\to+\infty}\frac{\sqrt{x}\times 1}{\sqrt{x\left(1+\frac{1}{\sqrt{x}}\right)}}\\ &= \lim_{x\to+\infty}\frac{\sqrt{x}\times 1}{\sqrt{x}\sqrt{1+\frac{1}{\sqrt{x}}}}\\ &= \lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{1}{\sqrt{x}}}}\\ &= 1\end{aligned}$$

4.

$$\begin{aligned}\lim_{x\to+\infty}\frac{2\ln(x)-\ln(3x^2-2)}{x\sin(\frac{1}{x})} &= \lim_{x\to+\infty}\frac{\ln(x^2)-\ln(3x^2-2)}{x\sin(\frac{1}{x})}\\ &= \lim_{x\to+\infty}\frac{\ln\left(\frac{x^2}{3x^2-2}\right)}{\frac{\sin(\frac{1}{x})}{\frac{1}{x}}}\end{aligned}$$

Since $\lim_{x\to+\infty}\ln\left(\frac{x^2}{3x^2-2}\right)=\ln\frac{1}{3}=-\ln 3 \wedge \lim_{x\to+\infty}\frac{\sin(\frac{1}{x})}{\frac{1}{x}}=1$, so

$$\lim_{x\to+\infty}\frac{2\ln(x)-\ln(3x^2-2)}{x\sin(\frac{1}{x})}=-\ln 3$$

Exercise 6

Let be the numerical function defined by:

$$f(x) = \begin{cases} 0, & si\ x \in]-\infty, 2] \\ a - \dfrac{b}{x}, & si\ x \in]2, 4] \\ 1, & si\ x \in]4, +\infty[\end{cases}$$

Determine the real parameters a and b so that the function f is continuous on $\mathbb{R}$. Then draw the graph of f.

Correction 6

It is clear that f is continuous on

1. $]-\infty, 0[$: $f(x) = 0 =$ Constant.
2. $]4, +\infty[$: $f(x) = 1 =$ Constant.
3. $]2, 4[$: $f(x) = a - \dfrac{b}{x}$, as the inverse of a strictly increasing and continuous function $f(x) = x$ whose denominator does not equal zero. There remain the points $x_1 = 2$, $x_2 = 4$, which means that:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2) = 0 \text{ et } \lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x) = f(4) = 1$$

We get:

$$\begin{cases} a - b/2 = 0 \\ a - b/4 = 1 \end{cases} \iff \begin{cases} a = b/2 \\ b/2 - b/4 = 1 \end{cases} \iff \begin{cases} a = b/2 \\ b/4 = 1 \end{cases} \iff \begin{cases} b = 4 \\ a = 2 \end{cases}$$

Exercise 7

Which of the following given functions $f_i : \mathbb{R}^* \to \mathbb{R}$ can be extended to become a continuous function at 0.

1. $f_1(x) = \begin{cases} x\sin(\dfrac{1}{x}), & si\ x > 0 \\ \dfrac{\sin(x)}{x}, & si\ x < 0 \end{cases}$

2. $f_2(x) = \begin{cases} \dfrac{1-\cos(x)}{x^2}, & si\ x > 0 \\ x\sin(\dfrac{1}{x}), & si\ x < 0 \end{cases}$

3. $f_2(x) = \begin{cases} x\cos(\dfrac{1}{x}), & si\ x > 0 \\ \dfrac{1-\cos(x)}{x^2}, & si\ x < 0 \end{cases}$

Correction 7

We have

a.

$$\begin{aligned} \forall x \in \mathbb{R}^*;\ -1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \ &\Rightarrow\ -x \leq x\sin\left(\frac{1}{x}\right) \leq x \\ &\Rightarrow\ \lim_{x\to 0} -x \leq \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0} x \\ &\Rightarrow\ 0 \leq \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \leq 0 \\ &\Rightarrow\ \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0 \end{aligned}$$

b. using the same reasoning, we find: $\lim\limits_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$

c. $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = 1$ et $\lim\limits_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$

From a,b, and c, we find the following results (1), (2), and (3):

$$.\begin{cases} \lim\limits_{x\to 0^+} f_1(x) &= \lim\limits_{x\to 0^+} x\sin {}^1\!/_x = 0 \\ \lim\limits_{x\to 0^-} f_1(x) &= \lim\limits_{x\to 0^-} {}^{\sin x}\!/_x = 1 \end{cases} \Longrightarrow \lim_{x\to 0^+} f_1(x) \neq \lim_{x\to 0^-} f_1(x). \tag{1}$$

This implies that f_1 cannot be extended by continuity at 0.

$$\begin{cases} \lim\limits_{x\to 0^+} f_2(x) &= \lim\limits_{x\to 0^+} \frac{1-\cos(x)}{x^2} = \frac{1}{2} \\ \lim\limits_{x\to 0^-} f_2(x) &= \lim\limits_{x\to 0^-} x\sin {}^1\!/_x = 0 \end{cases} \Longrightarrow \lim_{x\to 0^+} f_2(x) \neq \lim_{x\to 0^-} f_2(x). \tag{2}$$

This implies that f_2 cannot be extended by continuity at 0.

$$\begin{cases} \lim\limits_{x\to 0^+} f_3(x) &= \lim\limits_{x\to 0^+} x\cos {}^1\!/_x = 0 \\ \lim\limits_{x\to 0^-} f_3(x) &= \lim\limits_{x\to 0^-} \frac{1-\cos(x)}{x^2} = \frac{1}{2} \end{cases} \Longrightarrow \lim_{x\to 0^+} f_3(x) \neq \lim_{x\to 0^-} f_3(x). \tag{3}$$

This implies that f_3 cannot be extended by continuity at 0.

Exercise 8

Let $f : [0, 2] \to \mathbb{R}$ be a continuous function such that: $f(0) = f(2)$. Show that there exists at least one element α of $[0, 1]$ for which we have: $f(\alpha) = f(\alpha + 1)$.

Correction 8

From the question we can form an auxiliary function $g : [0, 1] \longrightarrow \mathbb{R}$ defined as suit: $g(x) = f(x + 1) - f(x)$, it is clear that if $x \in [0, 1]$ implies that $x + 1 \in [0, 2]$ which proves that g is well defined. Now we apply Bolzano-Cauchy's theorem:

1. since f is continuous on $[0, 2]$ implies that g is continuous on $[0, 1]$

2.

$$\begin{cases} g(0) = f(1) - f(0) \\ g(1) = f(2) - f(1) \\ f(0) = f(2) \end{cases} \Longrightarrow \begin{cases} g(0) = f(1) - f(0) \\ g(1) = f(0) - f(1) \end{cases} \Longrightarrow g(0)g(1) = -[f(1) - f(0)]^2$$

Which implies that $g(0)g(1) \leq 0$

3. According to Bolzano-Cauchy's theorem $\exists \alpha \in [0,1]$ such tnat: $g(\alpha) = 0$

4.

$$\begin{aligned} \text{the existence of } \alpha \in [0,1] : g(\alpha) = 0 &\iff \text{the existence of } \alpha \in [0,1] : f(\alpha+1) - f(\alpha) = 0 \\ &\iff \text{the existence of } \alpha \in [0,1] : f(\alpha+1) = f(\alpha). \end{aligned}$$

Exercise 9

Let $f, g : [0,1] \to \mathbb{R}$ be two continuous functions satisfying :

$$\begin{cases} f(0) = g(1) = 0 \\ \text{and} \\ f(1) = g(0) = 1 \end{cases}$$

Show that for every $\alpha \geq 0$, we can associate an element $x_\alpha \in [0,1]$: $f(x_\alpha) = \alpha g(x_\alpha)$.

Correction 9

Let $\alpha \geq 0$, from the question we can form an auxiliary function $h : [0,1] \longrightarrow \mathbb{R}$ defined as follows: $h(x) = f(x) - \alpha g(x)$, it is clear that h is well defined. Now we apply the Bolzano-Cauchy theorem:

1. h is continuous on $[0,1]$ by virtue of the continuity of f on $[0,1]$.

2.

$$\begin{cases} h(0) = f(0) - \alpha g(0) \\ h(1) = f(1) - \alpha g(1) \\ f(0) = g(1) = 0 \\ f(1) = g(0) = 1 \end{cases} \Longrightarrow \begin{cases} h(0) = -\alpha \\ h(1) = 1 \end{cases} \Longrightarrow h(0)h(1) = -\alpha$$

Which implies that $h(0)h(1) \leq 0$

3. According to Bolzano-Cauchy's theorem $\exists x_\alpha \in [0,1]$ tq: $h(x_\alpha) = 0$

4.

$$\begin{aligned} \text{the existence of } x_\alpha \in [0,1] : h(x_\alpha) = 0 &\iff \text{the existence of } x_\alpha \in [0,1] : f(x_\alpha) - \alpha g(x_\alpha) = 0 \\ &\iff \text{the existence of } x_\alpha \in [0,1] : f(x_\alpha) = \alpha g(x_\alpha). \end{aligned}$$

Exercise 10

Show that the function $f : [0, \frac{1}{4}] \to [0, \frac{1}{4}]$ defined by :

$$f(x) = \frac{1}{x^2 + 4}$$

is $\dfrac{1}{32}$-contraction.

Recall

A function f is said to be contracting on an interval I, if there exists a real $0 < k < 1$ such that: for all real x and y of the interval I we have :

$$|f(x) - f(y)| \leq k|x - y|$$

Correction 10

Let $x, y \in [0, \dfrac{1}{4}]$, we calculate the difference between $f(x)$ and $f(y)$:

$$\begin{aligned} |f(x) - f(y)| &= \left|\frac{1}{4+x^2} - \frac{1}{4+y^2}\right| \\ &= \left|\frac{4+y^2-4-x^2}{(4+y^2)(4+x^2)}\right| \\ &= \left|\frac{y^2-x^2}{(4+y^2)(4+x^2)}\right| \\ &= \left|\frac{(y-x)(y+x)}{(4+y^2)(4+x^2)}\right| \end{aligned}$$

And since $x, y \in [0, \dfrac{1}{4}]$ which implies that

$$\frac{(y+x)}{(4+y^2)(4+x^2)} \geq 0$$

This gives

$$|f(x) - f(y)| = \frac{(y+x)}{(4+y^2)(4+x^2)} |(y-x)|$$

On the other hand we have:

$$\begin{aligned} \begin{cases} 0 \leq x \leq \frac{1}{4} \\ 0 \leq y \leq \frac{1}{4} \end{cases} &\Rightarrow \begin{cases} 0 \leq x^2 \leq \frac{1}{16} \\ 0 \leq y^2 \leq \frac{1}{16} \end{cases} \\ &\Rightarrow \begin{cases} 4 \leq 4+x^2 \\ 4 \leq 4+y^2 \end{cases} \\ &\Rightarrow \begin{cases} \frac{1}{4+x^2} \leq \frac{1}{4} \\ \frac{1}{4+y^2} \leq \frac{1}{4} \\ x+y \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{cases} \\ &\Rightarrow \left(\frac{1}{4+x^2}\right)\left(\frac{1}{4+y^2}\right)(x+y) \leq \tfrac{1}{4} \times \frac{1}{4} \times \frac{1}{2} = \frac{1}{32} \end{aligned}$$

Therefore we find:

$$\forall x, y \in [0, \frac{1}{4}] : \quad |f(x) - f(y)| \leq \frac{1}{32} |(y-x)|$$

which proves that f is a k-contracting function

Chapter 5

Differential Calculus-Functions of One Variable-

5.1 The Derivative of a Function at a Point

Definition 5.1

Let f be a function defined in the neighborhood of x_0. We say that f is differentiable at a point x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists in $\mathbb{R}$. When this limit exists, it is denoted by $f'(x_0)$ and called the derivative of f at x_0.

Remark 5.1 *If we put $x - x_0 = h$, the quantity $\frac{f(x) - f(x_0)}{x - x_0}$ becomes $\frac{f(x_0 + h) - f(x_0)}{h}$. So we can define the notion of differentiability of f at x_0 in the following way:*

$$f \text{ is differentiable at the point } x_0 \Leftrightarrow \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ exists in } \mathbb{R}$$

Notations:

We can use the notations $f'(x_0)$, $Df(x_0)$, $\frac{df}{dx}(x_0)$ to designate the derivative of f at x_0.

Example 5.1

1. The function $f(x) = x^2$ is differentiable at any point $x_0 \in \mathbb{R}$ and the derivative $f'(x_0) = 2x_0$. As an explanation, given $x_0 \in \mathbb{R}$ we have:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0 + h)^2 - x_0^2}{h} = \lim_{h \to 0} (h + 2x_0) = 2x_0.$$

2. The function $f(x) = \sin(x)$ is differentiable at any point $x_0 \in \mathbb{R}$ and the derivative $f'(x_0) = \cos(x_0)$. As an explanation, given $x_0 \in \mathbb{R}$ we have:

$$\begin{aligned}\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} &= \lim_{h \to 0} \frac{\sin(x_0 + h) - \sin(x_0)}{h} \\ &= \lim_{h \to 0} \cos\left(\frac{2x_0 + h}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = \cos(x_0)\end{aligned}$$

Definition 5.2: (Left and right derivative)

1. Let f be a function defined on an interval of type $[x_0, x_0+\alpha[$ with $\alpha > 0$. We say that f is right-differentiable at x_0 iff:

$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$$

exists in $\mathbb{R}$. This limit is denoted by $f'_r(x_0)$ and is called the right derivative of f at x_0.

2. Let f be a function defined on an interval of type $]x_0-\alpha, x_0]$ with $\alpha > 0$. We say that f is left-differentiable at x_0 iff:

$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}$$

exists in $\mathbb{R}$. This limit is denoted by $f'_l(x_0)$ and is called the left derivative of f at x_0.

Proposition 5.1

Let f be a function defined in the neighborhood of x_0, we have:

$$f \text{ is differentiable at } x_0 \iff \begin{cases} f \text{ is differentiable on the right and left at } x_0 \\ \quad\text{and} \\ f'_r(x_0) = f'_l(x_0) \end{cases}$$

Example 5.2

Let $f(x) = |x|$, we have:

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} -\frac{h}{h} = -1 = f'_l(0)$$
$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1 = f'_r(0)$$

$\Longrightarrow$ The function f is differentiable on the right and on the left at $x_0 = 0$ and moreover $f'_r(0) = 1$ and

$$f'_l(0) = -1, \text{ so } f'_l(0) \neq f'_r(0) \implies f \text{ is not differentiable at } x_0 = 0$$

5.1.1 Geometrical interpretation

The figure below shows the graph of a function $y = f(x)$:

The ratio $\dfrac{f(x_0+h)-f(x_0)}{h} = \tan(\theta)$ is the slope of the straight line joining point $A(x_0, f(x_0))$ to point $B(x_0+h, f(x_0+h))$ on the graph. When $h \to 0$, this line tends towards the tangent (AC) to the curve at a point $A(x_0, f(x_0))$. So we get:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \tan(\alpha) = \frac{CD}{AD}$$

is the slope of the tangent to the curve at point $A(x_0, f(x_0))$.

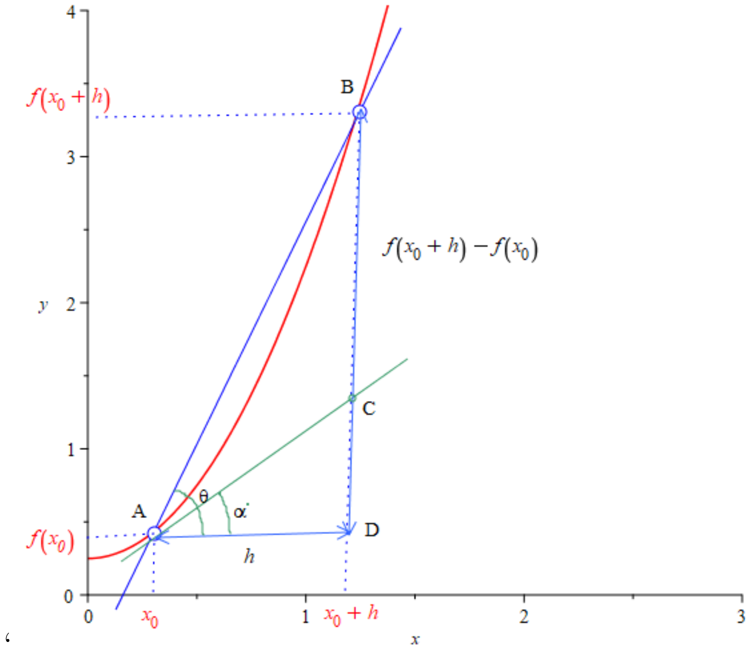


Figure 5.1: Geometrical Interpretation of Differentiability at a point x_0

Remark 5.2 *According to the figure above, the equation of the tangent to the curve* $y = f(x)$ *at the point* $A(x_0, f(x_0))$ *is* $y - f(x_0) = f'(x_0)(x - x_0)$

Proposition 5.2

Let f be a function differentiable at a point x_0, then f is continuous at x_0.

Proof:

We have: $\lim_{x\to x_0}(f(x) - f(x_0)) = \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}\right)(x - x_0)$

Since f is differentiable at x_0 we get:

$\lim_{x\to x_0}(f(x) - f(x_0)) = \lim_{x\to x_0} f'(x_0)(x - x_0) = 0 \implies f$ is continuous at x_0

Remark 5.3 *The opposite of this theorem is incorrect. A function can be continuous at a point* x_0 *without being differentiable at the same point. For example, the function* $x \mapsto |x|$ *is continuous at* $x_0 = 0$ *but not differentiable at the same point.*

5.2 Differential on an interval. Derivative function.

Definition 5.3

Let f be a function defined on an open interval I. We say that f is differentiable on I if: it is differentiable at any point on I. The function defined on I by: $x \mapsto f'(x)$ is called the derivative function or simply the derivative of the function f and is denoted by f' ou $\frac{df}{dx}$.

