Correction n°3

Real Functions of One Real Variable

Solution 1

Evaluate the limits :

1. $\lim\limits_{x\to+\infty} \dfrac{\sqrt{1+x^2}-\sqrt{1+x}}{x^2}$. At $+\infty, x>0$ hence $|x|=x$

$$\frac{\sqrt{1+x^2}-\sqrt{1+x}}{x^2} = \frac{\sqrt{x^2\left(\frac{1}{x^2}+1\right)}-\sqrt{x^2\left(\frac{1}{x^2}+\frac{1}{x}\right)}}{x^2} = \frac{|x|\sqrt{\frac{1}{x^2}+1}-|x|\sqrt{\frac{1}{x^2}+\frac{1}{x}}}{x^2}$$
$$= \frac{x\sqrt{\frac{1}{x^2}+1}-x\sqrt{\frac{1}{x^2}+\frac{1}{x}}}{x^2} = \frac{\sqrt{\frac{1}{x^2}+1}-\sqrt{\frac{1}{x^2}+\frac{1}{x}}}{x}$$

The numerator goes to 1 and the denominator goes to $+\infty$, then the limit of quotient goes to 0.

2. $\lim\limits_{x\to-\infty} \dfrac{4x^2-sin(5x)}{x^2+7}$. We know that

$$-1 \leqslant \sin(5x) \leqslant 1$$
$$-1 \leqslant -\sin(5x) \leqslant 1$$
$$4x^2-1 \leqslant 4x^2-\sin(5x) \leqslant 4x^2+1$$
$$\frac{4x^2-1}{x^2+7} \leqslant \frac{4x^2-\sin(5x)}{x^2+7} \leqslant \frac{4x^2+1}{x^2+7} \qquad \left(\frac{1}{x^2+7}>0\right)$$

Since $\lim\limits_{x\to-\infty} \dfrac{4x^2-1}{x^2+7} = \lim\limits_{x\to-\infty} \dfrac{4x^2}{x^2} = 4$ and $\lim\limits_{x\to-\infty} \dfrac{4x^2+1}{x^2+7} = \lim\limits_{x\to-\infty} \dfrac{4x^2}{x^2} = 4$

By squeeze theorem : $\lim\limits_{x\to-\infty} \dfrac{4x^2-sin(5x)}{x^2+7} = 4$

Solution 2

1. Show that f has a continuous extension to $x=2$, where $f(x) = \dfrac{x^2-x-2}{x^2-4}$, $\qquad x\neq 2$

Here $f(2)$ has not been defined.

$$\lim_{x\to2}\frac{x^2-x-2}{x^2-4} = \lim_{x\to2}\frac{(x-2)(x+1)}{(x-2)(x+2)} = \lim_{x\to2}\frac{x+1}{x+2} = \frac{3}{4}$$

Thus, $\lim\limits_{x\to2} f(x)$ exists, therefore f has a removable discontinuity at $x_0=2$.
Hence, The continuous extension is

$$\tilde{f}(x) = \begin{cases} \dfrac{x^2-x-2}{x^2-4} & \text{for } x\neq 2 \\ \dfrac{3}{4} & \text{for } x=2 \end{cases}$$

2. Determine the value of a and b for which the function g is continuous at $x = 0$:

   (a) we have $g(0) = b$

   (b) Determie Left hand limit

$$\begin{aligned}\lim_{x\to 0^-}\frac{\sin((a+1)x)+\ln(x+1)}{x} &= \lim_{x\to 0^-}\frac{\sin((a+1)x)}{x}+\lim_{x\to 0^-}\frac{\ln(x+1)}{x}\\ &= \lim_{x\to 0^-}\frac{(a+1)\sin((a+1)x)}{(a+1)x}+\lim_{x\to 0^-}\frac{\ln(x+1)}{x}\\ &= (a+1)+1 = a+2.\end{aligned}$$

