Correction n°1

Sets, Relations and Functions

Solution 1

1/ 1. By Roster method :

$$A = \left\{x \in \mathbb{Z},\ |x-1| < \frac{3}{2}\right\} = \left\{x \in \mathbb{Z},\ -\frac{3}{2} < x-1 < \frac{3}{2}\right\} = \left\{x \in \mathbb{Z},\ -\frac{1}{2} < x < \frac{5}{2}\right\} = \{0,1,2\}$$

$$C = \left\{x \in \mathbb{N},\ \frac{2x+3}{2} \leqslant 4\right\} = \{x \in \mathbb{N},\ 2x+3 \leqslant 8\} = \left\{x \in \mathbb{N},\ x \leqslant \frac{5}{2}\right\} = \{0,1,2\}$$

2. The relations of equality or subsets existing between these sets:

$$A = C, \quad A \subset D, \quad C \subset D, \quad B \subset E.$$

3. The cardinal of each of these sets:

$\mathrm{card}(A) = 3, \quad \mathrm{card}(B) = 2,$

$\mathrm{card}(A \times B) = \mathrm{card}(A) \times \mathrm{card}(B) = 3 \times 2 = 6, \quad \mathrm{card}(\mathcal{P}(B)) = 2^{\mathrm{card}(B)} = 2^2 = 4.$

4. $A \cap B = \varnothing$, $A \cup B = \{0,1,2,3,4\}$, $C \setminus E = \{0\}$, $\complement_D(A) = \{5\}$.
$A \times B = \{(0,3),(0,4),(1,3),(1,4),(2,3),(2,4)\}$
$\mathcal{P}(B) = \{\varnothing, \{3\}, \{4\}, \{3,4\}\}$.

2/ The complement in $\mathbb{R}$:

$$\complement_{\mathbb{R}}(A) = [1,2], \quad \complement_{\mathbb{R}}(B) = [1,+\infty[, \quad \complement_{\mathbb{R}}(C) =]-\infty, 2] \quad \complement_{\mathbb{R}}(B) \cap \complement_{\mathbb{R}}(C) = [1,2].$$

We conclude that, $\complement_{\mathbb{R}}(B) \cap \complement_{\mathbb{R}}(C) = \complement_{\mathbb{R}}(A)$.

Solution 2

Let $A, B, C \in \mathcal{P}(E)$, and $f : E \to F$ be a function,
1) Prove that $A \subseteq B \implies f(A) \subseteq f(B)$
Assume that $A \subseteq B$ and show that $f(A) \subseteq f(B)$. $\Big(y \in f(A) \iff \exists x \in A,\ y = f(x)\Big)$
Let $y \in F$,

$$\begin{aligned} y \in f(A) &\iff \exists x \in A,\ y = f(x) \\ &\implies \exists x \in B,\ y = f(x) \quad (\text{ because } A \subseteq B) \\ &\implies y \in f(B) \end{aligned}$$

Therefore, $f(A) \subseteq f(B)$

2) Prove that $\begin{cases} A \subseteq B \\ \wedge \\ B \cap C = \varnothing \end{cases} \implies A \cap C = \varnothing$

By contradiction, assume that $A \subseteq B \wedge B \cap C = \varnothing$ and $A \cap C \neq \varnothing$

Since $A \cap C \neq \varnothing$, let $x \in A \cap C$. Then

$$\begin{aligned} x \in A \cap C &\Longrightarrow x \in A \wedge x \in C \\ &\Longrightarrow x \in B \wedge x \in C \qquad (A \subseteq B) \\ &\Longrightarrow x \in B \cap C \\ &\Longrightarrow B \cap C \neq \varnothing \qquad (\text{Contradicion } B \cap C = \varnothing) \end{aligned}$$

Hence, $A \subseteq B \wedge A \cap B = \varnothing \implies A \cap C = \varnothing$.

Solution 3

Let $\mathcal{R}$ be the relation defined on $\mathbb{Z}$ by : $\forall n, m \in \mathbb{Z},\ n\mathcal{R}m \Longleftrightarrow \exists k \in \mathbb{Z}, n - m = 3k$

a) ($\mathcal{R}$ is reflexive)$\Longleftrightarrow (\forall n \in \mathbb{Z},\ n\mathcal{R}n)$

Let $n \in \mathbb{Z}$,

$$n - n = 3k \Longrightarrow k = 0 \in \mathbb{Z} \implies n\mathcal{R}n\,.$$

So $\mathcal{R}$ is reflexive.

b) ($\mathcal{R}$ is symmetric) $\Longleftrightarrow (\forall n, m \in \mathbb{Z}, n\mathcal{R}m \Longrightarrow m\mathcal{R}n)$

Let $n, m \in \mathbb{Z}$,

$$\begin{aligned} n\mathcal{R}m &\Longrightarrow \exists k \in \mathbb{Z}, n - m = 3k \\ &\Longrightarrow \exists k \in \mathbb{Z}, m - n = 3(-k) \\ &\Longrightarrow \exists k' = -k \in \mathbb{Z}, m - n = 3k' \\ &\Longrightarrow m\mathcal{R}n\,. \end{aligned}$$