Remark 5.4 *let* f *be a function defined on an interval* I *and* $a, b \in \mathbb{R} \cup \{+\infty, -\infty\}$ *then:*

- *We say that* f *is differentiable on* $I = [a, b]$ *iff: it is differentiable on the open interval* $]a, b[$ *and differentiable on the right at* a *and on the left at* b.

- *We say that f is differentiable on $I = [a, b[$ if: it is differentiable on the open interval $]a, b[$ and differentiable on the right at a.*

- *We say that f is differentiable on $I =]a, b]$ if: it is differentiable on the open interval $]a, b[$ and differentiable on the left at b.*

5.3 Operations on differentiable functions

Proposition 5.3: (At a point)

Let f, g be two functions differentiable at x_0, then we have:

- $f + g$ is differentiable at x_0 et $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
- $f.g$ is differentiable at x_0 et $(f.g)'(x_0) = f'(x_0).g(x_0) + f(x_0).g'(x_0)$
- If we have: $f(x_0) \neq 0$, alors $\frac{1}{f}$ is differentiable at x_0 et $\left(\frac{1}{f}\right)'(x_0) = -\frac{f'(x_0)}{f(x_0)^2}$
- If we have: $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0).g(x_0) - f(x_0).g'(x_0)}{g(x_0)^2}$$

Proposition 5.4: (On an interval)

Let f and g be two functions differentiable on an open interval I then:

- $f + g$ is differentiable on I and $(f + g)' = f' + g'$
- $f.g$ is differentiable on I and $(f.g)' = f'.g + f.g'$
- If $f \neq 0$ on I, $\frac{1}{f}$ is differentiable on I and $\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$
- If $g \neq 0$ on I, $\frac{f}{g}$ is differentiable on I and

$$\left(\frac{f}{g}\right)' = \frac{f'.g - f.g'}{g^2}$$

Proposition 5.5: Differentiability and composition

Let $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ be two functions where I and J are two open intervals such that: $f(I) \subset J$

- **Differentiability at a point:** If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0 and $(g \circ f)'(x_0) = f'(x_0).g'(f(x_0))$
- **differentiability on an interval:** If f is differentiable on I and g is differentiable on J, then $g \circ f$ is differentiable on I and $(g \circ f)' = f'.(g' \circ f)$

Proposition 5.6: Differentiability and inverse function

Let $f : I \longrightarrow J$ be a bijective and differentiable function at $x_0 \in I$. Then f^{-1} is differentiable at $y_0 = f(x_0)$ if and only if $f'(x_0) \neq 0$ and in this case: $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$.

Proposition 5.7

Let $f : I \longrightarrow J$ be a bijective and differentiable function on I. If $f' \neq 0$ on I, then f^{-1} is differentiable on J and we have : $(f^{-1})' = \dfrac{1}{f' \circ f^{-1}}$

5.4 Mean value Theorem

Theorem 5.1: (Rolle's theorem)

Let f be a function defined on $[a, b]$. If we have:

1. f is continuous on $[a, b]$.
2. f is differentiable on $]a, b[$
3. $f(a) = f(b)$

then there exists a real number $c \in]a, b[$ such that $f'(c) = 0$

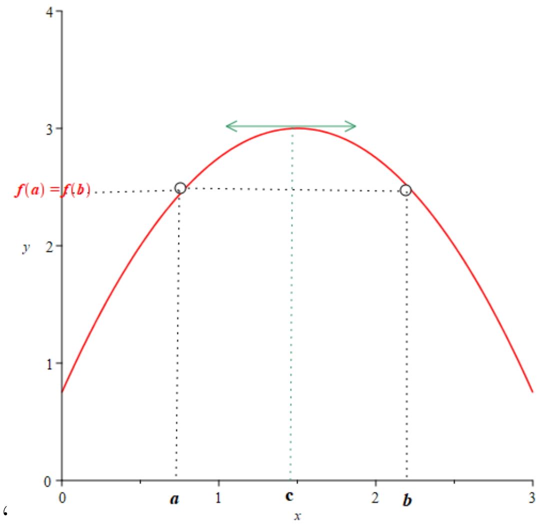


Figure 5.2: Geometrical interpretation of Rolle's theorem

Theorem 5.2: (Mean value Theorem)

Let f be a function defined on $[a, b]$, if we have:

1. f is continuous on $[a, b]$.
2. f is differentiable on $]a, b[$

then there exists a real number $c \in]a, b[$ such that:

$$f(b) - f(a) = f'(c)(b - a)$$

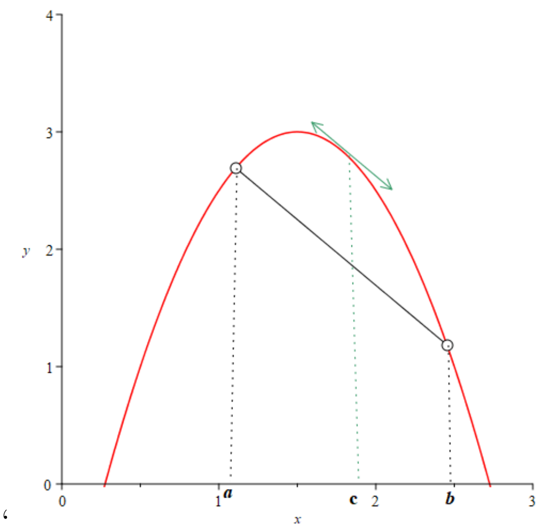


Figure 5.3: Geometrical interpretation of the mean value theorem

Consequence:(second form of the mean value theorem)

Let f be a function defined on I, $h > 0$ and $x_0 \in I$ such that $x_0 + h \in I$, then if we have:

1. f is continuous on $[x_0, x_0 + h]$.

2. f is derivable on $]x_0, x_0 + h[$

then there exists a $\theta \in]0, 1[$ such that:

$$f(x_0 + h) - f(x_0) = f'(x_0 + \theta.h)h$$

Example 5.3

By using the mean value theorem, show that:

$$\forall x > 0; \sin(x) \leq x$$

By putting $f(t) = t - \sin(t)$ we get:

$$\forall x > 0 \text{ we have:} \begin{cases} f \text{ is continuous on } [0, x] \\ \quad\text{and} \\ f \text{ is differentiable on }]0, x[\end{cases}$$

According to the mean value theorem, there exists $c \in]0, x[$ such that:

$$f(x) - f(0) = f'(c)(x - 0)$$

$$\iff x - \sin(x) = (1 - \cos(c))x \iff \sin(x) = \cos(c)x$$

$$\implies \sin(x) \leq x \text{ (as } \cos(c) \leq 1)$$

Theorem 5.3: Generalized mean value theorem

Let f and g be two real functions defined on $[a, b]$ such that:

1. f and g are continuous on $[a, b]$.
2. f and g are differentiable on $]a, b[$.

Then there exists a real number $c \in]a, b[$ such that:

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

Proposition 5.8: (Variations of a function)

Let f be a continuous function on $[a, b]$ and differentiable on $]a, b[$, we have:

1. If $f'(x) > 0$ on $]a, b[$, then f is strictly increasing on $[a, b]$.
2. If $f'(x) \geq 0$ on $]a, b[$, then f is increasing on $[a, b]$.
3. If $f'(x) < 0$ on $]a, b[$, then f is strictly decreasing on $[a, b]$.
4. If $f'(x) \leq 0$ on $]a, b[$, then f is decreasing on $[a, b]$.
5. If $f'(x) = 0$ on $]a, b[$, then f is constant on $[a, b]$.

5.4.1 L'Hôpital's rules

Theorem 5.4: (First rule of L'Hôpital)

Let f and g be two continuous functions on I (where I is a neighborhood of x_0), differentiable on $I - \{x_0\}$ and satisfying the following conditions:

1. $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$
2. $\forall x \in I - \{x_0\};\ g'(x) \neq 0$

Then:

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = l \implies \lim_{x\to x_0} \frac{f(x)}{g(x)} = l$$

Example 5.4

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1$$

Remark 5.5 *The converse is generally false. For example:* $f(x) = x^2\cos(\frac{1}{x})$, $g(x) = x$. *We have:* $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0} x\cos(\frac{1}{x}) = 0$. *While* $\lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to 0}(2x\cos(\frac{1}{x}) + \sin(\frac{1}{x}))$ *does not exist (since:* $\lim\limits_{x\to 0} \sin(\frac{1}{x})$ *does not exist)*

Remark 5.6 *Also, the Hopital's rules is true when* $x \to \pm\infty$

Theorem 5.5: (Second rule of L'Hôpital)

Let f and g be two functions defined on I (where I is a neighborhood of x_0), differentiable on $I - \{x_0\}$ and satisfying the following conditions:

1. $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \pm\infty$
2. $\forall x \in I - \{x_0\};\ g'(x) \neq 0$

Then:

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = l \implies \lim_{x\to x_0} \frac{f(x)}{g(x)} = l$$

Example 5.5

$$\lim_{x\to +\infty} \frac{x^n}{e^x} = \lim_{x\to +\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to +\infty} \frac{n(n-1)x^{n-2}}{e^x} = \ldots\ldots\ldots = \lim_{x\to +\infty} \frac{n!x^0}{e^x} = 0$$

5.5 Higher Order Derivatives

Definition 5.4

Let f be a function differentiable on I, then f' is called the 1st-order derivative of f; if f' is differentiable on I, then its derivative is called the 2nd-order derivative of f and is denoted by f'' or $f^{(2)}$. Recursively, we define the derivative of order n of f as follows:

$$\begin{cases} f^{(0)} = f \\ \\ (f^{(n-1)})' = f^{(n)} \end{cases}$$

Another notations used are: $D_n f, \dfrac{d^n f}{dx^n}$ for $f^{(n)}$

Example 5.6

$$\sin^{(n)}(x) = \sin(x + n\frac{\pi}{2}) \quad \text{and} \quad \cos^{(n)}(x) = \cos(x + n\frac{\pi}{2})$$

Definition 5.5: (Class Functions: C^n)

Let n be a non-zero natural number. A function f defined on I is said to be of class C^n or n times continuously differentiable if it is n times differentiable and $f^{(n)}$ is continuous on I, and we note $f \in C^n(I)$.

Remark 5.7 *A function f is said to be "of class C^0" if it is continuous on I.*

Definition 5.6: (Class Functions: C^∞)

A function f is said to be of class C^∞ on I if it is in the class C^n. $\forall n \in \mathbb{N}$

5.5.1 n-th derivative of a product (Leibniz rule)

Theorem 5.6

Let f and g be two functions n times differentiable on I, then fg is n times differentiable on I and we have:

$$\forall x \in I;\ (f.g)^n(x) = \sum_{k=0}^{n} C_n^k f^{(n-k)}(x) g^{(k)}(x)$$

$$\text{with: } C_n^k = \frac{n!}{k!(n-k)!}$$

Example 5.7

Compute $(x^2\sin(2x))^{(3)}$ According to Leibniz' formula, we have:

$$\begin{aligned}(x^2\sin(2x))^{(3)} =& \sum_{k=0}^{3} C_3^k (x^2)^{(3-k)}(\sin(2x))^{(k)}\\ =&C_3^0(x^2)^{(3)}(\sin(2x))^{(0)} + C_3^1(x^2)^{(2)}(\sin(2x))^{(1)}\\ &+ C_3^2(x^2)^{(1)}(\sin(2x))^{(2)} + C_3^3(x^2)^{(0)}(\sin(2x))^{(3)}\\ =&12\cos(2x) - 24x\sin(2x) - 8x^2\cos(2x)\end{aligned}$$

5.6 Taylor's formulas

Theorem 5.7: (Taylor's formula with Lagrange remainder)

Let $x_0 \in [a,b]$ et $f : [a,b] \longrightarrow \mathbb{R}$ be a function that checks:

1. $f \in C^n$ on $[a,b]$.
2. $f^{(n)}$ is differentiable on $]a,b[$.

then, $\forall x \in [a,b]$ (with $x \neq x_0$), $\exists c \in [a,b]$ such that:

$$\begin{aligned}f(x) =&f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\\ &+ \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}\end{aligned}$$

This expression is the Taylor formula of order n with the Lagrange remainder

$$R_n(x,x_0) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

Theorem 5.8: (Taylor Mac-Laurin formula)

If we set $x_0 = 0$ in the Taylor-Lagrange formula, we obtain:
$\exists \theta \in]0,1[$ such that:

$$f(x) =f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$$

This is Taylor Mac-Laurin's formula.

Remark 5.8 *In practice, the Taylor Mac-Laurin formula is used to calculate the approximate values.*

Example 5.8

Show that for every $x \in \mathbb{R}_+$:

$$x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$

Let $x \geq 0$, Applying the Taylor Mac-Laurin formula of order 2 to the function $f(x) = \ln(1+x)$, we find:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} \; / \theta \in]0,1[$$

Since $x \geq 0$ then,

$$x - \frac{x^2}{2} \leq x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$$

$$\Longrightarrow \; x - \frac{x^2}{2} \leq \ln(1+x) \tag{5.1}$$

On the other hand $\dfrac{x^3}{3(1+\theta x)^3} \leq \dfrac{x^3}{3}$

$$\Longrightarrow \; x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$\Longrightarrow \; \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} \tag{5.2}$$

from (5.1) and (5.2) we get:

$$x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$

Chapter's exercises with answers

Exercise 1

By calculating the right and left derivatives of the following functions, determine which one is differentiable at a:

1. $f_1(x) = x^2 + |x+1|,\ a = 1, -1$
2. $f_2(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & si\ x \in \mathbb{R}^* \\ 0, & si\ x = 0 \end{cases}$ $, a = 0$

Correction1

1. $f_1(x) = x^2 + |x+1| = \begin{cases} x^2 + x + 1 & \text{si} \quad x \geq -1 \\ x^2 - x - 1 & \text{si} \quad x < -1 \end{cases}$

   (a) For the point $a = 1$. $f(1) = 3$
   The right derivative is the following limit:
   $$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1^+} \frac{x^2+x-2}{x-1} = \lim_{x\to 1^+} \frac{(x-1)(x+2)}{x-1} = \lim_{x\to 1^+} x+2 = 3$$
   The right derivative is the following limit:
   $$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1^-} \frac{x^2+x-2}{x-1} = \lim_{x\to 1^-} \frac{(x-1)(x+2)}{x-1} = \lim_{x\to 1^-} x+2 = 3$$
   As a result
   $$f'_D(1) = f'_G(1) \Rightarrow f \text{ is differentiable at the point } a = 1$$

   (b) For $a = -1$. $f(-1) = 1$
   The right derivative is the following limit:
   $$\lim_{x\to -1^+} \frac{f(x) - f(-1)}{x-1} = \lim_{x\to 1^+} \frac{x^2+x}{x+1} = \lim_{x\to -1^+} \frac{x(x+1)}{x+1} = \lim_{x\to 1^+} x = -1$$
   The right derivative is the following limit:
   $$\lim_{x\to 1^-} \frac{f(x) - f(-1)}{x+1} = \lim_{x\to -1^-} \frac{x^2-x-2}{x+1} = \lim_{x\to -1^-} \frac{(x+1)(x-2)}{x+1} = \lim_{x\to -1^-} x-2 = -3$$
   As a result
   $$f'_D(-1) \neq f'_G(-1) \Rightarrow f \text{ is not differentiable at the point } a = -1$$

2. $f_2(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & si\ x \in \mathbb{R}^* \\ 0, & si\ x = 0 \end{cases}$ $, a = 0$; on a $f(0) = 0$
   The right derivative is the following limit:
   $$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^+} \frac{\frac{x}{1+e^{1/x}}}{x} = \lim_{x\to 0^+} \frac{1}{1+e^{1/x}} = 0$$
   On the other hand, the left derivative is the following limit:
   $$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^-} \frac{\frac{x}{1+e^{1/x}}}{x} = \lim_{x\to 0^-} \frac{1}{1+e^{1/x}} = 1$$
   Therefore
   $$f'_D(0) \neq f'_G(0) \Rightarrow f \text{ is not differentiable at the point } a = 0$$

Exercise 2

Compute the derivatives of the following functions and precise their domains of definition.

1. $\sqrt[4]{x^3}$
2. $\dfrac{x}{x^3+1}$
3. $\dfrac{(1+\sqrt{x})^3}{(x+1)^2}$
4. $x\sqrt[n]{x},\ n\in\mathbb{N}^*$
5. $x\ln|x+1|$
6. $x^2e^{1/x}$
7. $\sin(\cos(5x))$
8. $a^x\ ,a\in\mathbb{R}^{+*}$
9. $(x+\ln x)^n,\quad n\in\mathbb{N}^*$
10. $x^3\ln(x)$
11. x^2e^x

Correction2

We denote by D: the domain of derivation of f'.

