Correction n°4

Elementary Function and Application

Solution 1

Let

$$f :]1, +\infty[\longrightarrow]-1, +\infty[$$
$$x \longmapsto f(x) = x\ln(x) - x$$

1) Show that f admits an inverse function f^{-1} :

   (a) The function f is continuous on $]1, +\infty[$ as the product and sum of two functions $(x \longmapsto x\ln(x)$ and $x \longmapsto -x)$ which are continuous on $]1, +\infty[$.

   (b) The function f is differentiable on $]1, +\infty[$ and $f'(x) = \ln(x)$
   It is clear that, for $x \in]1, +\infty[$, $\ln(x) > 0$. Then, f is strictly increasing on $]1, +\infty[$.

   It follows that f admits an inverse function f^{-1} defined on $]-1, +\infty[$, indeed

$$J = f(]1, +\infty[) = \left] \lim_{x \longrightarrow 1} f(x), \lim_{x \longrightarrow +\infty} f(x) \right[=]-1, +\infty[$$

2) Find $f^{-1}(0)$ and $\left(f^{-1}\right)'(0)$:
   ▷ Since f^{-1} is the inverse function of f, we have:

$$\begin{array}{l} 0 \in]-1, +\infty[\\ f^{-1}(0) = x \end{array} \quad \Leftrightarrow \quad \begin{array}{l} x \in]1, +\infty[\,, \\ f(x) = 0 \end{array}$$

   So, we solve for $x \in]1, +\infty[$, $f(x) = 0$

$$\begin{aligned} f(x) = 0 &\Longleftrightarrow x\ln(x) - x = 0 \\ &\Longleftrightarrow x = 0 \text{ ou } \ln(x) = 1 \\ &\Longleftrightarrow x = 0 \text{ ou } x = e \end{aligned}$$

   Thus, $f(x) = 0$ for $x = e$ ($x = 0$ refused because $0 \notin]1, +\infty[$). Hence, $f^{-1}(0) = e$

   ▷ Find $(f^{-1})'(0)$: We have $f(e) = 0$ and $f'(e) = 1 \neq 0$, therefore f^{-1} is differentiable in 0 and

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{\ln(e)} = 1$$

Solution 2

1) For all $x \in \mathbb{R}$:

$$\frac{1 - \tan^2(x)}{1 + \tan^2(x)} = \frac{1 - \dfrac{\sin^2(x)}{\cos^2(x)}}{1 + \dfrac{\sin^2(x)}{\cos^2(x)}} = \frac{\cos^2(x) - \sin^2(x)}{\cos^2(x) + \sin^2(x)} = \cos^2(x) - \sin^2(x) = \cos(2x)\,.$$

Because, $\begin{cases} \cos^2(x) + \sin^2(x) = 1 \\ \cos(2x) = \cos(x+x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x) \end{cases}$

2) Show that : $\arccos\left(\frac{4}{5}\right) = 2\arctan\left(\frac{1}{3}\right)$.
According to the question 1, we have

$$\cos\left(2\arctan\left(\frac{1}{3}\right)\right) = \frac{1-\tan^2(\arctan\left(\frac{1}{3}\right))}{1+\tan^2(\arctan\left(\frac{1}{3}\right))} = \frac{1-\left(\frac{1}{3}\right)^2}{1+\left(\frac{1}{3}\right)^2} = \frac{4}{5}$$

Since $0 < \frac{1}{3}$ and arctan is increasing, then

$$\begin{aligned} \arctan(0) < \arctan\left(\frac{1}{3}\right) < \frac{\pi}{2} &\Longrightarrow 0 < \arctan\left(\frac{1}{3}\right) < \frac{\pi}{2} \\ &\Longrightarrow 0 < 2\arctan\left(\frac{1}{3}\right) < \pi \end{aligned}$$

Thus,

$$\cos\left(2\arctan\left(\frac{1}{3}\right)\right) = \frac{4}{5} \Longrightarrow 2\arctan\left(\frac{1}{3}\right) = \arccos\left(\frac{4}{5}\right)$$