Thus, $\mathcal{R}$ is symmetric.

c) ($\mathcal{R}$ is antisymmetric) $\Longleftrightarrow (\forall n, m \in \mathbb{Z}, n\mathcal{R}m \wedge m\mathcal{R}n \Longrightarrow n = m)$

$\mathcal{R}$ is not antisymmetric, because $\exists n = 6 \in \mathbb{Z},\ \exists m = 3 \in \mathbb{Z},\ (6\mathcal{R}3 \wedge 3\mathcal{R}6) \wedge (6 \neq 3)$.

d) ($\mathcal{R}$ is transitive) $\Longleftrightarrow (\forall n, m, w \in \mathbb{Z}, n\mathcal{R}m \wedge m\mathcal{R}w \Longrightarrow n\mathcal{R}w)$

Let $n, m, w \in \mathbb{Z}$,

$$\begin{cases} n\mathcal{R}m \Longrightarrow \exists k \in \mathbb{Z}\,,\ n - m = 3k \ldots\ldots (3) \\ \wedge \\ m\mathcal{R}w \Longrightarrow \exists k' \in \mathbb{Z}\,,\ m - w = 3k' \ldots\ldots (4) \end{cases}$$

From (3) et (4) we obtain : $\ n - w = 3(k' + k) \Longrightarrow \exists k'' = k + k' \in \mathbb{Z},\ n - w = 3k'' \Longrightarrow n\mathcal{R}w\,.$

Therefore, $\mathcal{R}$ is transitive.

Conclusion: Since $\mathcal{R}$ is reflexive, symmetric and tranditive, Then $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

- Find the equivalence class $\mathcal{C}(2)$:

$$\begin{aligned} \mathcal{C}(2) &= \{m \in \mathbb{Z}, \quad m\mathcal{R}2\} \\ &= \{m \in \mathbb{Z}, \quad \exists k \in \mathbb{Z}, m - 2 = 3k\} \\ &= \{m \in \mathbb{Z}, \quad \exists k \in \mathbb{Z}, m = 3k + 2\} \\ &= \big\{3k + 2,\ k \in \mathbb{Z}\big\} \end{aligned}$$

- Since $5\mathcal{R}2$, then $\mathcal{C}(5) = \mathcal{C}(2)$

Solution 4

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^2 - 4x + 5$

1/ Find $f^{-1}(\{5\})$:

$$\begin{aligned} f^{-1}(\{5\}) &= \big\{x \in \mathbb{R},\ f(x) \in \{5\}\,\big\} \\ &= \big\{x \in \mathbb{R},\ f(x) = 5\big\} \\ &= \big\{x \in \mathbb{R},\ x(x-4) = 0\big\} \\ &= \big\{x \in \mathbb{R},\ x = 0 \vee x = 4\big\} \\ &= \big\{0, 4\big\} \end{aligned}$$

2/ f is not injective because $\exists x_1 = 0 \in \mathbb{R},\ \exists x_2 = 4 \in \mathbb{R},\ (f(0) = f(4) = 5) \wedge (0 \neq 4)$

3/ Proving that $\forall x \in \mathbb{R},\ f(x) \geqslant 1$

Let $x \in \mathbb{R}$,

$$\begin{aligned} f(x) &= x^2 - 4x + 5 \\ &= (x-2)^2 + 1 \end{aligned}$$

Since $(x-2)^2 \geqslant 0$, then $(x-2)^2 + 1 \geqslant 1$.
Therefore, $\forall x \in \mathbb{R},\ f(x) \geqslant 1$

4/ f is not surjective because, $\exists y = 0 \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) \neq 0$

5/ Let $g : \big]-\infty, 2\big] \to \big[1, +\infty\big[$ be a function defined by $g(x) = f(x) = x^2 - 4x + 5$

- Proving that g is bijective : $\Big(\forall y \in \big[1, +\infty\big[,\ \exists! x \in \big]-\infty, 2\big],\ \ y = g(x)\Big)$

Let $y \in \big[1, +\infty\big[$,

$$\begin{aligned} y = g(x) &\iff y = x^2 - 4x + 5 \\ &\iff y = (x-2)^2 + 1 \\ &\iff (x-2)^2 = y - 1 \\ &\implies \sqrt{(x-2)^2} = \sqrt{y-1} \qquad \Big(\text{since } y \in \big[1, +\infty\big[,\ \sqrt{y-1} \text{ is well-defined}\Big) \\ &\implies |x-2| = \sqrt{y-1} \\ &\implies x - 2 = -\sqrt{y-1} \qquad \Big(\text{ for } x \in \big]-\infty, 2\big],\ |x-2| = -(x-2)\Big) \\ &\implies x = 2 - \sqrt{y-1} \end{aligned}$$

Therefore, g is bijective $\forall y \in \big[1, +\infty\big[,\ \exists! x = 2 - \sqrt{y-1} \in \big]-\infty, 2\big],\ \ y = g(x)$.

- Find g^{-1}

$$\begin{aligned} g^{-1} : \big[1, +\infty\big[&\longrightarrow \big]-\infty, 2\big] \\ x &\longmapsto 2 - \sqrt{x-1}. \end{aligned}$$