1. $D=]0,+\infty[$,

$$f(x)=\sqrt[4]{x^3}\Rightarrow f(x)=x^{\frac{3}{4}}\Rightarrow f'(x)=\frac{3}{4}x^{\frac{3}{4}-1}=\frac{3}{4}x^{\frac{-1}{4}}=\frac{3}{4\sqrt[4]{x}}$$

2. $D=\mathbb{R}\setminus\{-1\}$, we know that if f is in fractional form $f=\dfrac{g}{h}$, then $f'=\dfrac{g'h-gh'}{h^2}$. So

$$\begin{aligned} f(x)=\frac{x}{x^3+1} \quad &\Rightarrow\quad f'(x)=\frac{x^3+1-3x^3}{x}\\ &\Rightarrow\quad f'(x)=\frac{-2x^3+1}{x}\end{aligned}$$

3. $D=\mathbb{R}\setminus\{-1\}$,

$$\begin{aligned} f(x)=\frac{(1+\sqrt{x})^3}{(x+1)^2}\quad &\Rightarrow\quad f'(x)=\frac{\frac{3}{2\sqrt{x}}(1+\sqrt{x})^2(x+1)^2-2(1+x)(1+\sqrt{x})^3}{(x+1)^4}\\ &\Rightarrow\quad f'(x)=\frac{(1+\sqrt{x})^2(-x-4\sqrt{x}+3)}{2\sqrt{x}(x+1)^3}\end{aligned}$$

4. $f(x)=x^n\sqrt{x}=(x)^{1+\frac{1}{n}},n\in\mathbb{N}^*$. $\begin{cases} D &=\mathbb{R}\text{ if n is odd}\\ D &=\mathbb{R}^+\text{ if n is even}\end{cases}$

$$\begin{aligned} f(x)=x^{(1+\frac{1}{n})}\quad &\Rightarrow\quad f'(x)=\left(1+\tfrac{1}{n}\right)x^{\frac{1}{n}}\\ &\Rightarrow\quad f'(x)=\left(1+\tfrac{1}{n}\right)\sqrt[n]{x}\end{aligned}$$

5. $D=\mathbb{R}\setminus\{-1\}$,

$$f(x)=x\ln|1+x|\Rightarrow f'(x)=\ln|1+x|+\frac{x}{x+1}$$

6. $D=\mathbb{R}^*$,

$$\begin{aligned} f(x)=x^2e^{\frac{1}{x}}\quad &\Rightarrow\quad f'(x)=2xe^{\frac{1}{x}}-\tfrac{1}{x^2}e^{\frac{1}{x}}x^2\\ &\Rightarrow\quad f'(x)=(2x-1)e^{\frac{1}{x}}\end{aligned}$$

7. $D=\mathbb{R}$,

$$\begin{aligned} f(x)=\sin(\cos(5x))\quad &\Rightarrow\quad f'(x)=(\cos 5x)'\cos(\cos 5x)\\ &\Rightarrow\quad f'(x)=-5\sin(5x)\cos(\cos 5x)\end{aligned}$$

8. $D = \mathbb{R}$,

$$\begin{cases} a \in \mathbb{R}^+ \\ f(x) = a^x \end{cases} \Rightarrow f(x) = e^{x \ln a}$$
$$\Rightarrow f'(x) = (\ln a) e^{x \ln a}$$
$$\Rightarrow f'(x) = (\ln a) a^x$$

9. $D = \mathbb{R}^{+*}$,

$$f(x) = (x + \ln x)^n \Rightarrow f'(x) = n\left(1 + \tfrac{1}{x}\right)(x + \ln x)^{n-1}$$

10. $D = \mathbb{R}^{+*}$,

$$\begin{aligned} f(x) = x^3 \ln x &\Rightarrow f'(x) = 3x^2 \ln x + \tfrac{1}{x}(x^3) \\ &\Rightarrow f'(x) = 3x^2 \ln x + x^2 \\ &\Rightarrow f'(x) = x^2 (3 \ln x + 1) \end{aligned}$$

11. $D = \mathbb{R}$,

$$\begin{aligned} f(x) = x^2 e^x &\Rightarrow f'(x) = 2xe^x + e^x x^2 \\ &\Rightarrow f'(x) = (2 + x) x e^x \end{aligned}$$

Exercise 3

Study the differentiability on $\mathbb{R}$ of the following functions:

1. $f(x) = x|x|$
2. $g(x) = \dfrac{1}{2 + |x|}$
3. $h(x) = \begin{cases} x^2 \cos \dfrac{1}{x}, & si\ x \neq 0 \\ 0, & si\ x = 0 \end{cases}$

Correction3

1.

$$\begin{aligned} f(x) &= x|x| \\ &= \begin{cases} x^2 & \text{, si } x \geq 0 \\ -x^2 & \text{, si } x < 0 \end{cases} \end{aligned}$$

(a) It is clear that x^2 is differentiable on $\mathbb{R}^* =]-\infty, 0[\cup]0, +\infty[$, with $(x^2)' = 2x$

(b) There remains the point 0. For this point we have $f(0) = 0$ and

$$\begin{aligned} \lim_{x \to 0} \tfrac{f(x) - f(0)}{x} &= \lim_{x \to 0} \tfrac{x|x|}{x} \\ &= \lim_{x \to 0} |x| \\ &= 0 \end{aligned}$$

hence the function f is differentiable at the point $a = 0$..
Therefore f is differentiable on $\{0\} \cup]-\infty, 0[\cup]0, +\infty[= \mathbb{R}$.

2.

$$g(x) = \begin{cases} \frac{1}{2+x} & \text{, si } x \geq 0 \\ \frac{1}{2-x} & \text{, si } x < 0 \end{cases}$$

(a) $-2 \notin [0, +\infty[\Rightarrow f$ is differentiable on $[0, +\infty[$, the same reasoning with the point $a = 2 \notin [-\infty, 0[\Rightarrow f$ is differentiable on $[-\infty, 0[$, therefore f is differentiable on $]-\infty, 0[\cup]0, +\infty[$.

(b) There remains the point 0. For this point we have $f(0) = \frac{1}{2}$ and for calculating $\lim_{x \to 0} \frac{f(x) - f(0)}{x}$ we must calculate the left and right derivative of f at 0 because f changes its form in the neighbourhood of 0.

$$\begin{aligned}\lim_{x\to 0^-}\frac{f(x)-f(0)}{x} &= \lim_{x\to 0^-}\frac{\frac{1}{2-x}-\frac{1}{2}}{x} \\ &= \lim_{x\to 0^-}\frac{1}{x(2-x)}-\frac{1}{2x} \\ &= \lim_{x\to 0^-}\frac{2}{2x(2-x)}-\frac{(2-x)}{2x(2-x)} \\ &= \lim_{x\to 0^-}\frac{1}{2(2-x)} \\ &= \frac{1}{4}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+}\frac{f(x)-f(0)}{x} &= \lim_{x\to 0^+}\frac{\frac{1}{2+x}-\frac{1}{2}}{x} \\ &= \lim_{x\to 0^+}\frac{1}{x(2+x)}-\frac{1}{2x} \\ &= \lim_{x\to 0^+}\frac{2}{2x(2+x)}-\frac{(2+x)}{2x(2+x)} \\ &= \lim_{x\to 0^+}\frac{-1}{2(2+x)} \\ &= \frac{-1}{4}\end{aligned}$$

We have

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}=\frac{-1}{4}\neq \lim_{x\to 0^+}\frac{f(x)-f(0)}{x}=\frac{1}{4}\Rightarrow \text{ f is not differentiable at the point 0}$$

Therefore g is differentiable on $]-\infty,0[\cup]0,+\infty[=\mathbb{R}^*$.

3. For the function $h(x)=\begin{cases} x^2\cos\frac{1}{x}, & si\ x\neq 0\\ 0, & si\ x=0\end{cases}$. It is clear that the function $f(x)=x^2\cos\frac{1}{x}$ is differentiable on $\mathbb{R}^*=]-\infty,0[\cup]0,+\infty[$, with

$$\begin{aligned}f'(x) &= 2x\cos\frac{1}{x}+\frac{-1}{x^2}\left(-\sin\frac{1}{x}\right)x^2 \\ &= 2x\cos\frac{1}{x}+\sin\frac{1}{x}\end{aligned}$$

There remains the point 0. To calculate $\lim_{x\to 0}\frac{f(x)-f(0)}{x}$, we have:

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)-f(0)}{x} &= \lim_{x\to 0}\frac{x^2\cos\frac{1}{x}}{x} \\ &= \lim_{x\to 0}x\cos\frac{1}{x} \\ &= 0\end{aligned}$$

Explication

$$\begin{aligned}\forall x\in\mathbb{R}^*:\left|\cos\frac{1}{x}\right|\leq 1 &\Rightarrow |x|\left|\cos\frac{1}{x}\right|\leq |x| \\ &\Rightarrow \left|x\cos\frac{1}{x}\right|\leq |x| \\ &\Rightarrow -|x|\leq x\cos\frac{1}{x}\leq |x| \text{ since } \lim_{x\to 0}|x|=\lim_{x\to 0}-|x|=0 \\ &\Rightarrow \lim_{x\to 0}x\cos\frac{1}{x}=0\end{aligned}$$

Therefore f is differentiable on $\{0\}\cup]-\infty,0[\cup]0,+\infty[=\mathbb{R}$.

Exercise 4

Compute the nth derivative of the following functions

1. $x\sqrt{x}$
2. $\ln(x)$
3. e^{ax}
4. $\frac{1}{1-x}$

Correction4

1. $D = \mathbb{R}^{*+}$

$$f(x) = x\sqrt{x} = xx^{\frac{1}{2}} = x^{\frac{3}{2}}$$
$$f'(x) = \frac{3}{2}x^{\frac{3}{2}-1}$$
$$f''(x) = \frac{3}{2}\left(\frac{3}{2}-1\right)x^{\frac{3}{2}-2}$$
$$f^{(3)}(x) = \frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)x^{\frac{3}{2}-3}$$
$$f^{(4)}(x) = \frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\left(\frac{3}{2}-3\right)x^{\frac{3}{2}-4}$$
$$\vdots$$
$$f^{(n)}(x) = \frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\left(\frac{3}{2}-3\right)\cdots\cdots\left(\frac{3}{2}-n+1\right)x^{\frac{3}{2}-n}$$

The induction proof

(a) **Statement**.
$P_n : \forall n \in \mathbb{N}^* : f^{(n)}(x) = \frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\left(\frac{3}{2}-3\right)\cdots\cdots\left(\frac{3}{2}-n+1\right)x^{\frac{3}{2}-n}$.

(b) **Base Case**. For $n = 1$, we have $f^{(1)}(x) = f'(x)$, and $\left(\frac{3}{2}-1+1\right)x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{3}{2}-1} = f'(x)$ so the property P_1 holds.

(c) **Induction hypothesis**. Let $n \in \mathbb{N}^*$ such that P_n is true, i.e.

$$\forall n \in \mathbb{N}^* : f^{(n)}(x) = \frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\left(\frac{3}{2}-3\right)\cdots\cdots\left(\frac{3}{2}-n+1\right)x^{\frac{3}{2}-n},$$

then we have:

$$(f^{(n)})'(x) = \left(\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\left(\frac{3}{2}-3\right)\cdots\cdots\left(\frac{3}{2}-n+1\right)x^{\frac{3}{2}-n}\right)' =$$
$$\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\left(\frac{3}{2}-3\right)\cdots\cdots\left(\frac{3}{2}-n+1\right)\left(\frac{3}{2}-n\right)x^{\frac{3}{2}-n-1} =$$
$$\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\left(\frac{3}{2}-3\right)\cdots\cdots\left(\frac{3}{2}-n+1\right)\left(\frac{3}{2}-n\right)x^{\frac{3}{2}-(n+1)} =$$
$$f^{(n+1)}(x)$$

(d) **Conclusion**. The property P_n is true for all $n \in \mathbb{N}^*$.

2. $D=\mathbb{R}^{*+}$

$$\begin{aligned} f(x) &= \ln x \\ f'(x) &= \frac{1}{x} = x^{-1} \\ f''(x) &= -x^{-2} \\ f^{(3)}(x) &= 2x^{-3} \\ f^{(4)}(x) &= -3\times 2x^{-4} \\ f^{(5)}(x) &= 4\times 3\times 2x^{-5} \\ &\vdots \\ f^{(n)}(x) &= (-1)^{n-1}(n-1)\,!x^{-n} \end{aligned}$$

with induction proof here

3. $D=\mathbb{R}$

$$\begin{aligned} f(x) &= e^{ax} \\ f'(x) &= ae^{ax} \\ f''(x) &= a^2e^{ax} \\ f^{(3)}(x) &= a^3e^{ax} \\ f^{(4)}(x) &= a^4e^{ax} \\ &\vdots \\ f^{(n)}(x) &= a^ne^{ax} \end{aligned}$$

with induction proof here

4. $D=\mathbb{R}\backslash\{1\}$

$$\begin{aligned} f(x) &= \frac{1}{1-x} = (1-x)^{-1} \\ f'(x) &= (1-x)^{-2} \\ f''(x) &= 2(1-x)^{-3} \\ f^{(3)}(x) &= 3\times 2(1-x)^{-4} \\ f^{(4)}(x) &= 4\times 3\times 2(1-x)^{-5} \\ &\vdots \\ f^{(n)}(x) &= n\,!(1-x)^{-(n+1)} \end{aligned}$$

induction proof

(a) **Statement**.
$P_n : \forall n\in\mathbb{N}^* : f^{(n)}(x) = n\,!(1-x)^{-(n+1)}$

(b) **Base case**. For $n=1$, we have $f^{(1)}(x)=f'(x)$, and $1\,!(1-x)^{-2}=(1-x)^{-2}=f'(x)$ so the property P_1 holds.

(c) **Induction hypothesis**. Let $n\in\mathbb{N}^*$ such that P_n is true, i.e:

$$f^{(n)}(x) = n\,!(1-x)^{-(n+1)},$$

them we have:

$$\begin{aligned} f^{(n)}(x)=n(1-x)^{-(n+1)} \quad &\Rightarrow \quad \left(f^{(n)}\right)'(x) = n(1-x)^{-(n+1)} \\ &\Rightarrow \quad \left(f^{(n)}\right)'(x) = (-1)\left[-(n+1)\right]n(1-x)^{-n-2} \\ &\Rightarrow \quad \left(f^{(n)}\right)'(x) = (n+1)(1-x)^{-n-2} \\ &\Rightarrow \quad \left(f^{(n)}\right)'(x) = (n+1)(1-x)^{-(n+2)} \\ &\Rightarrow \quad f^{(n+1)}(x) = (n+1)(1-x)^{-(n+2)} \end{aligned}$$

(d) **Conclusion**. the property P_n is true for all $n\in\mathbb{N}^*$.

Exercise 5

Let a and b be two real numbers and f be a function defined on $[0,+\infty[$ by

$$f(x)=\begin{cases} 2\sqrt{x}, & si\ 0\leq x\leq 1\\ ax+b, & si\ x>1\end{cases}$$

Find a and b so that f is differentiable on $]0,+\infty[$

Correction5

1. It is clear that $2\sqrt{x}$ is differentiable on $]0,1[$, with $(2\sqrt{x})'=\dfrac{1}{\sqrt{x}}$
2. The same on $]1,+\infty[$ with $(ax+b)'=a$
3. There remains the point $a=1$, for this point we need to calculate the right and left derivatives as the function f changes its form at 1.

$$\begin{aligned}
\lim_{x\to1^-}\tfrac{f(x)-f(1)}{x-1} &= \lim_{x\to1^-}\tfrac{2\sqrt{x}-2}{x-1}=\lim_{x\to1^-}\tfrac{2\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{(x-1)\left(\sqrt{x}+1\right)}\\
&= \lim_{x\to1^-}\tfrac{2(x-1)}{(x-1)\left(\sqrt{x}+1\right)}=\lim_{x\to1^-}\tfrac{2}{\sqrt{x}+1}=1.\\
\lim_{x\to1^+}\tfrac{f(x)-f(1)}{x-1} &= \lim_{x\to1^+}\tfrac{ax+b-2}{x-1}=\lim_{x\to1^+}\tfrac{ax+b-(a+b)}{x-1}\\
&= \lim_{x\to1^+}\tfrac{a(x-1)}{x-1}=a
\end{aligned}$$

Since f is continuous at the point $a=1$, $\lim_{x\to1^-}f(x)=2=\lim_{x\to1^+}f(x)=a+b$

From the previous section we find

$$\begin{cases} a &=1\\ a+b &=2\end{cases}\Rightarrow\begin{cases} a &=1\\ b &=1\end{cases}$$

Exercise 6

Show that:

1. $\forall x\in]0,\pi[\,:x\cos(x)-\sin(x)<0$
2. $\forall x\in]0,\frac{\pi}{2}[\,:\dfrac{2x}{\pi}<\sin(x)<x$

Correction6

1. we use the Mean Value Theorem (MVT) with the function $h(t)=t\cos t-\sin t$ on the interval $[0,x]\in[0,\pi]$, which explains the existence of $c\in]0,x[$ such that:

$$\begin{aligned}
h(x)-h(0)=h'(c)x &\Leftrightarrow x\cos x-\sin x=(\cos c-c\sin c-\cos c)\,x\\
&\Leftrightarrow x\cos x-\sin x=-cx\,(\sin c)\\
&\Rightarrow x\cos x-\sin x>0 \text{ since } 0<c<x<\pi \text{ with } \sin c>0
\end{aligned}$$

2. We are studying the function $g(x) = \dfrac{\sin x}{x}$ on the interval $]0, \frac{\pi}{2}[$, we have:

$$g'(x) = \frac{x\cos x - sinx}{x^2}.$$

From the previous question, g is strictly decreasing on $]0, \frac{\pi}{2}[\in]0, \pi[$, which imlies that:

$$\begin{aligned} 0 < x < \tfrac{\pi}{2} &\Rightarrow \tfrac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} < \tfrac{\sin x}{x} < \lim_{x\to 0^+} \tfrac{\sin x}{x} \\ &\Rightarrow \tfrac{2}{\pi} < \tfrac{\sin x}{x} < 1 \\ &\Rightarrow \tfrac{2x}{\pi} < \sin x < x, \end{aligned}$$

which completes the demonstration

Exercise 7

In which of the following functions Rolle's theorem is applicable?

1. $x^2 - 2,$ sur $[-2, 2]$
2. $|x - 2|,$ sur $[1, 3]$
3. $\sqrt{1 - x^2},$ sur $[-1, 1]$
4. $\tan(x),$ sur $[\dfrac{\pi}{4}, \dfrac{\pi}{3}]$

Correction7

To answer this question, we need to check the Rolle conditions for each function.

1. For the first function $f(x) = x^2 - 2,$ sur $[-2, 2]$, we have:
   (a) f is continuous on $[-2, 2]$,
   (b) f is differentiable on $]-2, 2[$ with $f'(x) = 2x$
   (c) $f(-2) = f(2) = 2$

   So the answer is "Yes".

