University of Batna 2

Institute of Industrial Hygiene and Safety

Module : Math 1 (L1)

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Chapter 4 : Elementary Functions and Applications

Introduction

This chapter is devoted to the elementary functions which appear naturally in the resolution of simple problems, especially physics issues. In this regard, we present the foundations of these functions, and some of their properties.

Contents

The Inverse Function Theorem

Theorem 1. *Let f be a continuous function and strictly monotonic on an interval I. Then*

1. *$f(I)$ is an interval denoted J of the same nature as I (closed, open or semi-open) and its ends are the limits of f at the endpoints of I.*

2. *The function f admits an inverse function defined on $J = f(I)$; more precisely, f defines a bijection of the interval I onto the interval J, so there exists a function denoted f^{-1} from J into I, such that*

$$x \in I,\ y = f(x) \quad \Leftrightarrow \quad y \in J,\ x = f^{-1}(y)$$

3. *The inverse function f^{-1} is continuous and strictly monotonic on J, with the same sense of monotonicity as f.*

4. *If f is differentiable at x_0 in I and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{-1}(y_0)\right)}$$

5. *The graphs of the functions f and f^{-1} are symmetric with respect to the first bisector $y = x$.*

Example :

We consider the real function f defined on $I = \left]0, \sqrt{2}\right[$ by $x \mapsto f(x) = \dfrac{x^3}{4 - x^4}$

The function f is the quotient of the two functions g and h defined on I by

$$x \mapsto g(x) = x^3, \qquad x \mapsto h(x) = 4 - x^4.$$

► The functions g and h are continuous on I and h is not zero on I, we deduce that f is continuous on I.

► On I, the function g is strictly increasing and positive, the function h is strictly decreasing and positive, then the function $\frac{1}{h}$ is strictly increasing and positive. Therefore f is strictly increasing as the product of two strictly positive increasing functions.

As the function f satisfies the assumptions of the inverse function theorem, we have the following results :

1. $f\left(\left]0, \sqrt{2}\right[\right) = \left]\lim_{x \to 0} f(x), \lim_{x \to \sqrt{2}} f(x)\right[$ d'où $f\left(\left]0, \sqrt{2}\right[\right) = \left]0, +\infty\right[$.

2. The function f admits an inverse function, denoted f^{-1}, defined on $J = \left]0, +\infty\right[$.

- The function f^{-1} is continuous on $J = \left]0, +\infty\right[$.

- The function f^{-1} is strictly increasing on J
- $f^{-1}\left(]0,+\infty[\right) = \left]0,\sqrt{2}\right[$ with $\lim\limits_{x\to 0} f^{-1}(x) = 0$ and $\lim\limits_{x\to+\infty} f^{-1}(x) = \sqrt{2}$.

Furthermore, the functions g and h being differentiable on I, f is differentiable as the quotient of two differentiable functions, the denominator function is not zero on I. Its derivative, after calculations, is the function $f' : x \mapsto \dfrac{x^2(12+x^4)}{(4-x^2)^2}$.
Then, the function f^{-1} is differentiable at any point image of a x such that $f'(x) \neq 0$.
Hence f^{-1} is differentiable on $]0,+\infty[$ and $\forall a \in]0,+\infty[, (f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$.
For example, f is defined at 1 and $f(1) = \dfrac{1}{3}$, so f^{-1} is differentiable at $\dfrac{1}{3}$ and

$$(f^{-1})'(\frac{1}{3}) = \frac{1}{f'(1)} = \frac{9}{13}$$

Parity :

> Let I be an interval of $\mathbb{R}$ symmetric with respect to 0 (i.e. of the form $]-a,a[$ or $[-a,a]$ or $\mathbb{R}$).
> Let $f : I \to \mathbb{R}$ be a function defined on I. We say that :
>
> - f is even if $\forall x \in I, \quad f(-x) = f(x)$.
> - f is odd if $\forall x \in I, \quad f(-x) = -f(x)$.

Example :

1)The function f is defined on $\mathbb{R}$ by $f(x) = 4x^2 + 5$ is even, indeed
For all real x , the real $-x$ also belongs to $\mathbb{R}$ and

$$f(-x) = 4(-x)^2 + 5 = 4x^2 + 5 = f(x)$$

2) The function g is defined on $\mathbb{R}^*$ by $g(x) = \dfrac{4}{x}$ is odd, indeed
For all real x , the real $-x$ also belongs to $\mathbb{R}^*$ and

$$g(-x) = \frac{4}{-x} = -\frac{4}{x} = -g(x)$$

Periodicity :

> Let $f : I \to \mathbb{R}$ be a function and T be a real number, $T > 0$.
> The function f is said to be periodic with period T if $\forall x \in \mathbb{R}, f(x+T) = f(x)$.

Example :

The cosine and sine functions are periodic with period 2π. In other words,

$$\forall x \in \mathbb{R}, \forall k \in \mathbb{Z}, \qquad cos(x+2k\pi) = cos(x) \text{ and } sin(x+2k\pi) = sin(x).$$

1 Exponential and Logarithmic Functions

Exponential and logarithmic functions are useful for modeling many phenomena. Logarithmic functions are particularly involved in evaluating decisions in the presence of risks.

