Course : Algebra 3
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Department of Computer Science

Chapter 4 : Vector spaces

1 Maps on vector spaces

Definition 1.1 *Let V be a vector space over a field K and let $f : V \times V \longrightarrow K$ be a function. Supppose that the following two conditions hold, for $\alpha, \beta \in K$.*

a. $f(\alpha x + \beta x^{'}, y) = \alpha f(x, y) + \beta f(x^{'}, y),\ x, x^{'}, y \in V.$

b. $f(x, \alpha y + \beta y^{'}) = \alpha f(x, y) + \beta f(x, y^{'}),\ x, y, y^{'} \in V.$

Then, f is called a bilinear map on V.

Example 1.1 *Consider the function $f : V \times V \longrightarrow K$ where*

$$f(x, y) = xAy^T, \tag{1}$$

with $V = \mathbb{R}^n$ and $K = \mathbb{R}$ and where A is an $n \times n$ matrix. Then , f represents a bilinear map.

Definition 1.2 *Let V be a vector space over a field K and let f be a bilinear map on V. Take that $g : V \longrightarrow K$ is a map, having*

$$g(x) = f(x, x). \tag{2}$$

Then, g is called a quadratic map on V.

Example 1.2 *Consider*

$$f(x, y) = a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2. \tag{3}$$

Then, we have

$$g(x) = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2, \tag{4}$$

this map represents a quadratic map.

Definition 1.3 *Suppose that the quadratic map* $g : V \longrightarrow K$ *satisfies, for* $x \neq 0$,

$$g(x) = f(x,x) \succ 0, \tag{5}$$

where f *is a bilinear map. Then,* g *and* f *are positive definite.*

Definition 1.4 *Let* V *be a vector space over* K *and let* S *be a subset of* V. *Suppose that* $x_1, x_2, \cdots, x_n$ *is a finite list of vectors with* $x_1, x_2, \cdots, x_n \in S$. *Then,* S *spans* V *iff* $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$, *for* $\alpha_1, \alpha_2, \cdots, \alpha_n \in K$.

Definition 1.5 *Let* V *be a vector space and let* $S \subseteq V$. *Assume that* S *spans* V *and* S *must be linearly independent. Then,* S *is called a basis of* V.

Definition 1.6 *Suppose that* V *is a vector space and that* $(x_1, x_2, \cdots, x_n)$ *is an ordered basis for* V. *Take that*

$$a_{ij} = f(x_i, x_j), \tag{6}$$

where f *is a bilinear map on* V. *Then,* $A = (a_{ij})$ *is said to be the matrix for* f *with respect to* $(x_1, x_2, \cdots, x_n)$.

2 Inner product spaces

Definition 2.1 *Suppose that* $Y = \mathbb{R}$ *or* $Y = \mathbb{C}$ *and that* V *is a vector space over* Y. *Take that* $<,>: V \times V \longrightarrow Y$ *is a function satisfies, for* $x, y, z \in V$,

1. $< x, x > \succcurlyeq 0$ *and* $< x, x >= 0 \Leftrightarrow x = 0$.

2. .

$< x, y >= \overline{< y, x >}$ *when* $Y = \mathbb{C}$.

$< x, y >=< y, x >$ *when* $Y = \mathbb{R}$.

3. $< \alpha x + \beta y, z >= \alpha < x, z > + \beta < y, z >$, *for* $\alpha, \beta \in Y$.

Then, the function $<,>: V \times V \longrightarrow Y$ *is called an inner product on V.*

Definition 2.2 *Suppose that V is a vector space over Y. Take that $<,>: V \times V \longrightarrow Y$ is an inner product on V.*

1. *When V is a real or complex vector space, V is said to be a real or complex inner product space.*

2. *When V is a real vector space, V is said to be a Euclidean space.*

3. *When V is a complex vector space, V is said to be a unitary space.*

Definition 2.3 *Let $d : X \times X \longrightarrow \mathbb{R}$ be a function where X is a nonempty set and assume that, for $x, y, z \in X$,*

1. $d(x, y) = 0$ *if and only if* $x = y$.

2. $0 \preccurlyeq d(x, y) \prec \infty$.

3. $d(x, y) = d(y, x)$.

4. $d(x, z) \preccurlyeq d(x, y) + d(y, z)$.

Then, $d(x, y)$ is said to be the distance from x to y or a metric on X.

Definition 2.4 *Suppose that X is a nonempty set and that $d : X \times X \longrightarrow \mathbb{R}$ is a metric on X. Then, X is said to be a metric space.*

Remark 2.1 .
For $x \in V$, the norm of x can be represented as

$$\| x \| = \sqrt{< x, x >}, \tag{7}$$

where V is an inner product space.

The polarization identities are presented in the following two theorems where V is a real or complex inner product space.

Theorem 2.1 *Let V be a real inner product space and let $x, y \in V$. Then, we have*

$$< x, y > = \frac{1}{4}(\| x + y \|^2 - \| x - y \|^2). \tag{8}$$

Theorem 2.2 *Let V be a complex inner product space and let $x, y \in V$. Then, we get*

$$< x, y > = \frac{1}{4}(\| x + y \|^2 - \| x - y \|^2) + \frac{1}{4}i(\| x + iy \|^2 - \| x - iy \|^2). \tag{9}$$

Definition 2.5 *Suppose that X is a metric space and that $x \in X$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of points in X. Then, we can say that $\{x_n\}_{n \in \mathbb{N}}$ converges to x when*

$$\lim_{n \longrightarrow \infty} d(x_n, x) = 0, \tag{10}$$

which means that for $\varepsilon \succ 0$ we find an integer $N \succ 0$ with $n \succcurlyeq N \Longrightarrow d(x_n, x) \prec \varepsilon$.