2. For the second function $f(x) = |x-2| = \begin{cases} x-2 & : x \geq 2 \\ -x+2 & : x < 2 \end{cases}$, on $[1, 3]$, we have:
   (a) f is continuous on $[1, 3]$,
   (b) $f(1) = f(3) = 1$
   (c) It is clear that f is differentiable on $]1, 2[\cup]2, 3[$, there remains the point $a = 2$. For this point we have:

   $$\begin{aligned} \lim_{x\to 2^-} \tfrac{f(x)-f(2)}{x-2} &= \lim_{x\to 2^-} \tfrac{-x+2}{x-2} = -1. \\ &\text{et} \\ \lim_{x\to 2^+} \tfrac{f(x)-f(2)}{x-2} &= \lim_{x\to 2^-} \tfrac{x-2}{x-2} = 1 \end{aligned}$$

   Since $\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} \neq \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2}$ so f is not differentiable at the point $2 \in]1, 3[$.

   So the answer is "No".

3. For the third function $f(x) = \sqrt{1 - x^2},$ on $[-1, 1]$, we have:

(a) f is continuous on $[-1,1]$,

(b) f is differentiable on $]-1,1[$ with $f'(x) = \frac{-x}{\sqrt{1-x^2}}$

(c) $f(-1) = f(1) = 0$

So the answer is "Yes".

4. For the fourth function $f(x) = \tan(x)$, on $[\frac{\pi}{4}, \frac{\pi}{3}]$, we have:

(a) f is continuous on $[\frac{\pi}{4}, \frac{\pi}{3}]$,

(b) f is differentiable on $]\frac{\pi}{4}, \frac{\pi}{3}[$ with $f'(x) = \frac{1}{\sqrt{1+\tan^2(x)}}$

(c) $f(\frac{\pi}{4}) = 1$ but , $f(\frac{\pi}{3}) = \sqrt{3} \neq f(\frac{\pi}{4})$

So the answer is "No".

Exercise 8

Let f be a function defined by

$$f(x) = e^{x^2}\cos(x)$$

Show that for all $a > 0$, the equation $f'(x) = 0$ has at least one solution on $[-a, a]$.

Correction8

By applying Rolle's theorem on $[-a, a]$ with the function f.

1. f is continuous on $[-a, a]$,
2. f is differentiable on $]-a, a[$ with $f'(x) = e^{x^2}(2x\cos x - \sin x)$
3. $f(-a) = e^{(-a)^2}\cos(-a) = e^{(a)^2}\cos(a) = f(a)$

Consequently, there is at least $c \in]-a, a[$ such that: $f'(c) = 0$

Exercise 9

1. apply the Mean value Theorem for the function $f : x \to x - x^3$ on the segment $[-2, 1]$ and compute the value $c \in]-2, 1[$ appearing in this formula.
2. apply the Mean value Theorem for the function $f : x \to x^2$ on the segment $[a, b]$ and compute the value $c \in]a, b[$ appearing in this formula.

Correction9

1. f is a polynomial function, so f is continuous on $[-2, 1]$ and differentiable on $]-2, 1[$, so according to the Mean value Theorem (MVT), there exist $c \in]-2, 1[$ such that:

$$\begin{aligned} f(1) - f(-2) = f'(c)\,(1 - (-2)) \;&\Leftrightarrow\; f(1) - f(-2) = 3f'(c) \\ &\Leftrightarrow\; 0 - 6 = 3(1 - 3c^2) \\ &\Leftrightarrow\; c^2 = 1 \end{aligned}$$

And since $c \in]-2, 1[\Rightarrow c = -1$

2. f is continuous and differentiable on $]a,b[$, then from (MVT), there exist $c \in]a,b[$ such that:

$$\begin{aligned} f(b)-f(a)=f'(c)\,(b-a) &\Leftrightarrow b^2-a^2=2c(b-a)\\ &\Leftrightarrow (b-a)(a+b)=2c(b-a)\\ &\Leftrightarrow c=\frac{a+b}{2}\end{aligned}$$

Exercise 10

1. Using the Mean value Theorem, show that: $\dfrac{1}{1+x}<\ln(1+x)-\ln(x)<\dfrac{1}{x}$
2. Compute $\lim\limits_{x\to+\infty} x[\ln(1+x)-\ln(x)]$
3. Deduce that: $\lim\limits_{x\to+\infty}(1+\dfrac{1}{x})^x=e$
4. Compute: $\lim\limits_{x\to-\infty}(1+\dfrac{1}{x})^x$.

Correction10

1. By applying the MVT theorem on the interval $[x,x+1], x>0$ with the function $\ln x$. Hence the existence of $x<c<x+1$ such that:

$$\begin{aligned} \ln(x+1)-\ln(x) &= \frac{1}{c}\\ \text{and since } x<c<x+1 &\Rightarrow \frac{1}{x+1}<\frac{1}{c}<\frac{1}{x}\\ \text{which implies that}&\\ \frac{1}{x+1} &< \ln(x+1)-\ln(x)<\frac{1}{x}\end{aligned}$$

2. From the previous question we have:

$$\begin{aligned}\forall x>0:\frac{1}{x+1}<\ln(x+1)-\ln(x)<\frac{1}{x} &\Rightarrow \frac{x}{x+1}<x\left[\ln(x+1)-\ln(x)\right]<\frac{x}{x}=1\\ &\Rightarrow \lim_{x\to+\infty}\frac{x}{x+1}<\lim_{x\to+\infty}x\left[\ln(x+1)-\ln(x)\right]<1\\ &\Rightarrow 1\le\lim_{x\to+\infty}x\left[\ln(x+1)-\ln(x)\right]\le 1\\ &\Rightarrow \lim_{x\to+\infty}x\left[\ln(x+1)-\ln(x)\right]=1\end{aligned}$$

3. From the previous question we have::

$$\begin{aligned}\lim_{x\to+\infty}x\left[\ln(x+1)-\ln(x)\right]=1 &= \lim_{x\to+\infty}x\left[\ln\frac{x+1}{x}\right]=1\\ &= \lim_{x\to+\infty}x\left[\ln\left(1+\frac{1}{x}\right)\right]=1\\ &= \lim_{x\to+\infty}\ln\left(1+\frac{1}{x}\right)^x=1\end{aligned}$$

and since the function $\ln$ is continuous so

$$\begin{aligned}\lim_{x\to+\infty}\ln\left(1+\frac{1}{x}\right)^x &= \ln\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x\\ &= 1\\ \text{as a result }\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x &= e\end{aligned}$$

4. by setting $y = \dfrac{1}{x} \Rightarrow (x \to -\infty \Leftrightarrow y \to 0^-)$, so

$$\begin{aligned}
\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x &= \lim_{y \to 0^-} (1+y)^{\dfrac{1}{y}} \\
&= \lim_{y \to 0^-} e^{\ln(1+y)^{\frac{1}{y}}} \\
&= \lim_{y \to 0^-} e^{\dfrac{\ln(1+y)}{y}} \\
&= e^1 = e
\end{aligned}$$

Chapter 6

Usual functions

6.1 An overview of inverse function

Let I be an interval of $\mathbb{R}$, f a function defined on I and $J = f(I)$. Our interest lies in the existence of the inverse function of f, i.e the existence of a function f^{-1} from J into I such that:

$$\forall x \in I, f^{-1}(f(x)) = x \quad \text{and} \quad \forall y \in J, f(f^{-1}(y)) = y$$

Proposition 6.1: Existence of an inverse function

Let I be an interval and f a function defined on I. If f is continuous and strictly monotone on I then f is a bijection from I to $J = f(I)$ and admits a reciprocal function f^{-1} from J to I which has the following properties:

1. f^{-1} is continuous on J.
2. f^{-1} is strictly monotonic on J and has the same direction of monotonicity as f.
3. f^{-1} is bijective.

Remark 6.1 *The graphical representations of f and f^{-1} are symmetrical with respect to the line with equation $y = x$.*

Example 6.1

Let f be a function defined by:

$$f: \begin{array}{rcl} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \ln(x) \end{array}$$

We have:

x	0	$+\infty$
$f(x)$	$\Vert$ $-\infty$	$\nearrow$ $+\infty$

Set $I = \mathbb{R}_+^*$, then $J = f(I) =]-\infty, +\infty[= \mathbb{R}$
From the table of variations of f we have:

1. f is continuous on I
2. f is strictly increasing on I

then f admits an inverse function f^{-1} denoted by e^x or $\exp(x)$ defined by:

$$f^{-1}: \begin{array}{l} \mathbb{R} \to]0, +\infty[\\ x \mapsto f^{-1}(x) = e^x \end{array}$$

Proposition 6.2: (Differentiability at a point)

Let $f : I \to J$ be a bijective and differentiable function at $x_0 \in I$.
If we have $f'(x_0) \neq 0$ then f^{-1} is differentiable at $y_0 = f(x_0)$ and moreover:

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Proposition 6.3: (Differentiability on an interval)

Let $f : I \to J$ be a bijective and differentiable function on I (with I is an open interval).
If we have: $\forall x \in I;\ f'(x) \neq 0$, then f^{-1} is differentiable on J and moreover:

$$\forall y \in J;\ (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Example 6.2

Let $f(x) = \ln(x)$ and $I = \mathbb{R}_+^*$, then $J = f(I) = \mathbb{R}$. From the previous example, f is bijective from I into J and admits an inverse function $f^{-1}(x) = e^x$.
We have: for all $x \in \mathbb{R}_+^*$, $f(x)$ is differentiable and moreover $f'(x) = \dfrac{1}{x} \neq 0$. According to proposition (5.3) f^{-1} is differentiable on $J = \mathbb{R}$ and

$$\forall y \in \mathbb{R};\ (f^{-1})'(y) = (e^y)' = \frac{1}{f'(f^{-1}(y))} = \frac{1}{\frac{1}{e^y}} = e^y$$

Remark 6.2 *In the previous formula, we can replace y by x and write:*

$$\forall x \in \mathbb{R};\ (e^x)' = e^x$$

6.2 Logarithmic Functions

6.2.1 The neperian logarithm function

Definition 6.1

The function that satisfies the following two conditions is called the neperian logarithm function and is denoted by ln:

1. $\forall x \in \mathbb{R}_+^*;\ (\ln(x))' = \dfrac{1}{x}$
2. $\ln(1) = 0$

Remark 6.3 *(Properties of derivatives)*

1. *According to the previous definition, the function* $\ln(x)$ *is differentiable on* $\mathbb{R}_+^*$ *and* $\forall x \in \mathbb{R}_+^*$; $(\ln(x))' = \dfrac{1}{x}$.
2. *The function* $\ln(|x|)$ *is differentiable on* $\mathbb{R}^*$ *and* $\forall x \in \mathbb{R}^*$; $(\ln(|x|))' = \dfrac{1}{x}$
3. *Let g be a function differentiable and non-zero on I then the function* $\ln(|g(x)|$ *is differentiable on I and its derivative:* $(\ln(|g(x)|)' = \dfrac{g'(x)}{g(x)}$

Proposition 6.4: (Limits and classical inequalities)

1. $\lim\limits_{x\to+\infty} \ln(x) = +\infty$
2. $\lim\limits_{x\to 0^+} \ln(x) = -\infty$
3. $\lim\limits_{x\to+\infty} \dfrac{\ln(x)}{x} = 0$
4. $\lim\limits_{x\to+\infty} \dfrac{\ln(x)}{x^\alpha} = 0$ (with $\alpha \in \mathbb{R}_+^*$).
5. $\lim\limits_{x\to 0^+} x\ln(x) = 0$
6. $\lim\limits_{x\to 0} \dfrac{\ln(x+1)}{x} = 1$
7. $\forall x \in]-1, +\infty[;\ \ln(x+1) \leq x$

'

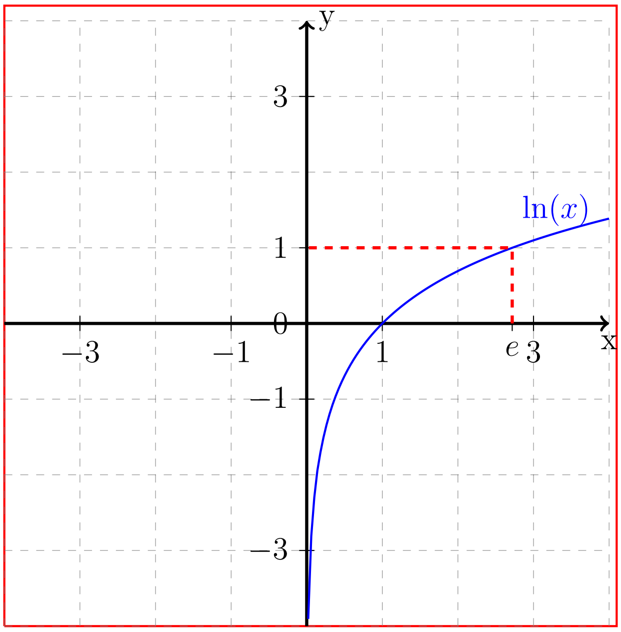


Figure 6.1 – *Graphical representation of the function* $\ln(x)$

Proposition 6.5: (Algebraic properties of the function $\ln(x)$)

For all $x,y \in \mathbb{R}_+^*$ and $\alpha \in \mathbb{Q}$, we have the following properties:

1. $\ln(x \times y) = \ln(x) + \ln(y)$
2. $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
3. $\ln\left(\frac{1}{x}\right) = -\ln(x)$
4. $\ln(x^\alpha) = \alpha \ln(x)$

6.2.2 The logarithmic function with base a

Definition 6.2

Let $a \in]0,1[\cup]1,+\infty[$.
We call the logarithm function with base a and denote $\log_a$, the function defined by:

$$\forall x \in]0,+\infty[;\ \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Remark 6.4 *(Properties of the function* $\log_a$*)*

1. *We have:* $\ln(x) = \log_e(x)$ *i.e., the neperian logarithm function is the logarithm function with base* e*.*

2. *The logarithm function with base* a *verifies relations analogous to those stated for the neperian logarithm function.*

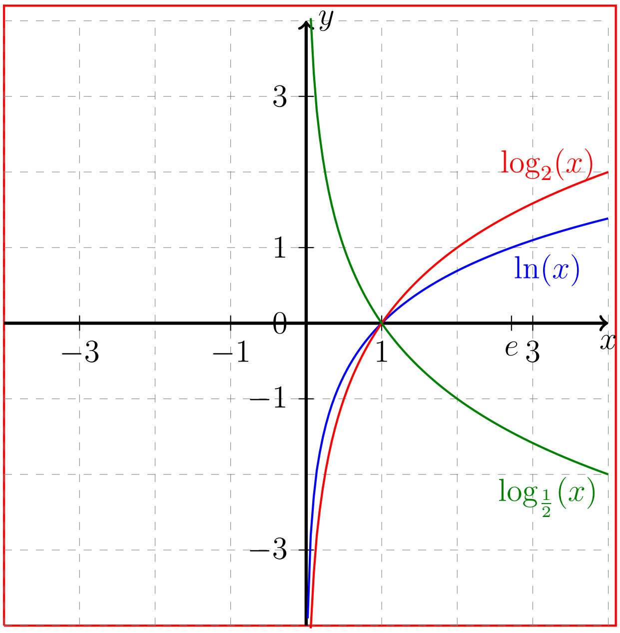


Figure 6.2 – *Graphical representation of the logarithmic functions and logarithms with base a for* $a = \frac{1}{2}$, $a = 2$

6.3 Exponential Functions

6.3.1 The exponential function

Definition 6.3

The inverse function of the function $\ln(x)$ is called the exponential function and is denoted by: $\exp(x)$ or e^x, and satisfies the following properties:

1. $\forall x \in]0, +\infty[;\ x = e^{\ln(x)}$
2. $\forall y \in \mathbb{R};\ y = \ln(e^y)$

Proposition 6.6

1. The function e^x is continuous and strictly increasing on $\mathbb{R}$.
2. The function e^x is differentiable on $\mathbb{R}$ and we have: $\forall x \in \mathbb{R};\ (e^x)' = e^x$
3. If u is differentiable on I then: the function $e^{u(x)}$ is differentiable on I and its derivative defined by: $\forall x \in I;\ (e^{u(x)})' = u'(x).e^{u(x)}$

Proposition 6.7: (Limits and inequalities)

1. $\lim\limits_{x \to -\infty} e^x = 0$
2. $\lim\limits_{x \to +\infty} e^x = +\infty$
3. $\lim\limits_{x \to +\infty} xe^{-x} = 0, \; \lim\limits_{x \to +\infty} \dfrac{x^\alpha}{e^x} = 0, \; \lim\limits_{x \to +\infty} \dfrac{e^x}{x^\alpha} = +\infty$ (with $\alpha \in \mathbb{R}$)
4. $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$
5. $\forall x \in \mathbb{R};\ e^x \geq 1 + x$

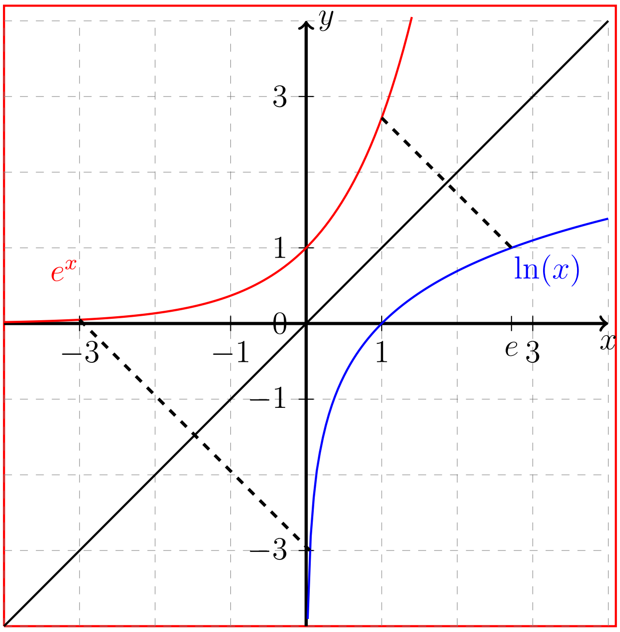


Figure 6.3 – *Graphical representation of the function* e^x

Proposition 6.8: (Algebraic properties of the function e^x)

For all $x, y \in \mathbb{R}$ and $\alpha \in \mathbb{Q}$, we have:

1. $e^{x+y} = e^x \times e^y$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $e^{x-y} = \dfrac{e^x}{e^y}$
4. $e^{\alpha x} = (e^x)^\alpha$

6.3.2 The exponential function with base a

Definition 6.4

Let $a \in]0,1[\cup]1,\infty[$.
The inverse function of the function $\log_a(x)$ is called the exponential function with base a and is denoted a^x:

1. $\forall x \in \mathbb{R};\ a^x = e^{x\ln(a)}$
2. $\forall x \in \mathbb{R};\ \log_a(a^x) = \log_a(e^{x\ln(a)}) = \dfrac{\ln(e^{x\ln(a)})}{\ln(a)} = x$

Remark 6.5 *The function a^x is differentiable on $\mathbb{R}$ and we have:*

$$\forall x \in \mathbb{R};\ (a^x)' = \ln(a)a^x$$

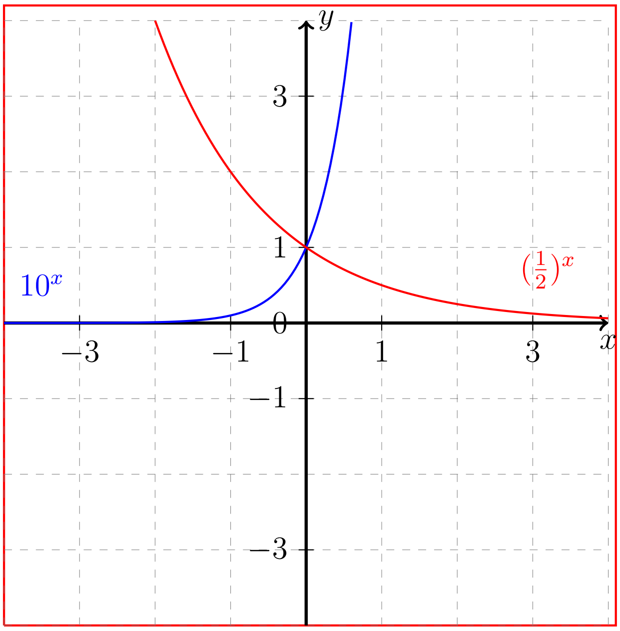


Figure 6.4 – *Graphical representation of functions* 10^x *et* $\left(\dfrac{1}{2}\right)^x$

Remark 6.6 *The exponential function with base a verifies similar properties to those of the exponential function.*

6.4 Power functions

Definition 6.5

Let $\alpha \in \mathbb{R}$, we name power function of exponent α, the function defined by:

$$\forall x \in]0,+\infty[;\ x^\alpha = e^{\alpha\ln(x)}$$

Remark 6.7 *If $n \in \mathbb{N}^*$, we have :*

$$e^{n\ln(x)} = e^{\sum_{k=1}^{n}\ln(x)} = \prod_{k=1}^{k=n} e^{\ln(x)} = \prod_{k=1}^{k=n} x = \underbrace{x \times x \times .... \times x}_{n fois} = x^n$$

Proposition 6.9

1. For $\alpha \in \mathbb{R}^*$, the power function with exponent α is a continuous function on $]0, +\infty[$ and strictly monotonic (strictly increasing if $\alpha > 0$ and strictly decreasing if $\alpha < 0$).

2. It is differentiable on $]0, +\infty[$ with derivative : $(x^\alpha)' = \alpha x^{\alpha-1}$, $\forall x \in]0, +\infty[$

3. We have:

$$\lim_{x\to+\infty} x^\alpha = \begin{cases} 0, & si\ \alpha < 0 \\ 1, & si\ \alpha = 0 \\ +\infty, & si\ \alpha > 0 \end{cases} \quad \text{and} \quad \lim_{x\to 0^+} x^\alpha = \begin{cases} +\infty, & si\ \alpha < 0 \\ 1, & si\ \alpha = 0 \\ 0, & si\ \alpha > 0 \end{cases}$$

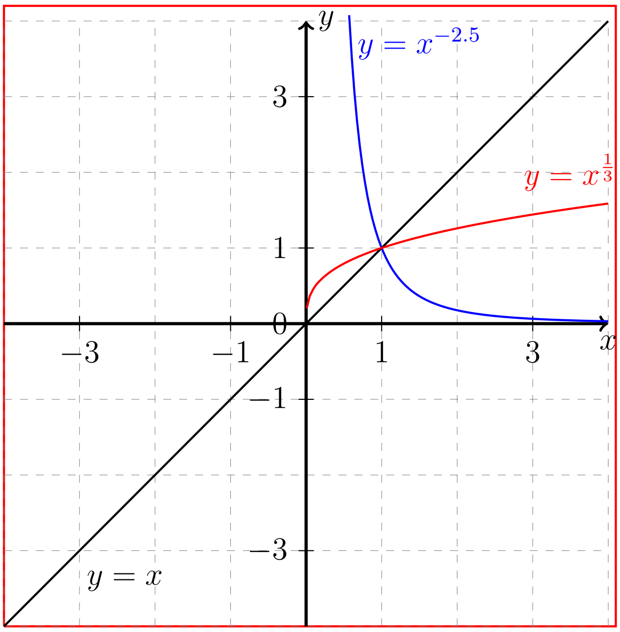


Figure 6.5 – *Graphical representation of functions* x^α*, with* $\alpha = -2.5, 1, \frac{1}{3}$

Proposition 6.10

For $x \in \mathbb{R}_+^*$ and $\alpha, \beta \in \mathbb{R}$ we have the following relationships:

1. $x^{\alpha+\beta} = x^\alpha x^\beta$.
2. $x^{-\alpha} = \dfrac{1}{x^\alpha}$.
3. $x^{\alpha-\beta} = \dfrac{x^\alpha}{x^\beta}$.
4. $x^{\alpha\beta} = (x^\alpha)^\beta = (x^\beta)^\alpha$.

6.5 Circular (or trigonometric) functions

6.5.1 Recalls on the functions $cos(x)$ and $sin(x)$.

Proposition 6.11

The functions $\begin{cases} x \longmapsto \cos(x) \\ \quad \text{and} \\ x \longmapsto \sin(x) \end{cases}$ are defined on $\mathbb{R}$ and satisfy the following properties:

1. $\forall x \in \mathbb{R};\ |\cos(x)| \leq 1 \wedge |\sin(x)| \leq 1$
2. $\cos(x)$ and $\sin(x)$ are 2π-periodic i.e.:

$$\forall x \in \mathbb{R}; \begin{cases} \cos(x+2\pi) = \cos(x) \\ \quad \text{and} \\ \sin(x+2\pi) = sin(x) \end{cases}$$

3. The function $\cos(x)$ is even and the function $\sin(x)$ is odd, i.e.:

$$\forall x \in \mathbb{R}; \begin{cases} \cos(-x) = \cos(x) \\ \quad \text{and} \\ \sin(-x) = -\sin(x) \end{cases}$$

4. The functions $\cos(x)$ and $\sin(x)$ belong to $C^\infty(\mathbb{R})$ and we have:

   a $\forall x \in \mathbb{R}; \begin{cases} (\cos(x))' = -\sin(x) \\ \quad \text{and} \\ (\sin(x))' = \cos(x) \end{cases}$

   b $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}; \begin{cases} \cos^{(n)}(x) = \cos(x + n\frac{\pi}{2}) \\ \quad \text{and} \\ \sin^{(n)}(x) = \sin(x + n\frac{\pi}{2}) \end{cases}$

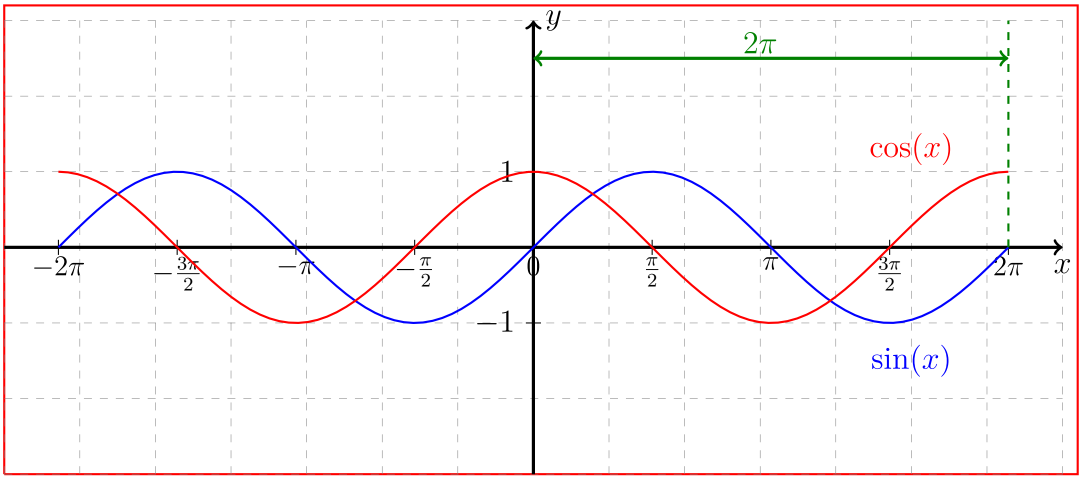


Figure 6.6 – *Graphical representation of functions* $\sin(x)$ *and* $\cos(x)$

Proposition 6.12: (Formules d'addition)

For all $(x,y) \in \mathbb{R}^2$, we have the following formulas:

- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
- $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
- $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$
- $\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

6.5.2 Recall about the function $\tan(x)$

Definition 6.6

The tangent function is one of the main trigonometric functions and defined by:

$$\begin{array}{rccl} \tan: & \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \tan(x) = \dfrac{\sin(x)}{\cos(x)} \end{array}$$

Proposition 6.13

The function $\tan(x)$ is differentiable on $\mathbb{R}\backslash\{\frac{\pi}{2} + k\pi/k \in \mathbb{Z}\}$ and we have:

$$\forall x \in \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi/k \in \mathbb{Z}\};\ (\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$$

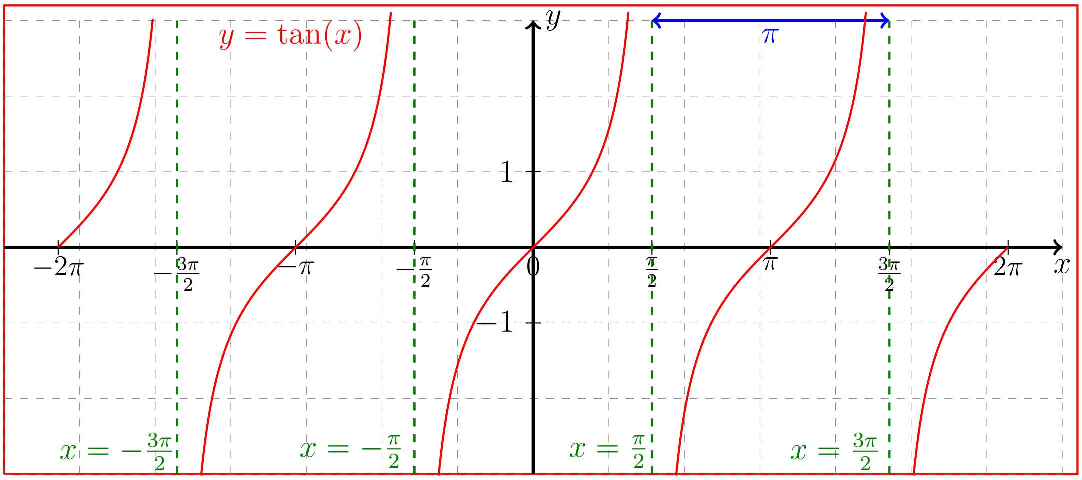


Figure 6.7 – *Graphical representation of the function* $\tan(x)$

Proposition 6.14

The function $\tan(x)$ checks the following properties:

1. The function $\tan(x)$ is π-periodic i.e :

$$\forall x \in \mathbb{R} \backslash \{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\};\ \tan(x+\pi) = \tan(x)$$

2. For any $x,y \in \mathbb{R} \backslash \{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\}$ we have:

$$\begin{cases} \tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \\ \qquad \text{and} \\ \tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)} \end{cases}$$

3. $\forall x \in \mathbb{R} \backslash \{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\};\ \tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}$

Proposition 6.15: (Some usual limits)

1. $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1$
2. $\lim\limits_{x \to 0} \dfrac{1-\cos(x)}{x^2} = \dfrac{1}{2}$
3. $\lim\limits_{x \to 0} \dfrac{\cos(x)-1}{x} = 0$
4. $\lim\limits_{x \to -\frac{\pi}{2}} \tan(x) = -\infty$
5. $\lim\limits_{x \to +\frac{\pi}{2}} \tan(x) = +\infty$
6. $\lim\limits_{x \to 0} \dfrac{\tan(x)}{x} = 1$

6.6 Hyperbolic Functions

6.6.1 Hyperbolic cosine, sine and tangent functions

Any function f defined on $\mathbb{R}$ can be uniquely decomposed into a sum of two functions f_{ev} and f_{od} where f_{ev} is an even function and f_{od} is an odd function. This means for every $x \in \mathbb{R}$ we can write

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

and we choose

$$\begin{cases} f_p(x) = \dfrac{f(x)+f(-x)}{2} \\ \qquad et \\ f_i(x) = \dfrac{f(x)-f(-x)}{2} \end{cases}$$

Remark 6.8 *We can easily check that this decomposition is unique, and f_{ev} is an even function and f_{od} is an odd function.*

Definition 6.7: (Hyperbolic cosine)

We call the hyperbolic cosine function and denoted (ch or cosh), the even part of the exponential function defined by:

$$\begin{aligned} \text{ch} : \ \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \text{ch}(x) = \frac{e^x + e^{-x}}{2} \end{aligned}$$

Definition 6.8: (Hyperbolic sine)

The hyperbolic sine function, denoted by (sh or sinh), is the odd part of the exponential function defined by:

$$\begin{aligned} \text{sh} : \ \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \text{sh}(x) = \frac{e^x - e^{-x}}{2} \end{aligned}$$

Definition 6.9: (Hyperbolic tangent)

The hyperbolic tangent function, denoted by (th or tanh), is the quotient of the hyperbolic sine function with the hyperbolic cosine function and defined by:

$$\begin{aligned} \text{th} : \ \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \text{th}(x) = \frac{\text{sh}(x)}{\text{ch}(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \end{aligned}$$

Proposition 6.16

- The function $\mathrm{ch}(x)$ is a function defined on $\mathbb{R}$, continuous and even.
- The function $\mathrm{sh}(x)$ is a function defined on $\mathbb{R}$, continuous and odd.
- The function $\mathrm{th}(x)$ is a function defined on $\mathbb{R}$, continuous and odd.
- The functions $\mathrm{ch}(x)$, $\mathrm{sh}(x)$ and $\mathrm{th}(x)$ are differentiable on $\mathbb{R}$ and their derivatives are defined by:
$$\forall x \in \mathbb{R};\ \begin{cases} (\mathrm{ch}(x))' = \mathrm{sh}(x) \\ (\mathrm{sh}(x))' = \mathrm{ch}(x) \\ (\mathrm{th}(x))' = \dfrac{1}{\mathrm{ch}(x)^2} = 1 - \mathrm{th}(x)^2 \end{cases}$$

Proof

These properties can be verified using the properties of the e^x function. In our proof, we're interested with the function $\mathrm{th}(x)$.
We have:
$$\forall x \in \mathbb{R};\ \mathrm{th}(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

- The continuity: The functions $(e^x - e^{-x})$ and $(e^x + e^{-x})$ are continuous on $\mathbb{R}$, with $e^x + e^{-x} \neq 0$ then the quotient function $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is continuous on $\mathbb{R} \implies \mathrm{th}(x)$ is continuous on $\mathbb{R}$
- The parity: We have:
$$\forall x \in \mathbb{R};\ \mathrm{th}(-x) = \frac{e^{-x} - e^x}{e^{-x} + e^x} = -\frac{e^x - e^{-x}}{e^x + e^{-x}} = -\mathrm{th}(x)$$
So $\mathrm{th}(x)$ is odd.
- The differentiability: The functions $(e^x - e^{-x})$ et $(e^x + e^{-x})$ are differentiable on $\mathbb{R}$, with $e^x + e^{-x} \neq 0$ then the quotient function $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is differentiable on $\mathbb{R} \implies \mathrm{th}(x)$ is differentiable on $\mathbb{R}$ and we have:
$$\forall x \in \mathbb{R};\ (\mathrm{th}(x))' = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)' = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$
$$\iff \mathrm{th}(x)' = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = 1 - \mathrm{th}(x)^2$$
also $\mathrm{th}(x)' = \dfrac{4}{(e^x + e^{-x})^2} = \dfrac{1}{\mathrm{ch}(x)^2}$.

