Correction n°5

Limited Development

Solution 1

Use the Taylor's formula to give the limited development of order 4 at $x_0 = 0$ for $f(x) = \ln(1+x)$
By Taylor's formula, we have

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + o(x^4)$$

with,

$$\begin{aligned}
&f(0) = 0 \\
&f^{(1)}(x) = \frac{1}{1+x} \implies f^{(1)}(0) = 1 \\
&f^{(2)}(x) = \frac{-1}{(1+x)^2} \implies f^{(2)}(0) = -1 \\
&f^{(3)}(x) = \frac{2}{(1+x)^3} \implies f^{(3)}(0) = 2 \\
&f^{(4)}(x) = \frac{-6}{(1+x)^4} \implies f^{(4)}(0) = -6
\end{aligned}$$

Therefore, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$

Solution 2

The limited development for :

1. $g(x) = x^2 ln(x)$ of order 3 at $x_0 = 1$.
   We put $t = x - 1$, then $t \to 0$ when $x \to 1$. Thus, $x = t + 1$

   $$g(x) = x^2 ln(x) = (t+1)^2 \ln(t+1) = (1 + 2t + t^2)\ln(t+1)$$

   The limited development for $1 + 2t + t^2$ of order 3 is $1 + 2t + t^2$.
   The limited development for $\ln(t+1)$ of order 3 is $t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3)$

   $$\begin{aligned}
   g(x) &= (1 + 2t + t^2)(t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3)) \\
   &= t + \left(-\frac{1}{2} + 2\right)t^2 + \left(\frac{1}{3} - 1 + 1\right)t^3 + o(t^3) \\
   &= t + \frac{3}{2}t^2 + \frac{1}{3}t^3 + o(t^3) \\
   &= x - 1 + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + o((x-1)^3)
   \end{aligned}$$

2. $f(x) = ln(\cosh(x))$ of order 4 at $x_0 = 0$

Since the limited development for $\cosh(x)$ of order 4 is $1 + \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$

Then,

$$\begin{aligned} f(x) = ln(\cosh(x)) &= \ln(1 + \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)) \\ &= \ln(1+Y), \quad \text{with } Y = \frac{x^2}{2} + \frac{x^4}{24} + o(x^4) \\ &= Y - \frac{Y^2}{2} + o(Y^2) \\ &= \frac{x^2}{2} + \frac{x^4}{24} - \frac{1}{2}\left(\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)^2 + o(x^4) \\ &= \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} + o(x^4) \\ &= \frac{x^2}{2} - \frac{x^4}{12} + o(x^4) \end{aligned}$$

3. $h(x) = \dfrac{ln(1+x)}{\sin(x)}$ of order 3 at $x_0 = 0$

We have : $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$ and $\sin(x) = x - \frac{x^3}{6} + (x^4)$

$$\frac{ln(1+x)}{\sin(x)} = \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)}{x - \frac{x^3}{6} + o(x^4)} = \frac{x\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + o(x^3)\right)}{x\left(1 - \frac{x^2}{6} + o(x^3)\right)} = \frac{1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + o(x^3)}{1 - \frac{x^2}{6} + o(x^3)}$$

Apply the division according to the increasing degrees,

$$\begin{array}{r|l}
1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + o(x^3) & 1 - \frac{x^2}{6} + o(x^3) \\ \cline{2-2}
1 \qquad - \frac{x^2}{6} \qquad + o(x^3) & 1 - \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{3} \\ \hline
- \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{4} + o(x^3) & \\
- \frac{x}{2} \qquad + \frac{x^3}{12} + o(x^3) & \\ \hline
\frac{x^2}{2} - \frac{x^3}{3} + o(x^3) & \\
\frac{x^2}{2} \qquad + o(x^3) & \\ \hline
- \frac{x^3}{3} + o(x^3) & \\
- \frac{x^3}{3} + o(x^3) & \\ \hline
o(x^3) &
\end{array}$$

Therefore,

$$\frac{ln(1+x)}{\sin(x)} = 1 - \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{3} + o(x^3)$$

Solution 3

Compute the following limits using limited development:

1. $\lim_{x\to 0} \dfrac{\sinh(x)}{\sin(x)}$

   we have $\sin(x) = x + o(x)$ and $\sinh(x) = x + o(x)$

   Then,

   $$\frac{\sinh(x)}{\sin(x)} = \frac{x+o(x)}{x+o(x)} = \frac{x(1+o(1))}{x(1+o(1))} = \frac{1+o(1)}{1+o(1)}$$

   Hence, $\lim\limits_{x\to 0} \dfrac{\sinh(x)}{\sin(x)} = \lim\limits_{x\to 0} \dfrac{1+o(1)}{1+o(1)} = 1$

2. $\lim_{x\to 0} \dfrac{\cos(x)\sqrt{1+x}-1}{x}$

   We have $\cos(x) = 1 - \dfrac{x^2}{2} + o(x^2)$ and $\sqrt{1+x} = 1 + \dfrac{x}{2} - \dfrac{x^2}{8} + o(x^2)$

   Thus,

   $$\begin{aligned}\cos(x)\sqrt{1+x} &= \left(1 - \frac{x^2}{2} + o(x^2)\right)\left(1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)\right)\\ &= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^2}{2} + o(x^2)\\ &= 1 + \frac{x}{2} - \frac{5}{8}x^2 + o(x^2)\end{aligned}$$

   $$\begin{aligned}\lim_{x\to 0} \frac{\cos(x)\sqrt{1+x}-1}{x} &= \lim_{x\to 0} \frac{1 + \frac{x}{2} - \frac{5}{8}x^2 + o(x^2) - 1}{x}\\ &= \lim_{x\to 0} \frac{\frac{x}{2} - \frac{5}{8}x^2 + o(x^2)}{x}\\ &= \lim_{x\to 0} \left(\frac{1}{2} - \frac{5}{8}x + o(x)\right)\\ &= \frac{1}{2}\end{aligned}$$