Definition 2.6 *Assume that X is a metric space and that $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of points in X. Then, $\{x_n\}_{n \in \mathbb{N}}$ is called a Cauchy sequence when we have that for $\varepsilon \succ 0$ we find an integer $N \succ 0$ with $m, n \succcurlyeq N \Longrightarrow d(x_m, x_n) \prec \varepsilon$.*

Theorem 2.3 *Let $x, y, z \in V$. Then, we have*

1. $\| x+y \| \preccurlyeq \| x \| + \| y \|$, *(The triangle inequality).*

2. $\| x+y \|^2 + \| x-y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$, *(The parallelogram law).*

3. $\| x-y \| \preccurlyeq \| x-z \| + \| z-y \|$.

4. $| < x, y > | \preccurlyeq \| x \| \| y \|$, *(The Cauchy-Schwarz inequality).*

5. $\| x \| \succcurlyeq 0$ *and* $\| x \| = 0$ *if and only if* $x = 0$.

Lemma 2.1 *Let X be a metric space and let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence in X. Then, $\{x_n\}_{n\in\mathbb{N}}$ is said to be a Cauchy sequence.*

Definition 2.7 *Suppose that X is a metric space and that x is an element of X. Take that each Cauchy sequence in X converges to x. Then, X is said to be complete.*

Remark 2.2 .
Let V be a real or complex vector space and let

$$\| x \| = \sqrt{< x, x >}. \tag{11}$$

Then, a complete metric space $(V, \| x-y \|)$ is said to be a Hilbert space.

3 Orthogonal sets

Definition 3.1 *Suppose that V is an inner product space and that x and y are vectors. Let $< x, y >= 0$ for $x, y \in V$. Then, the vectors x and y are called orthogonal and denoted by $x \perp y$.*

Definition 3.2 *Suppose that V is an inner product space and that A_1 and A_2 are subsets with $A_1, A_2 \subseteq V$. Let $x \perp y$ for every $x \in A_1$ and $y \in A_2$. Then, A_1 and A_2 are said to be orthogonal.*

Definition 3.3 *Let V be an inner product space and let A be a nonempty set of vectors where*

$$A = \{x_i \setminus i \in K\}. \tag{12}$$

1. *When we have $x_i \perp x_j$ for $i \neq j$, A is called orthogonal.*

2. *When we have*

$$< x_i, x_j >= \delta_{i,j}, \tag{13}$$

A is called orthonormal such that $\delta_{i,j}$ represents the Kronecker delta function with

$$\delta_{i,j} := \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

Theorem 3.1 *(Pythagoras) Let V be a real or complex inner product space and let $x \perp y$. Then, we have*

$$\parallel x+y \parallel^2 = \parallel x \parallel^2 + \parallel y \parallel^2 . \tag{14}$$

Theorem 3.2 *(Gram-Schmidt) Suppose that V is a real or complex inner product space and that $\{v_1, v_2, \cdots, v_n\}$ is a basis for V. Then, we say that $\{u_1, u_2, \cdots, u_n\}$ represents an orthogonal basis for V where*

$$u_1 = v_1, \tag{15}$$

and

$$u_j = v_j - \sum_{i=1}^{j-1} \frac{< v_j, u_i >}{< u_i, u_i >} u_i, j = 2, \cdots, n. \tag{16}$$

An orthonormal basis for V is given by

$$\{\frac{u_1}{\parallel u_1 \parallel}, \frac{u_2}{\parallel u_2 \parallel}, \cdots, \frac{u_n}{\parallel u_n \parallel}\}. \tag{17}$$

4 Orthogonal matrices and their properties

Definition 4.1 *Let A be an $n \times n$ matrix over $\mathbb{R}$ and let*

$$A^T A = A A^T = I_n. \tag{18}$$

Then, we say that A is an orthogonal matrix.

Theorem 4.1 *Let A be an orthogonal matrix. Then, we have*

a. *A^{-1} is an orthogonal matrix.*

b. *A^T is an orthogonal matrix.*

Theorem 4.2 *Let A and B be two matrices of order n such that A and B are orthogonal matrices. Then, the product AB represents an orthogonal matrix, on the other hand, the product BA is also orthogonal.*

Theorem 4.3 *Let A be an orthogonal matrix. Then, the determinant of A is equal to ± 1.*

Remark 4.1 .
The group which is denoted by $GL_n(\mathbb{R})$ is said to be the general linear group of degree n over $\mathbb{R}$ if $GL_n(\mathbb{R})$ is the group of $n \times n$ matrices that are real and nonsingular such that this group is the group under matrix multiplication. On he other hand, the general linear group of degree n over $\mathbb{C}$ is denoted by $GL_n(\mathbb{C})$.
The group which id denoted by $O_n(\mathbb{R})$ is said to be the orthogonal group if $O_n(\mathbb{R})$ is the group of $n \times n$ orthogonal matrices over $\mathbb{R}$ such that this group is the group under multiplication.

5 Unitary matrices and their properties

Definition 5.1 *Let A be an $n \times n$ matrix over $\mathbb{C}$ and let*

$$A^*A = AA^* = I_n. \tag{19}$$

Then, we say that A is a unitary matrix such that A^* is the conjugate transpose of A.

Theorem 5.1 *Let A be a unitary matrix. Then, we have that A^{-1} is a unitary matrix.*

Theorem 5.2 *Suppose that A and B are two matrices of the same order such that A and B are unitary. Then, AB is a unitary matrix.*

Remark 5.1 .
The multiplicative group of $n \times n$ unitary matrices over $\mathbb{C}$ is the so-called unitary group and is denoted by $U_n(\mathbb{C})$.

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