Remark 6.9 *The functions $ch(x)$,$sh(x)$ and $th(x)$ have the following properties:*

1. $\mathrm{ch}(0) = 1$, $\mathrm{sh}(0) = 0$ *and* $\mathrm{th}(0) = 0$.
2. $\lim_{x \to -\infty} sh(x) = -\infty$, $\lim_{x \to -\infty} ch(x) = +\infty$ *and* $\lim_{x \to -\infty} th(x) = -1$

3. $\lim_{x \to +\infty} sh(x) = +\infty$, $\lim_{x \to +\infty} ch(x) = +\infty$ *and* $\lim_{x \to +\infty} th(x) = 1$

Therefore, the above results can be grouped together in tabular form.

x	$-\infty$	0	$+\infty$
$\text{sh}(x)' = \text{ch}(x)$		$+$	
$\text{sh}(x)$	$-\infty$ ↗	0 ↗	$+\infty$

(a) *Function sh(x)*

x	$-\infty$	0	$+\infty$
$\text{ch}(x)' = \text{sh}(x)$	$-$	0	$+$
$\text{ch}(x)$	$+\infty$ ↘	1 ↗	$+\infty$

(b) *Function ch(x)*

Figure 6.8 – *Functions sh(x) and ch(x)*

x	$-\infty$	0	$+\infty$
$\text{th}(x)' = \dfrac{1}{\text{ch}(x)^2}$		$+$	
$\text{th}(x)$	-1 ↗	0 ↗	1

Figure 6.9 – *Function th(x)*

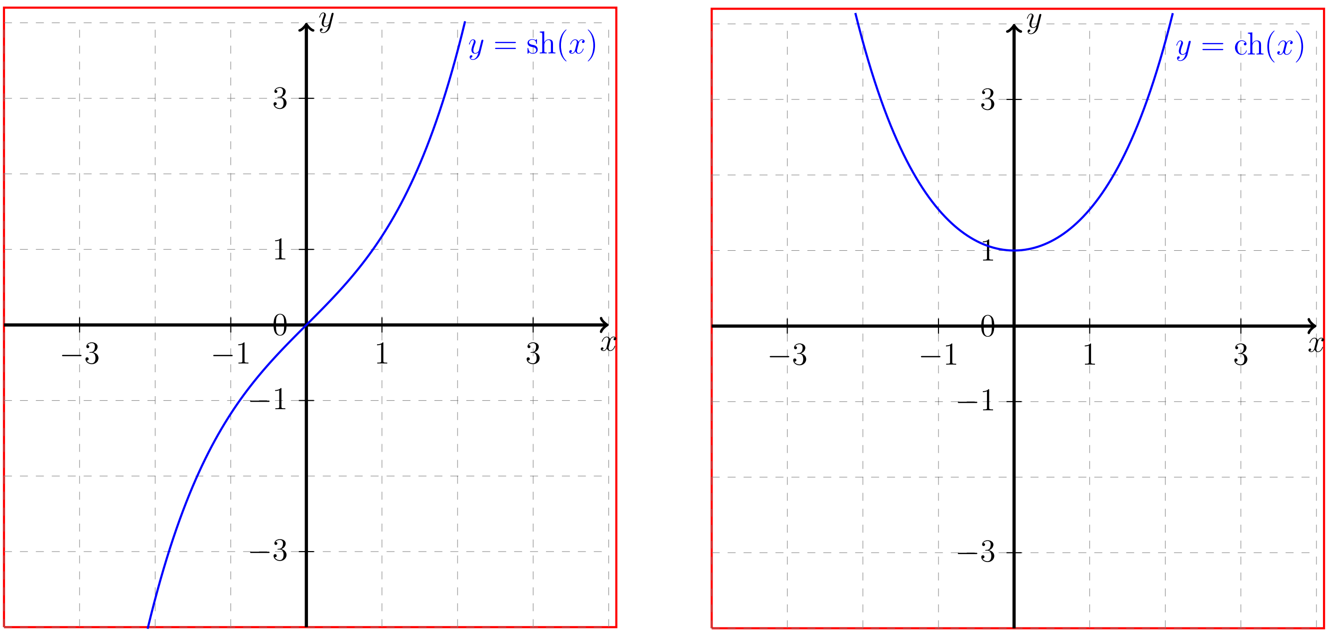


Figure 6.10 – *Graphical representation of functions sh(x) et ch(x)*

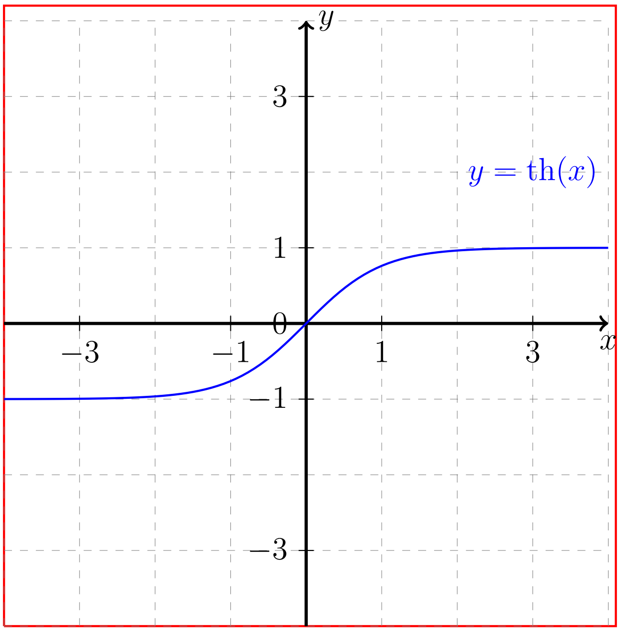


Figure 6.11 – *Graphical representation of the function th(x)*

Proposition 6.17

For every real x, we have:

- $\text{ch}(x) + \text{sh}(x) = e^x$
- $\text{ch}(x) - \text{sh}(x) = e^{-x}$
- $\text{ch}(x)^2 - \text{sh}(x)^2 = 1$

Proposition 6.18: (Addition formulas)

For all $(x,y) \in \mathbb{R}^2$, we have the following formulas:

- $\text{ch}(x+y) = \text{ch}(x)\text{ch}(y) + \text{sh}(x)\text{sh}(y)$
- $\text{ch}(x-y) = \text{ch}(x)\text{ch}(y) - \text{sh}(x)\text{sh}(y)$
- $\text{sh}(x+y) = \text{sh}(x)\text{ch}(y) + \text{ch}(x)\text{sh}(y)$
- $\text{sh}(x-y) = \text{sh}(x)\text{ch}(y) - \text{ch}(x)\text{sh}(y)$
- $\text{th}(x+y) = \dfrac{\text{th}(x) + \text{th}(y)}{1 + \text{th}(x)\text{th}(y)}$
- $\text{th}(x-y) = \dfrac{\text{th}(x) - \text{th}(y)}{1 - \text{th}(x)\text{th}(y)}$

Proof

We prove these formulas by using the expressions of hyperbolic functions with the exponential function. We have:

$$\begin{aligned}
\mathrm{ch}(x)\mathrm{ch}(y) + \mathrm{sh}(x)\mathrm{sh}(y) &= \frac{1}{4}\left((e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})\right) \\
&= \frac{1}{4}\left(2e^x e^y + 2e^{-x}e^{-y}\right) \\
&= \frac{1}{2}\left(e^{(x+y)} + e^{-(x+y)}\right) \\
&= \mathrm{ch}(x+y).
\end{aligned}$$

The other relations are shown using the same method.

Proposition 6.19: (Some usual limits of hyperbolic functions)

1. $\lim\limits_{x \to +\infty} \dfrac{\mathrm{ch}(x)}{e^x} = \dfrac{1}{2}$
2. $\lim\limits_{x \to +\infty} \dfrac{\mathrm{sh}(x)}{e^x} = \dfrac{1}{2}$
3. $\lim\limits_{x \to 0} \dfrac{\mathrm{sh}(x)}{x} = 1$
4. $\lim\limits_{x \to 0} \dfrac{\mathrm{ch}(x) - 1}{x^2} = \dfrac{1}{2}$

6.7 Inverse Trigonometric Functions

6.7.1 The function arc-sinus

According to the variation table below, we have:
The function $\sin(x)$ is continuous and strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then the function $\sin(x)$ represents a bijection from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $[-1,1]$.

x	$-\frac{\pi}{2}$	0	$+\frac{\pi}{2}$
$\sin(x)' = \cos(x)$		$+$	
$\sin(x)$	-1 ↗	0 ↗	1

Figure 6.12 – *Function* $\sin(x)$

Definition 6.10

The inverse function of the restriction of $\sin(x)$ on $[-\frac{\pi}{2},\frac{\pi}{2}]$ is called the arcsine function and is denoted by $\arcsin(x)$ or $\sin^{-1}(x)$:

$$\begin{array}{rrcl} \arcsin: & [-1,1] & \longrightarrow & [-\frac{\pi}{2},\frac{\pi}{2}] \\ & x & \longmapsto & \arcsin(x) \end{array}$$

Proposition 6.20

The function *arcsin*(x) has the following properties:

1. The function $\arcsin(x)$ is continuous and strictly increasing on $[-1,1]$. (According to the inverse function theorem)
2. $\forall x \in [-\frac{\pi}{2},\frac{\pi}{2}]$; $\arcsin(\sin(x)) = x$.
3. $\forall y \in [-1,1]$; $\sin(\arcsin(y)) = y$.
4. $\forall x \in [-\frac{\pi}{2},\frac{\pi}{2}], \forall y \in [-1,1]$; $(\sin(x) = y \iff x = \arcsin(y))$.
5. The function *arcsin*(x) is odd.

Proof

Let's prove property (5).

1. The function *arcsin*(x) is defined on $[-1,1]$, so in this case the domain of definition is symmetric about 0.
2. Let $x \in [-1,1]$ and:
$$\arcsin(-x) = y \tag{6.1}$$
$$\Leftrightarrow -x = \sin(y) \Leftrightarrow x = -\sin(y) \Leftrightarrow x = \sin(-y) \text{ (Since } \sin(x) \text{ is odd)}$$
We have: $y \in [-\frac{\pi}{2},\frac{\pi}{2}] \implies -y \in [-\frac{\pi}{2},\frac{\pi}{2}]$
So we obtain: $\arcsin(x) = -y \Leftrightarrow -\arcsin(x) = y$
From equation (6.1) we get: $\arcsin(-x) = -\arcsin(x)$

$\implies$ The function $\arcsin(x)$ is odd.

Remark 6.10 *The following table contains some usual values for the function* $\arcsin(x)$

$\sin(0) = 0$	$\arcsin(0) = 0$
$\sin(\frac{\pi}{6}) = \frac{1}{2}$	$\arcsin(\frac{1}{2}) = \frac{\pi}{6}$
$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$	$\arcsin(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$
$\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$	$\arcsin(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$
$\sin(\frac{\pi}{2}) = 1$	$\arcsin(1) = \frac{\pi}{2}$

Proposition 6.21

The arcsine function is differentiable on $]-1,1[$ and verifies:

$$\forall x \in]-1,1[;\ (\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$$

Proof

The function $\sin(x)$ has the following two properties:

1. $\sin(x)$ is differentiable on $]-\frac{\pi}{2},\frac{\pi}{2}[$.
2. $\forall x \in]-\frac{\pi}{2},\frac{\pi}{2}[;\ (\sin(x))' = \cos(x) \neq 0$

$\Longrightarrow$ (from proposition (5.3)), the function $\arcsin(x)$ is differentiable on $]-1,1[$ and we have:

$$\forall x \in]-1,1[;\ (\arcsin(x))' = \frac{1}{\cos(\arcsin(x))} \tag{6.2}$$

Let $x \in]-1,1[$, and $y = \arcsin(x)$

$$\Longrightarrow\ y \in]-\frac{\pi}{2},\frac{\pi}{2}[\wedge \cos(y) > 0$$

Based on the relationship $\cos^2(y) + \sin^2(y) = 1$, we deduce that: $\cos(y) = \sqrt{1-\sin^2(y)}$. Since for all $x \in]-1,1[$ we have: $\sin(\arcsin(x)) = x$

$$\Longrightarrow\ \cos(\arcsin(x)) = \sqrt{1-x^2}$$

From equation (6.2) we obtain:

$$\forall x \in]-1,1[;\ (arcsin(x))' = \frac{1}{\sqrt{1-x^2}}.$$

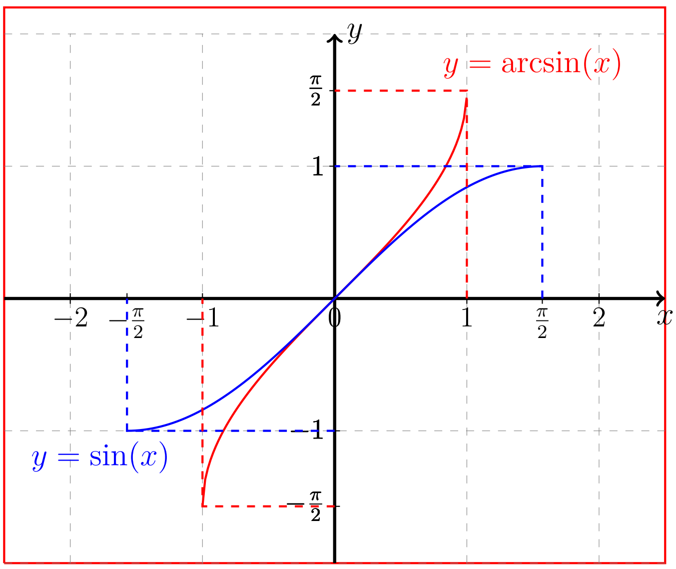


Figure 6.13 – *Graphical representation of the function* $\arcsin(x)$

6.7.2 The Arccosine Function

In the variation table below, we have:
The function $\cos(x)$ is continuous and strictly decreasing on $[0,\pi]$, so the function $cos(x)$ makes a bijection from $[0,\pi]$ into $[-1,1]$.

x	0 $\qquad\qquad$ π
$(\cos(x))' = -\sin(x)$	$-$
$\cos(x)$	$1 \searrow -1$

Figure 6.14 – *The function* $\cos(x)$

Definition 6.11

The inverse function of the restriction of $\cos(x)$ on $[0,\pi]$ is called the arccosine function and is denoted by $\arccos(x)$ or $\cos^{-1}(x)$:

$$\begin{array}{rccl} \arccos : & [-1,1] & \longrightarrow & [0,\pi] \\ & x & \longmapsto & \arccos(x) \end{array}$$

Proposition 6.22

The function $\arccos(x)$ has the following properties:

1. The function $\arccos(x)$ is continuous and strictly decreasing on $[-1,1]$. (From the inverse function theorem)
2. $\forall x \in [0,\pi];\ \arccos(\cos(x) = x$.
3. $\forall y \in [-1,1];\ \cos(\arccos(y) = y$.
4. $\forall x \in [0,\pi], \forall y \in [-1,1];\ (\cos(x) = y \Longleftrightarrow x = \arccos(y))$.
5. The function $\arccos(x)$ is neither even nor odd.

Remark 6.11 *The table below shows some usual values for the function* $\arccos(x)$.

$\cos(0) = 1$	$\arccos(1) = 0$
$\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$	$\arccos(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$
$\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$	$\arccos(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$
$\cos(\frac{\pi}{3}) = \frac{1}{2}$	$\arccos(\frac{1}{2}) = \frac{\pi}{3}$
$\cos(\frac{\pi}{2}) = 0$	$\arccos(0) = \frac{\pi}{2}$

Proposition 6.23

The arccosine function is differentiable on $]-1,1[$ and verifies:

$$\forall x \in]-1,1[;\ (\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}$$

Proof

We have the function $\cos(x)$ satisfying the following two properties:

1. $\cos(x)$ is differentiable on $]0,\pi[$.
2. $\forall x \in]0,\pi[;\ (\cos(x))' = -\sin(x) \neq 0$

$\Longrightarrow$ (from proposition (6.3)), the function $\arccos(x)$ is differentiable on $]-1,1[$ and we have:

$$\forall x \in]-1,1[;\ (\arccos(x))' = \frac{1}{-\sin(\arccos(x))} \tag{6.3}$$

Let $x \in]-1,1[$, and $y = \arccos(x)$

$$\Longrightarrow\ y \in]0,\pi[\wedge \sin(y) > 0$$

Using the relationship $\cos^2(y) + \sin^2(y) = 1$, we deduce that $\sin(y) = \sqrt{1-\cos^2(y)}$.
Since for any $x \in]-1,1[$ we have: $\cos(\arccos(x)) = x$, then we get:

$$\sin(\arccos(x)) = \sqrt{1-x^2}$$

From equation (6.3) we obtain:

$$\forall x \in]-1,1[;\ (arccos(x))' = -\frac{1}{\sqrt{1-x^2}}.$$

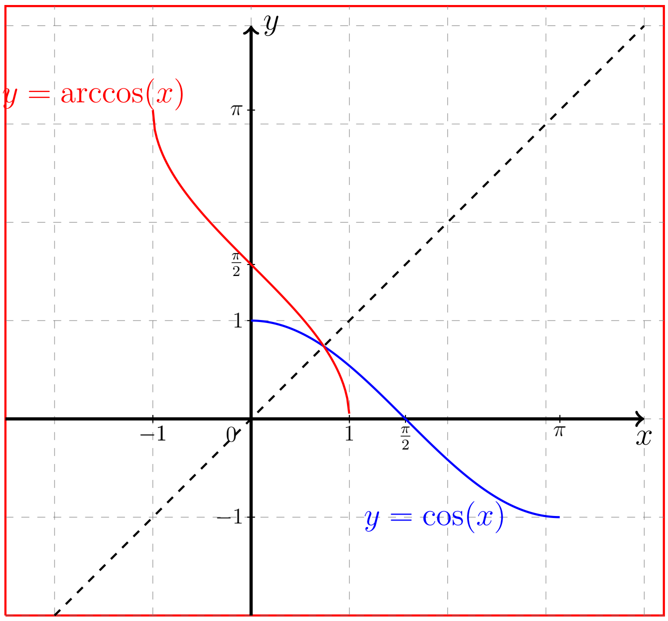


Figure 6.15 – *Graphical representation of the function* $\arccos(x)$

6.7.3 The Arctangent function

The function $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ is defined on $D = \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. It is continuous and differentiable on its domain of definition and for all $x \in D$ we have:

$$(\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$$

Consider the restriction of the function $\tan(x)$ on the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$, from the table of variation below we have: the function $\tan(x)$ is continuous and strictly increasing on $]-\frac{\pi}{2}, \frac{\pi}{2}[$, then the function $\tan(x)$ makes a bijection from $]-\frac{\pi}{2}, \frac{\pi}{2}[$ into $\mathbb{R}$.

x	$-\frac{\pi}{2}$		$\frac{\pi}{2}$
$(\tan(x))' = \dfrac{1}{\cos^2}$	‖	$+$	‖
$\tan(x)$	$-\infty$	↗	$+\infty$

Figure 6.16 – *The function* $\tan(x)$

Definition 6.12

We call the arctangent function $\arctan(x)$ or $\tan^{-1}(x)$ the inverse of the tangent function on $]-\frac{\pi}{2}, \frac{\pi}{2}[$ defined by:

$$\begin{array}{rrcl} \arctan : &]-\infty, +\infty[& \longrightarrow &]-\frac{\pi}{2}, \frac{\pi}{2}[\\ & x & \longmapsto & \arctan(x) \end{array}$$

Proposition 6.24

The function $\arctan(x)$ has the following properties:

1. The function $\arctan(x)$ is continuous and strictly increasing on $\mathbb{R}$, with values in $]-\frac{\pi}{2}, \frac{\pi}{2}[$
2. $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[;\ \arctan(\tan(x)) = x$
3. $\forall y \in \mathbb{R};\ \tan(\arctan(y)) = y$.
4. $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[, \forall y \in \mathbb{R};\ \tan(x) = y \Longleftrightarrow x = \arctan(y)$
5. The function *arctan*(x) is odd.