Solution 3

We have :

$$f(x) = \arcsin\left(\frac{1-x^2}{1+x^2}\right).$$

1) f is well defined if and only if, $-1 \leqslant \frac{1-x^2}{1+x^2} \leqslant 1$
note that:

$$\begin{aligned} 1 - \left(\frac{1-x^2}{1+x^2}\right)^2 &= \frac{(1+x^2)^2 - (1-x^2)^2}{(1+x^2)^2} \\ &= \frac{4x^2}{(1+x^2)^2} \geqslant 0 \end{aligned}$$

Thus,

$$\begin{aligned} \forall x \in \mathbb{R}, \quad \left(\frac{1-x^2}{1+x^2}\right)^2 \leqslant 1 &\Longrightarrow \left|\frac{1-x^2}{1+x^2}\right| \leqslant 1 \\ &\Longrightarrow -1 \leqslant \frac{1-x^2}{1+x^2} \leqslant 1 \end{aligned}$$

Hence, f is well defined and continuous on $\mathbb{R}$.

2) f is differentiable if and only if, $-1 < \frac{1-x^2}{1+x^2} < 1$
From previous question, we have

$$1 - \left(\frac{1-x^2}{1+x^2}\right)^2 = \frac{4x^2}{(1+x^2)^2} > 0, \qquad \text{for } x \in \mathbb{R}^*$$

Thus,

$$\forall x \in \mathbb{R}^*, -1 < \frac{1-x^2}{1+x^2} < 1$$

Hence f is differentiable on $\mathbb{R}^*$.
To simplify the calculation of the derivative, we put: $u(x) = \frac{1-x^2}{1+x^2}$, so $f(x) = \arcsin(u(x))$

we have

$$f'(x) = \frac{u'(x)}{\sqrt{1-(u(x))^2}}$$

On the other hand,

$$u'(x) = \frac{-4x}{(1+x^2)^2}$$

Thus,

$$f'(x) = \frac{-4x}{(1+x^2)^2} \cdot \frac{1}{\sqrt{1-(\frac{1-x^2}{1+x^2})^2}} = \frac{-4x}{(1+x^2)^2} \cdot \frac{1+x^2}{2\,|x|}$$

Then, $f'(x) = \dfrac{-2x}{|x|\,(1+x^2)}, \quad x \neq 0.$

3) Find $\lim_{x\to+\infty} f(x)$:we have

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \arcsin\left(u(x)\right)$$

such that $\lim_{x\to+\infty} u(x) = \lim_{x\to+\infty} \dfrac{1-x^2}{1+x^2} = -1$

Thus,

$$\lim_{x\to+\infty} f(x) = \arcsin(-1) = \frac{-\pi}{2}\,.$$

<u>Solution</u> 4

1) Find:

$$\cosh\left(\frac{1}{2}\ln(3)\right) \quad \text{and} \quad \sinh\left(\frac{1}{2}\ln(3)\right).$$

We have : $\cosh(x) = \frac{e^x+e^{-x}}{2}$, so

$$\cosh\left(\frac{1}{2}\ln(3)\right) = \frac{e^{\frac{1}{2}\ln(3)} + e^{-\frac{1}{2}\ln(3)}}{2} = \frac{\sqrt{3}+\frac{1}{\sqrt{3}}}{2} = \frac{2\sqrt{3}}{3}$$

and $\sinh x = \frac{e^x-e^{-x}}{2}$, hence

$$\sinh\left(\frac{1}{2}\ln(3)\right) = \frac{e^{\frac{1}{2}\ln(3)} - e^{-\frac{1}{2}\ln(3)}}{2} = \frac{\sqrt{3}-\frac{1}{\sqrt{3}}}{2} = \frac{\sqrt{3}}{3}.$$

2) Using the formula : $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y).$