   (c) Determie Right hand limit

$$\begin{aligned}\lim_{x\to 0^+}\frac{\sqrt{x+x^2}-\sqrt{x}}{x\sqrt{x}} &= \lim_{x\to 0^+}\frac{\sqrt{x+x^2}-\sqrt{x}}{x\sqrt{x}}\\ &= \lim_{x\to 0^+}\frac{\sqrt{x}\sqrt{x+1}-\sqrt{x}}{x\sqrt{x}}\\ &= \lim_{x\to 0^+}\frac{\sqrt{x+1}-1}{x}\\ &= \lim_{x\to 0^+}\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)}\\ &= \lim_{x\to 0^+}\frac{x}{x(\sqrt{x+1}+1)}\\ &= \lim_{x\to 0^+}\frac{1}{\sqrt{x+1}+1} = \frac{1}{2}\end{aligned}$$

   From $(a), (b)$ and (c), g is continuous if $b = a + 2 = \frac{1}{2}$. Therefore $a = -\frac{3}{2}$ and $b = \frac{1}{2}$

<u>Solution</u> 3

1. Examine the differentiability :

   ♦• f is differentiable on ℝ-{0}, because it is a product and a composite of differentiable function on ℝ-{0}

   • Show that f is differentiable at 0,

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x^2\sin(\frac{1}{x})}{x} &= \lim_{x\to 0} x\sin(\frac{1}{x})\\ &= \lim_{t\to +\infty}\frac{1}{t}\sin(t)\\ &= 0\end{aligned}$$

   Hence f is differentiable at 0. Therefore, f is differentiable on ℝ

   ♦• g is differentiable at 0: we have $\frac{g(x)-g(0)}{x-0} = \frac{\ln(1+|x|)}{x}$ and we know that

$$\lim_{t\to 0}\frac{\ln(1+t)}{t} = 1$$

   For $x < 0$, since $t = -x$ we obtain

$$\lim_{x\to 0^-}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0^-}\frac{\ln(1-x)}{x} = -1$$

For $x > 0$,

$$\lim_{x\to 0^+} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = 1$$

The Right hand limit and Left hand limit are not equal. Thus, g is not differentiable at 0.

Solution 4

1. Let f be the function defined by : $\quad f(x) = 2x^2 - 16x + 1$

   (a) Find the extremum of f on $\left[0, 9\right]$
   First, we find all possible critical points. Since f is differentiable :

$$\begin{aligned} f'(x) = 0 &\iff 4x - 16 = 0 \\ &\implies x = 4 \end{aligned}$$

   for $x \in [0, 4[$, we have $f'(x) < 0$ and for $x \in]4, 9]$, we have $f'(x) > 0$ Then $f(4) = -31$ is the minimum value of f on $[0, 9]$.

   (b) Show that the equation $f(x) = 0$ has a unique solution α on $\left[0, 3\right]$.
   - f is continuous on $\mathbb{R}$ because it is a polynomial function, then f is continuous on $\left[0, 3\right]$.
   - From (a) f is strictly decreasing on $[0, 4[$, then on $[0, 3]$.
   - $f([0, 3]) = [f(3), f(0)] = [-29, 1]$

   Hence $0 \in f([0, 3])$. (ie. $f(0).f(3) < 0$)
   Therefore, by IVT the equation $f(x) = 0$ has unique solution α on $[0, 3]$.

2. The function g is continuous on $[-1; 0[\cup]0, 1]$ and differentiable on $[-1; 0[\cup]0, 1]$

   - Show that g is differentiable at 0

$$\lim_{x\to 0} \frac{g(x)-g(0)}{x} = \lim_{x\to 0} \frac{1-\cos(2\pi x)}{(2\pi x)^2} 4\pi^2 = 2\pi^2 \in \mathbb{R}$$

   Hence: g is differentiable at 0

   - we know that if g is differentiable at 0, then g is continuous at 0. Indeed,

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{1-\cos(2\pi x)}{(2\pi x)^2} 4\pi^2 x = 0 = g(0)$$

   It follows that g is continuous on $[-1; 1]$ and differentiable on $[-1; 1]$ and we have $g(-1) = g(1)$
   By Rolle's Theorem, there exist $c \in]-1, 1[$ such that: $g'(c) = 0$