Remark 6.12 *The table below shows some usual values for the function* $\arctan(x)$.

$\tan(0) = 0$	$\arctan(0) = 0$
$\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$	$\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$
$\tan(\frac{\pi}{4}) = 1$	$\arctan(1) = \frac{\sqrt{2}}{2}$
$\tan(\frac{\pi}{3}) = \sqrt{3}$	$\arctan(\sqrt{3}) = \frac{\pi}{3}$

Proposition 6.25

The function $\arctan(x)$ is differentiable on $\mathbb{R}$ and verifies:

$$\forall x \in \mathbb{R};\ (\arctan(x))' = \frac{1}{1+x^2}$$

Proof

The function $\tan(x)$ has the following two properties:

1. The function $\tan(x)$ is differentiable on $]-\frac{\pi}{2}, \frac{\pi}{2}[$.
2. $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[;\ (\tan(x))' = \dfrac{1}{\cos^2(x)} = 1 + \tan^2(x) \neq 0$

From proposition (6.3), the function $\arctan(x)$ is differentiable on $]-\frac{\pi}{2}, \frac{\pi}{2}[$ and we have:

$$\forall x \in \mathbb{R};\ (\arctan(x))' = \frac{1}{1+\tan^2(\arctan(x))} = \frac{1}{1+x^2}$$

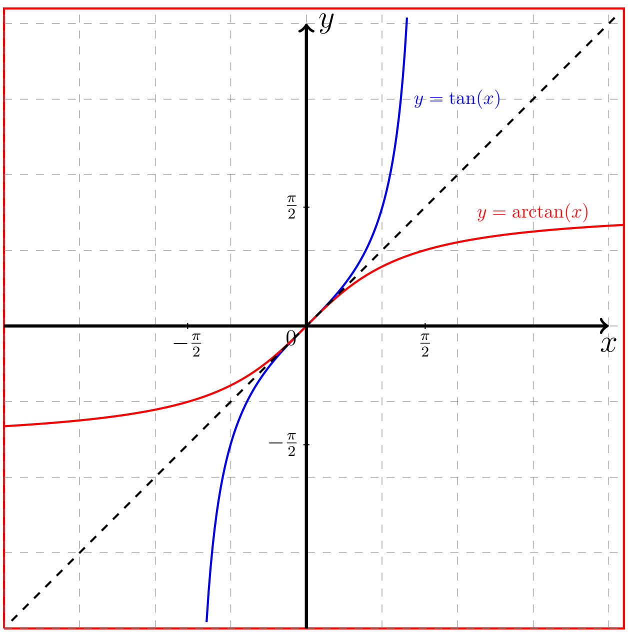


Figure 6.17 – *Graphical representation of the function* $\arctan(x)$

Proposition 6.26: (Some properties)

1. For any $x \in [-1,1]$ we have:

$$\arccos(x) + \arcsin(x) = \frac{\pi}{2}$$

2. For all $x \in \mathbb{R}_-^*$ we have:

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}$$

3. For every $x \in \mathbb{R}_+^*$ we have:

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

Proof

We'll show properties (2) and (3).
Set $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$.
Since the functions $\dfrac{1}{x}$ and $\arctan(x)$ are differentiable on $\mathbb{R}^*$), the function f is differentiable on $\mathbb{R}^*$ and we have:

$$f'(x) = \frac{1}{1+x^2} + \left(\frac{1}{x}\right)' \frac{1}{1+\left(\frac{1}{x}\right)^2} = \frac{1}{1+x^2} - \frac{1}{x^2}\left(\frac{x^2}{1+x^2}\right) = 0$$

From this we deduce that f is a constant function on each of the intervals $]-\infty,0[$ and $]0,+\infty[$.
On the other hand, we have:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right) = -\frac{\pi}{2}$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right) = \frac{\pi}{2}$$

so f can't be extended by continuity at 0. So we deduce that:

$$\exists C_1,C_2 \in \mathbb{R} \text{ tq:} f(x) = \begin{cases} C_1 & \text{if } x \in]0,+\infty[\\ C_2 & \text{if } x \in]-\infty,0[\end{cases}$$

Since $f(1) = 2\arctan(1) = 2\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{2} = C_1$
and $f(-1) = 2\arctan(-1) = 2\left(-\dfrac{\pi}{4}\right) = -\dfrac{\pi}{2} = C_2$

$$\Longrightarrow f(x) = \begin{cases} \frac{\pi}{2} & \text{if } x \in]0,+\infty[\\ -\frac{\pi}{2} & \text{if } x \in]-\infty,0[\end{cases}$$

So $\forall x \in \mathbb{R}_-^*;\ \arctan(x) + \arctan\left(\dfrac{1}{x}\right) = -\dfrac{\pi}{2}$ and $\forall x \in \mathbb{R}_+^*;\ \arctan(x) + \arctan\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$

6.8 The inverse hyperbolic functions

6.8.1 The inverse of hyperbolic Sine function

From the above table of variation of $\text{sh}(x)$ we have: $\text{sh}(x)$ is continuous and strictly increasing on $\mathbb{R}$. Hence, it realizes a bijection from $\mathbb{R}$ into $\mathbb{R}$.

Definition 6.13

The inverse function of the hyperbolic sine function on $\mathbb{R}$ is denoted $\text{argsh}(x)$ or $\text{sh}^{-1}(x)$.

$$\begin{array}{rccc} \text{argsh}: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \text{argsh}(x) \end{array}$$

Proposition 6.27

The function $\text{argsh}(x)$ has the following properties:

1. The function $\text{argsh}(x)$ is defined on $\mathbb{R}$, it is continuous and strictly increasing on $\mathbb{R}$.
2. $\forall x \in \mathbb{R}$; $\text{argsh}(\text{sh}(x))=x$.
3. $\forall y \in \mathbb{R}$; $\text{sh}(\text{argsh}(y))=y$.
4. $\forall (x,y) \in \mathbb{R}^2$; $y = \text{sh}(x) \iff x = \text{argsh}(y)$.
5. $\text{argsh}(x)$ is odd function.

Proof

We'll show that $\text{argsh}(x)$ is odd.
Let $x \in \mathbb{R}$, and

$$y = \text{argsh}(-x) \tag{6.4}$$

(6.4)$\iff \text{sh}(y) = -x \iff \text{sh}(-y) = x$ (Since $\text{sh}(x)$ is odd)
$\implies -y = \text{argsh}(x) \iff y = -\text{argsh}(x)$.
From (6.4), we get: $\text{argsh}(-x) = -\text{argsh}(x)$.
So, $\forall x \in \mathbb{R}$; $\text{argsh}(-x) = -\text{argsh}(x) \implies \text{argsh}(x)$ is odd.

Proposition 6.28

The function $\text{argsh}(x)$ is differentiable on $\mathbb{R}$ and verifies:

$$\forall x \in \mathbb{R};\ (\text{argsh}(x))' = \frac{1}{\sqrt{1+x^2}}.$$

Proof

The $\text{sh}(x)$ function verifies the following two properties:

1. $\text{sh}(x)$ is differentiable on $\mathbb{R}$.
2. $\forall x \in \mathbb{R}; (\text{sh}(x))' = \text{ch}(x) = \dfrac{e^x + e^{-x}}{2} \neq 0$

From proposition (6.3), the function $\text{argsh}(x)$ is differentiable on $\mathbb{R}$:

$$\forall x \in \mathbb{R};\ (\text{argsh}(x))' = \frac{1}{\text{sh}'(\text{argsh}(x))} = \frac{1}{\text{ch}(\text{argsh}(x))}$$

On the other hand, we have: $\text{ch}(x)^2 - \text{sh}(x)^2 = 1 \implies \text{ch}(x) = \sqrt{1 + \text{sh}^2(x)}$ because $\text{ch}(x)$ is positive function.

$$\implies \forall x \in \mathbb{R};\ \text{ch}(\text{argsh}(x)) = \sqrt{1 + (\text{sh}(\text{argsh}(x))^2} = \sqrt{1 + x^2}$$

$$\implies \forall x \in \mathbb{R};\ (\text{argsh}(x))' = \frac{1}{\sqrt{1 + x^2}}$$

.

Proposition 6.29

$$\forall x \in \mathbb{R};\ \text{argsh}(x) = \ln\left(x + \sqrt{1 + x^2}\right)$$

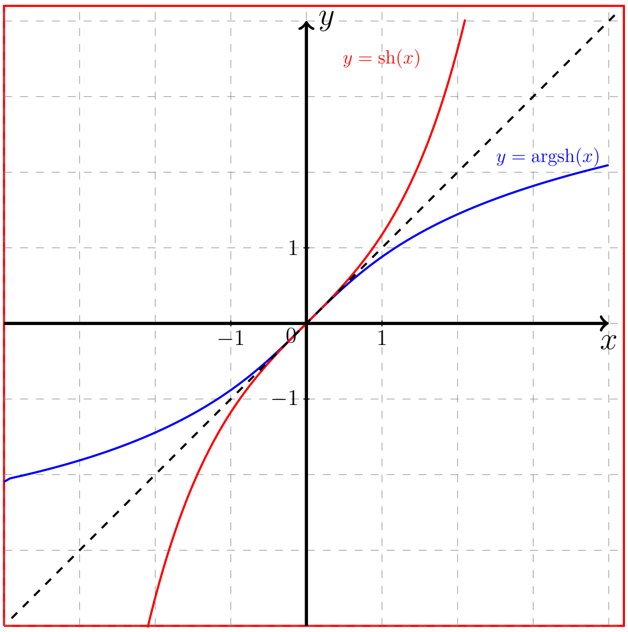


Figure 6.18 – *Graphical representation of the function argsh(x)*

6.8.2 The inverse hyperbolic cosine function

From the table of variation of the function $\mathrm{ch}(x)$ above we have:
$\mathrm{ch}(x)$ is continuous and strictly increasing on $[0,+\infty[$. So it forms a bijection from $[0,+\infty[$ into $[1,+\infty[$.

Definition 6.14

The inverse function of the restriction of $\mathrm{ch}(x)$ on $[0,+\infty[$ is denoted by $\mathrm{argch}(x)$ or $\mathrm{ch}^{-1}(x)$

$$\begin{array}{rccl} \mathrm{argch}: & [1,+\infty[& \longrightarrow & [0,+\infty[\\ & x & \longmapsto & \mathrm{argch}(x) \end{array}$$

Proposition 6.30

The $\mathrm{argch}(x)$ function has the following properties:

1. The function $\mathrm{argch}(x)$ is defined on $[1,+\infty[$, it is continuous and strictly increasing on $[1,+\infty[$.
2. $\forall x \in [0,+\infty[$; $\mathrm{argch}(\mathrm{ch}(x))=x$.
3. $\forall y \in [1,+\infty[$; $\mathrm{ch}(\mathrm{argch}(y))=y$.
4. $\forall x \in [0,+\infty[, \forall y \in [1,+\infty[$; $y=\mathrm{ch}(x) \Longleftrightarrow x=\mathrm{argch}(y)$.

Proposition 6.31

The inverse hyperbolic cosine function is differentiable on $]1,+\infty[$ and verifies:

$$\forall x \in]1,+\infty[;\ (\mathrm{argch}(x))' = \frac{1}{\sqrt{x^2-1}}$$

Remark 6.13 *The proof of proposition (6.31) is similar to the proof of proposition (6.28).*

Proposition 6.32

$$\forall x \in]1,+\infty[;\ \mathrm{argch}(x) = \ln\left(x+\sqrt{x^2-1}\right)$$

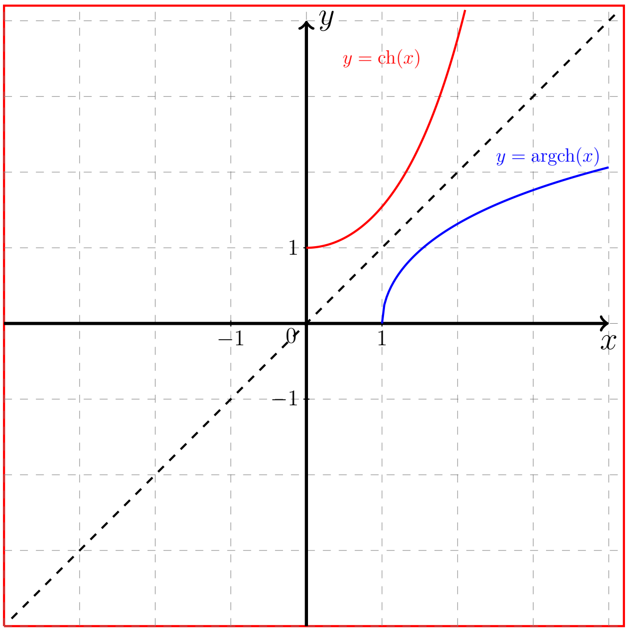


Figure 6.19 – *Graphical representation of the function argch*(x)

6.8.3 The inverse hyperbolic tangent function

From the table of variation of the function $\text{th}(x)$ above we have: $\text{th}(x)$ is continuous and strictly increasing on $\mathbb{R}$. So it makes is a bijection from $\mathbb{R}$ into $]-1{,}1[$.

Definition 6.15

The inverse function of the function $\text{th}(x)$ on $\mathbb{R}$ is denoted by $\text{argth}(x)$ or $\text{th}^{-1}(x)$

$$\begin{array}{rccc} \text{argth}: &]-1{,}1[& \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \text{argth}(x) \end{array}$$

Proposition 6.33

The function $\text{argth}(x)$ has the following properties:

1. The function $\text{argth}(x)$ is defined on $]-1{,}1[$, it is continuous and strictly increasing on $]-1{,}1[$.
2. $\forall x \in \mathbb{R}; \text{argth}(\text{th}(x)){=}x$.
3. $\forall y \in]-1{,}1[; \text{th}(\text{argth}(y)){=}y$.
4. $\forall x \in \mathbb{R}, \forall y \in]-1{,}1[; y = \text{th}(x) \iff x = \text{argth}(y)$.
5. The $\text{argth}(x)$ function is odd.

Proposition 6.34

The function $\text{argth}(x)$ is differentiable on $]-1{,}1[$ and verifies:

$$\forall x \in]-1{,}1[;\ (\text{argth}(x))' = \frac{1}{1-x^2}.$$

Proposition 6.35

$$\forall x \in]-1;1[;\ \operatorname{argth}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Proof

Let $x \in]-1;1[$, and $y = \operatorname{argth}(x)$.
We have:

$$\operatorname{th}(x) = \frac{e^{y}-e^{-y}}{e^{y}+e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}$$

$$\Longrightarrow\ e^{2y} = \frac{1+\operatorname{th}(y)}{1-\operatorname{th}(y)} = \frac{1+\operatorname{th}(\operatorname{argth}(x))}{1-\operatorname{th}(\operatorname{argth}(x))} = \frac{1+x}{1-x}$$

$$\Longleftrightarrow e^{2y} = \frac{1+x}{1-x} \Longleftrightarrow 2y = \ln\left(\frac{1+x}{1-x}\right) \Longleftrightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\Longrightarrow\ \forall x \in]-1,1[;\ \operatorname{argth}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

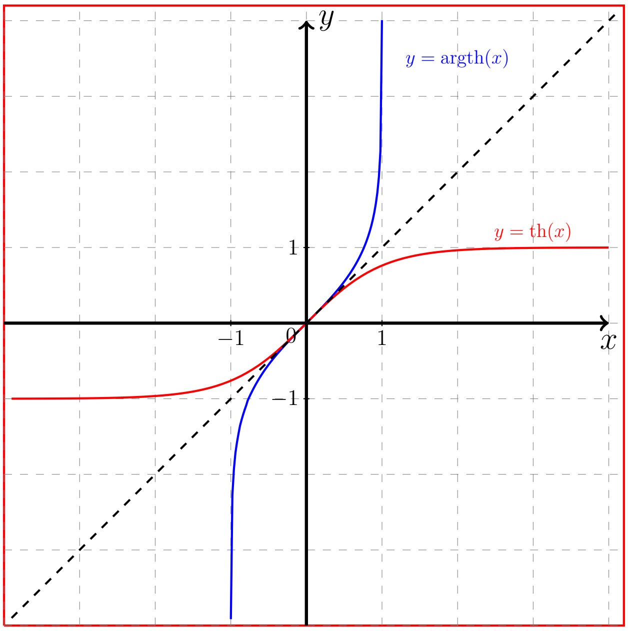


Figure 6.20 – *Graphical representation of the function argth*(x)

Chapter's exercises with answers

Exercice 1

Show that for all real numbers x and y :

$$e^{\frac{x+y}{2}} \leq \frac{e^x + e^y}{2}$$

Corrigé 1

Let $x, y \in \mathbb{R}$, we have:

$$\begin{aligned}\left(e^{\frac{x}{2}} - e^{\frac{x}{2}}\right)^2 \geq 0 \quad &\Rightarrow \quad e^x + e^y - 2e^{\frac{x+y}{2}} \geq 0 \\ &\Rightarrow \quad 2e^{\frac{x+y}{2}} \leq e^x + e^y \\ &\Rightarrow \quad e^{\frac{x+y}{2}} \leq \frac{e^x+e^y}{2}\end{aligned}$$

Exercice 2

Simplify the following expressions :

1. $\cos(\arcsin x)$
2. $\sin(\arccos x)$
3. $\tan(arcsinx)$
4. $cos(2\arctan x)$

Corrigé 2

We know that :

1. $\arcsin : [-1, 1] \to [\frac{-\pi}{2}, \frac{\pi}{2}]$
2. $\arccos : [-1, 1] \to [0, \pi]$
3. $\arctan : \mathbb{R} \to]\frac{-\pi}{2}, \frac{\pi}{2}[$

1. For the first expression we have:

$$\begin{aligned}\forall x \in [-1, 1] : \cos^2(\arcsin x) + \sin^2(\arcsin x) = 1 \quad &\Rightarrow \quad \cos^2(\arcsin x) = 1 - x^2 \\ &\Rightarrow \quad \cos(\arcsin x) = \pm\sqrt{1 - x^2}\end{aligned}$$

and since $\arcsin x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ hence: $\cos x \geq 0$ so

$$\cos(\arcsin x) = \sqrt{1 - x^2}$$

2. For the second expression we have:

$$\begin{aligned}\forall x \in [-1, 1] : \cos^2(\arccos x) + \sin^2(\arccos x) = 1 \quad &\Rightarrow \quad \sin^2(\arccos x) = 1 - x^2 \\ &\Rightarrow \quad \sin(\arccos x) = \pm\sqrt{1 - x^2}\end{aligned}$$

and since $\arccos x \in [0, \pi]$ hence: $\sin x \geq 0$ so

$$\sin(\arccos x) = \sqrt{1 - x^2}$$

3. Let $x \in]-1, 1[$:

$$\tan(\arcsin x) = \frac{\sin(\arcsin x)}{\cos(\arcsin x)}$$

From the previous question:

$$\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$$

4. It is known that $\forall\theta \in \mathbb{R} : \cos^2\theta = \dfrac{1}{1+\tan^2\theta}$

For $\theta = \arctan x$, we obtain $\forall x \in \mathbb{R} : \cos^2 \arctan x = \dfrac{1}{1+x^2} \Rightarrow \cos\arctan x = \dfrac{1}{\sqrt{1+x^2}}$.