1. Solve the equation: $2\cosh(x) + \sinh(x) = \sqrt{3}\cosh(5x)$

$$\begin{aligned} 2\cosh(x) + \sinh(x) = \sqrt{3}\cosh(5x) &\iff 2\frac{\sqrt{3}}{3}\cosh(x) + \frac{\sqrt{3}}{3}\sinh(x) = \sqrt{3}\frac{\sqrt{3}}{3}\cosh(5x) \\ &\iff \cosh\left(\frac{1}{2}\ln(3)\right)\cosh(x) + \sinh\left(\frac{1}{2}\ln(3)\right)\sinh(x) = \cosh(5x) \\ &\iff \cosh\left(\frac{1}{2}\ln(3) + x\right) = \cosh(5x) \end{aligned}$$

Therefore, since cosh is even function :

$$\begin{cases} \frac{1}{2}\ln(3) + x = 5x \\ \frac{1}{2}\ln(3) + x = -5x \end{cases} \iff \begin{cases} 4x = \frac{1}{2}\ln(3) \\ 6x = -\frac{1}{2}\ln(3) \end{cases} \iff \begin{cases} x = \frac{1}{8}\ln(3) \\ x = -\frac{1}{12}\ln(3) \end{cases}$$

2. Simplify the expression :
$$\cosh(2\,\text{arsinh}(x))\,.$$

$$\begin{aligned}\cosh(2\,\text{arsinh}(x)) &= \cosh(\text{arsinh}(x)+\text{arsinh}(x))\\ &= \cosh(\text{arsinh}(x))\,\cosh(\text{arsinh}(x))+\sinh(\text{arsinh}(x))\,\sinh(\text{arsinh}(x))\\ &= \cosh^2(\text{arsinh}(x))+\sinh^2(\text{arsinh}(x))\end{aligned}$$

We know that : $\forall x\in\mathbb{R},\;\sinh(\text{arsinh}(x))=x$ which implies that

$$\sinh^2(\text{arsinh}(x))=x^2\,\ldots\ldots\ldots(1)$$

Furthermore, we have $\cosh^2(u)-\sinh^2(u)=1\Longrightarrow\cosh^2(u)=1+\sinh^2(u)$ hence,

$$\cosh^2(\text{arsinh}(x))=1+\sinh^2(\text{arsinh}(x))=1+x^2\,\ldots\ldots\ldots(2)$$

From (1) and (2), we obtain : $\cosh(2\,\text{arsinh}(x))=1+2x^2\,.$

Solution 5

1. Prove that : $\text{artanh}(x)=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$ for $x\in\,]-1,1[$
   Let $y=\text{artanh}(x)$, then

$$\begin{aligned} y=\text{artanh}(x) &\Longrightarrow \tanh(y)=x\\ &\Longrightarrow \frac{e^y-e^{-y}}{e^y+e^{-y}}=x\\ &\Longrightarrow x(e^y+e^{-y})=e^y-e^{-y}\\ &\Longrightarrow x(e^{2y}+1)=e^{2y}-1\quad(\text{multiply both sides by } e^y)\\ &\Longrightarrow e^{2y}(1-x)=(1+x)\\ &\Longrightarrow e^{2y}=\frac{1+x}{1-x}\\ &\Longrightarrow y=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\end{aligned}$$

   Therefore, $\text{artanh}(x)=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$ for $x\in\,]-1,1[$

2. Solve the equation $\text{artanh}(x)=\ln(3)$

$$\begin{aligned}\text{artanh}(x)=\ln(3) &\Longrightarrow \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)=\ln(3)\\ &\Longrightarrow \ln\left(\frac{1+x}{1-x}\right)=\ln(9)\\ &\Longrightarrow \frac{1+x}{1-x}=9\\ &\Longrightarrow 10x=8\\ &\Longrightarrow x=\frac{4}{5}\end{aligned}$$

   Therefore, $\text{artanh}(\frac{4}{5})=\ln(3)$