Since $\arctan x \in]\dfrac{-\pi}{2}, \dfrac{\pi}{2}[$ and $\forall z \in]\dfrac{-\pi}{2}, \dfrac{\pi}{2}[: \cos z \geq 0$.

Now we use the formula $\forall\theta \in \mathbb{R} : \cos 2\theta = 2\cos^2\theta - 1$ which gives the required result.

$$\begin{aligned} \cos 2\theta = 2\cos^2\theta - 1 \quad &\Rightarrow \quad \text{For } \theta = \arctan(x) : \cos 2(\arctan x) = 2\cos^2(\arctan x) - 1 \\ &\Rightarrow \quad \cos 2(\arctan x) = \frac{2}{1+x^2} - 1 \\ &\Rightarrow \quad \cos 2(\arctan x) = \frac{1-x^2}{1+x^2} \end{aligned}$$

Exercice 3

According to the values of x, find the limits of x^n when $n \to +\infty$

Corrigé 3

For $x \in]0, +\infty[$, $x^n = e^{n\ln x}$ and since $\begin{cases} \ln x \geq 0 & \text{si} \quad x \geq 1 \\ \ln x < 0 & \text{si} \quad 0 < x < 1 \end{cases}$ This gives us three cases to look at:

1. Case where: $x = 1$.

$$\begin{aligned} x = 1 \quad &\Rightarrow \quad x^n = 1 \\ &= \quad \lim_{n\to+\infty} x^n = 1 \end{aligned}$$

2. Case where: $0 < x < 1$.

$$\begin{aligned} 0 < x < 1 \quad &\Rightarrow \quad \ln x < 0 \\ &= \quad \lim_{n\to+\infty} e^{n\ln x} = 0 \\ &\Rightarrow \quad \lim_{n\to+\infty} x^n = 0 \end{aligned}$$

3. Case where: $x > 1$.

$$\begin{aligned} x > 1 \quad &\Rightarrow \quad \ln x > 0 \\ &= \quad \lim_{n\to+\infty} e^{n\ln x} = +\infty \\ &\Rightarrow \quad \lim_{n\to+\infty} x^n = +\infty \end{aligned}$$

For $x \in]-\infty, 0[$ which gives the existence of $y \in]0, +\infty[$ tq: $x = -y$ hence $x^n = (-1)^n y^n$. According to the previous results we can say that:

1. Case: $x = -1$.

$$\begin{aligned} x = -1 \quad &\Rightarrow \quad v_n = x^n = (-1)^n \\ &\Rightarrow \quad v_n \text{ diverges} \end{aligned}$$

2. Case: $-1 < x < 0$.

$$\begin{array}{rcl} -1 < x < 0 & \Rightarrow & x^n = (-1)^n (y)^n \text{ with } y \in]0,1[\\ \text{And since } \lim\limits_{n \to +\infty} y^n = 0 & \Rightarrow & \lim\limits_{n \to +\infty} x^n = 0 \\ \text{as} & & \\ -1 \leq (-1)^n \leq 1 & \Rightarrow & -(y)^n \leq (-1)^n (y^n) \leq y^n \\ & \Rightarrow & \lim\limits_{n \to +\infty} -(y)^n \leq \lim\limits_{n \to +\infty} (-1)^n (y^n) \leq \lim\limits_{n \to +\infty} y^n \\ & \Rightarrow & 0 \leq \lim\limits_{n \to +\infty} (-1)^n (y^n) \leq 0 \\ & \Rightarrow & \lim\limits_{n \to +\infty} (-1)^n (y^n) = 0 \end{array}$$

3. Case where $x < -1 \Rightarrow v_n = x^n = (-1)^n (y)^n$ with $y > 1 \Rightarrow |v_n| = (y)^n$. Based on previous results $|v_n| \to +\infty \Rightarrow v_n$ diverges

Conclusion

Let $x \in \mathbb{R}$, then if:

1. $x \leq -1 \Rightarrow x^n$ diverges
2. $-1 < x < 1 \Rightarrow x^n \to 0$
3. $x = 1 \Rightarrow x^n \to 1$
4. $x > 1 \Rightarrow x^n$ diverges

Exercice 4

1. Show that for all $x \in]0, \frac{\pi}{2}[$:

   a) $\sin(x) = \dfrac{\tan(x)}{\sqrt{1+\tan^2(x)}}$

   b) $\cos(x) = \dfrac{1}{\sqrt{1+\tan^2(x)}}$

   c) $0 < \arctan(\frac{1}{2}) + \arctan(\frac{3}{2}) < \frac{\pi}{2}$

2. Solve $\arcsin(x) = \arctan(\frac{1}{2}) + \arctan(\frac{3}{2})$

Corrigé 4

Let $x \in]0, \frac{\pi}{2}[$, then:

a)

$$\frac{\tan(x)}{\sqrt{1+\tan^2(x)}} = \frac{\dfrac{\sin(x)}{\cos(x)}}{\sqrt{1+\dfrac{\sin^2(x)}{\cos^2(x)}}} = \frac{\dfrac{\sin(x)}{\cos(x)}}{\sqrt{\dfrac{1}{\cos^2(x)}}} = \frac{\dfrac{\sin(x)}{\cos(x)}}{\dfrac{1}{\cos(x)}} = \sin(x)$$

b)

$$\frac{1}{\sqrt{1+\tan^2(x)}} = \frac{1}{\sqrt{1+\dfrac{\sin^2(x)}{\cos^2(x)}}} = \frac{1}{\sqrt{\dfrac{1}{\cos^2(x)}}} = \frac{1}{\dfrac{1}{\cos(x)}} = \cos(x)$$

c) Let's put $\begin{cases} \arctan\left(\frac{1}{2}\right) & = \alpha \\ \arctan(\frac{3}{2}) & = \beta \end{cases} \Rightarrow \begin{cases} \frac{1}{2} & = \tan\alpha \\ \frac{3}{2} & = \tan\beta \end{cases}$. According to the previous question

$$\begin{aligned}
\sin(\alpha) = \frac{\tan(\alpha)}{\sqrt{1+\tan^2(\alpha)}} &\Rightarrow \sin(\alpha) = \frac{1}{\sqrt{5}} < \frac{1}{2} = \sin(\frac{\pi}{6}) \\
&\Rightarrow \arcsin(\sin(\alpha)) < \arcsin(\sin(\frac{\pi}{6})) \\
&\Rightarrow \alpha < \frac{\pi}{6}. \\
\sin(\beta) = \frac{\tan(\beta)}{\sqrt{1+\tan^2(\beta)}} &\Rightarrow \sin(\beta) = \frac{3}{\sqrt{5}} < \frac{\sqrt{3}}{2} = \sin(\frac{\pi}{3}) \\
&\Rightarrow \arcsin(\sin(\beta)) < \arcsin(\sin(\frac{\pi}{6})) \\
&\Rightarrow \beta < \frac{\pi}{3}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\cos(\alpha) = \frac{1}{\sqrt{1+\tan^2(\alpha)}} &\Rightarrow \cos(\alpha) = \frac{2}{\sqrt{5}}, \text{ as } \sin(\alpha) > 0 \Rightarrow \alpha > 0 \\
\cos(\beta) = \frac{1}{\sqrt{1+\tan^2(\beta)}} &\Rightarrow \cos(\beta) = \frac{2}{\sqrt{13}}, \text{ and since } \sin(\beta) > 0 \Rightarrow \beta > 0
\end{aligned}$$

As a result

$$\begin{cases} 0 < \alpha < \dfrac{\pi}{6} \\ 0 < \beta < \dfrac{\pi}{3} \end{cases} \Rightarrow 0 < \alpha+\beta < \frac{\pi}{2} \Rightarrow 0 < \arctan(\frac{1}{2}) + \arctan(\frac{3}{2}) < \frac{\pi}{2}$$

2 We know that: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, hence:

$$\begin{aligned}
\arcsin(x) = \arctan(^1/_2) + \arctan(^3/_2) &\Leftrightarrow \sin[\arcsin(x)] = \sin[\arctan(^1/_2) + \\
&+ \arctan(^3/_2)] \\
&\Leftrightarrow x = \sin(\arctan(\tfrac{1}{2}))\cos(\arctan(^3/_2)) \\
&+ \sin(\arctan(^3/_2))\cos(\arctan(^1/_2)) \\
&\Leftrightarrow x = \frac{^1/_2}{\sqrt{1+\frac{1}{4}}}\frac{1}{\sqrt{1+\frac{9}{4}}} + \frac{1}{\sqrt{1+\frac{1}{4}}}\frac{^3/_2}{\sqrt{1+\frac{9}{4}}} \\
&\Leftrightarrow x = \frac{1}{\sqrt{5}}\frac{2}{\sqrt{13}} + \frac{2}{\sqrt{5}}\frac{3}{\sqrt{13}} \\
&\Leftrightarrow x = \frac{8}{\sqrt{65}}
\end{aligned}$$

Exercice 5

Show the following assertions:

1. $\forall x \in \mathbb{R} : argsh(x) = \ln(x + \sqrt{x^2 + 1})$
2. $\forall x \in [1, +\infty[: argch(x) = \ln(x + \sqrt{x^2 - 1})$
3. $\forall x \in]-1, 1[: argth(x) = \frac{1}{2} \ln(\frac{1+x}{1-x})$

Corrigé 5

1.

$$\forall x \in \mathbb{R} : \sinh(x) = \frac{e^{2x} - 1}{2e^x} \Rightarrow \sinh\left(\ln(x + \sqrt{1+x^2})\right) = \frac{e^{2\ln(x+\sqrt{1+x^2})} - 1}{2e^{\ln(x+\sqrt{1+x^2})}}$$
$$\Rightarrow \sinh\left(\ln(x + \sqrt{1+x^2})\right) = \frac{e^{\ln(x+\sqrt{1+x^2})^2} - 1}{2e^{\ln(x+\sqrt{1+x^2})}}$$
$$\Rightarrow \sinh\left(\ln(x + \sqrt{1+x^2})\right) = \frac{(x+\sqrt{1+x^2})^2 - 1}{2(x+\sqrt{1+x^2})}$$
$$\Rightarrow \sinh\left(\ln(x + \sqrt{1+x^2})\right) = \frac{2x^2 + 2x\sqrt{1+x^2}}{2(x+\sqrt{1+x^2})}$$
$$\Rightarrow \sinh\left(\ln(x + \sqrt{1+x^2})\right) = x$$
$$\Rightarrow \text{argsinh}(\sinh\left(\ln(x + \sqrt{1+x^2})\right)) = \text{argsinh}(x)$$
$$\Rightarrow \ln(x + \sqrt{1+x^2}) = \text{argsinh}(x)$$

2.

$$\forall x \in [1, +\infty[: \cosh(x) = \frac{e^{2x} + 1}{2e^x} \Rightarrow \cosh\left(\ln(x + \sqrt{x^2-1})\right) = \frac{e^{2\ln(x+\sqrt{x^2-1})} + 1}{2e^{\ln(x+\sqrt{x^2-1})}}$$
$$\Rightarrow \cosh\left(\ln(x + \sqrt{x^2-1})\right) = \frac{e^{\ln(x+\sqrt{x^2-1})^2} + 1}{2e^{\ln(x+\sqrt{x^2-1})}}$$
$$\Rightarrow \cosh\left(\ln(x + \sqrt{x^2-1})\right) = \frac{(x+\sqrt{x^2-1})^2 + 1}{2(x+\sqrt{x^2-1})}$$
$$\Rightarrow \cosh\left(\ln(x + \sqrt{x^2-1})\right) = \frac{2x^2 + 2x\sqrt{x^2-1}}{2(x+\sqrt{x^2-1})}$$
$$\Rightarrow \cosh\left(\ln(x + \sqrt{x^2-1})\right) = x$$
$$\Rightarrow \text{argcosh}(\sinh\left(\ln(x + \sqrt{x^2-1})\right)) = \text{argcosh}(x)$$
$$\Rightarrow \ln(x + \sqrt{x^2-1}) = \text{argsinh}(x)$$

3.

$$\forall x \in]-1,1[: \tanh(x) = \frac{e^{2x}-1}{e^{2x}+1} \Rightarrow \tanh(\frac{1}{2}\ln\frac{1+x}{1-x}) = \frac{e^{2[\frac{1}{2}\ln\frac{1+x}{1-x}]}-1}{e^{2[\frac{1}{2}\ln\frac{1+x}{1-x}]}+1}$$

$$\Rightarrow \tanh(\frac{1}{2}\ln\frac{1+x}{1-x}) = \frac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1}$$

$$\Rightarrow \tanh(\frac{1}{2}\ln\frac{1+x}{1-x}) = \frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1}$$

$$\Rightarrow \tanh(\frac{1}{2}\ln\frac{1+x}{1-x}) = \frac{2x}{2}$$

$$\Rightarrow \tanh(\frac{1}{2}\ln\frac{1+x}{1-x}) = x$$

$$\Rightarrow \text{argtanh}(\tanh(\frac{1}{2}\ln\frac{1+x}{1-x})) = \text{argtanh}(x)$$

$$\Rightarrow \frac{1}{2}\ln\frac{1+x}{1-x} = \text{argtanh}(x)$$

Exercice 6

1. Compute: $\cosh(\frac{1}{2}\ln(3))$ et $\sinh(\frac{1}{2}\ln(3))$
2. Show that: $\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$
3. Deduce the solutions of the equation: $2\cosh(x) + \sinh(x) = \sqrt{3}\cosh(5x)$

Corrigé 6

1.

$$\cosh(\frac{1}{2}\ln(3)) = \cosh(\ln\sqrt{3}) = \frac{e^{2\ln\sqrt{3}}+1}{2e^{\ln\sqrt{3}}} = \frac{3+1}{2\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

et

$$\sinh(\frac{1}{2}\ln(3)) = \sinh(\ln\sqrt{3}) = \frac{e^{2\ln\sqrt{3}}-1}{2e^{\ln\sqrt{3}}} = \frac{3-1}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

2. Let $a, b \in \mathbb{R}$, then we have:

$$\begin{aligned}
\cosh a \cosh b + \sinh a \sinh b = & \quad (\frac{e^{2a}+1}{2e^a})(\frac{e^{2b}+1}{2e^b}) + (\frac{e^{2a}-1}{2e^a})(\frac{e^{2b}-1}{2e^b}) \\
= & \quad \frac{e^{2(a+b)} + e^{2a} + e^{2b} + 1 + e^{2(a+b)} - e^{2a} - e^{2b} + 1}{4e^{(a+b)}} \\
= & \quad \frac{2e^{2(a+b)} + 2}{4e^{(a+b)}} \\
= & \quad \frac{e^{2(a+b)} + 1}{2e^{(a+b)}} \\
= & \quad \cosh(a+b)
\end{aligned}$$

3.
$$\begin{aligned}
2\cosh(x) + \sinh(x) = \sqrt{3}\cosh(5x) \quad & \Leftrightarrow \quad \tfrac{2}{\sqrt{3}}\cosh(x) + \tfrac{1}{\sqrt{3}}\sinh(x) = \cosh(5x) \\
& \Leftrightarrow \quad \cosh(\frac{1}{2}\ln(3))\cosh(x) \\
& + \quad \sinh(\frac{1}{2}\ln(3))\sinh(x) = \cosh(5x) \\
& \Leftrightarrow \quad \cosh(\ln\sqrt{3} + x) = \cosh(5x) \\
& \Leftrightarrow \quad \begin{cases} \ln\sqrt{3} + x & = 5x \\ \text{ ou bien} & \\ \ln\sqrt{3} + x & = -5x \end{cases} \\
& \Leftrightarrow \quad \begin{cases} 4x & = \ln\sqrt{3} \\ \text{ou bien} & \\ 6x & = -\ln\sqrt{3} \end{cases} \\
& \Leftrightarrow \quad \begin{cases} x & = \dfrac{\ln\sqrt{3}}{4} \\ \text{ou bien} & \\ x & = \dfrac{-\ln\sqrt{3}}{6} \end{cases}
\end{aligned}$$

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