1.1 Exponential function

Definition 1.1

We call exponential function the unique differentiable function on $\mathbb{R}$ such that $f' = f$ and $f(0) = 1$. We denote this function by

$$\exp : \mathbb{R} \longrightarrow]0, +\infty[$$
$$x \longmapsto \exp(x)$$

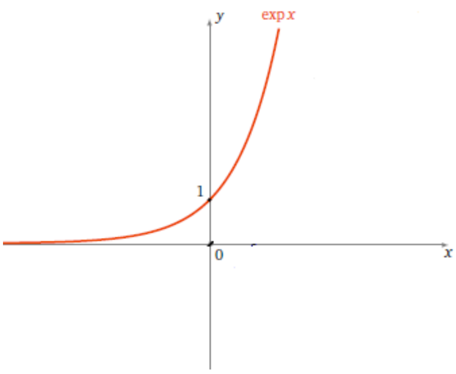
Remark 1. *For $x \in \mathbb{R}$ we also write e^x for* $\exp(x)$.

Operating properties : $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{Z}$:

$$e^{x+y} = e^x e^y, \qquad e^{x-y} = \frac{e^x}{e^y}, \qquad e^{xn} = (e^x)^n.$$

The exponential function satisfies the following properties :

- The exponential function is continuous and differentiable on $\mathbb{R}$ and $(e^x)' = e^x$.
- The exponential function is strictly increasing on $\mathbb{R}$.
- $\lim\limits_{x \longrightarrow -\infty} e^x = 0$ and $\lim\limits_{x \longrightarrow +\infty} e^x = +\infty$
- $\forall x \in \mathbb{R}$, $e^x > 0$.



Graph of the function exp

1.2 Logarithm function

Definition 1.2

The function exp *performs a bijection of $\mathbb{R}$ on $]0, +\infty[$. It is called a logarithm function* ln, *and defined as*

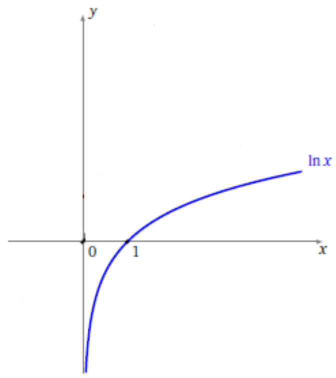
$$\ln :]0, +\infty[\longrightarrow \mathbb{R}$$
$$x \longmapsto \ln(x)$$

Operating properties : $\forall x, y > 0, \forall n \geqslant 1$:

$$\ln(xy) = \ln(x) + \ln(y), \qquad \ln(\frac{x}{y}) = \ln(x) - \ln(y), \qquad \ln(x^n) = n\ln(x).$$

The logarithm function checks the following properties :

- The function ln is continuous and differentiable on $]0, +\infty[$ and for all $x > 0, (\ln)'(x) = \frac{1}{x}$.
- The function ln is strictly increasing on $]0, +\infty[$.
- $\lim_{x \longrightarrow 0^+} \ln(x) = -\infty$ and $\lim_{x \longrightarrow +\infty} \ln(x) = +\infty$.



Graph of the function ln

Remark : As exp is the inverse of ln. Likewise ln is the inverse of exp, we have:

$$\forall x \in \mathbb{R}, \ \ln(\exp(x)) = x$$

$$\forall x \in \,]0; +\infty[\, , \ \exp(\ln(x)) = x$$

1.3 Exponentials and Logarithms of base a

The exponential function of base a is a bijection of $\mathbb{R}$ on $\mathbb{R}_+^*$. Its inverse is given a bijection of $\mathbb{R}_+^*$ on $\mathbb{R}$. It is called logarithm of base a.

Exponential of base a

For any positive number $a > 0$, and $a \neq 1$. An exponential function of base a is defined as

$$f: \begin{array}{l} \mathbb{R} \to \mathbb{R}_+^* \\ x \longmapsto a^x = e^{x\ln(a)} \end{array}$$

The function $x \longmapsto a^x$ defined on $\mathbb{R}$ to $\mathbb{R}_+^*$, continuous, differentiable and satisfies:

1. $(a^x)' = a^x \ln(a)$ for all $x \in \mathbb{R}$
2. For $a > 1$, the function $x \longmapsto a^x$ is strictly increasing and $\lim_{x \longrightarrow +\infty} a^x = +\infty$.
3. For $0 < a < 1$, the function $x \longmapsto a^x$ is strictly decreasing and $\lim_{x \longrightarrow +\infty} a^x = 0$.

Logarithm of base a

For $a > 0$, and $a \neq 1$. A logarithm function of base a is defined as

$$\log_a : \mathbb{R}_+^* \to \mathbb{R}$$
$$x \longmapsto \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

The function $x \longmapsto \log a(x)$ defined on $\mathbb{R}_+^*$ to $\mathbb{R}$, continuous, differentiable and satisfies:

1. $(\log_a)'(x) = \dfrac{1}{x\ln(a)}$ for all $x > 0$.
2. For $a > 1$, the function $x \longmapsto \log_a(x)$ is strictly increasing.
3. For $0 < a < 1$, the function $x \longmapsto \log_a(x)$ is strictly decreasing.

1.4 Power Function

Definition 1.3

For all $\alpha \in \mathbb{R}$, We call power function any function defined on $\mathbb{R}_+^*$ in $\mathbb{R}_+^*$ by

$$x \longmapsto x^\alpha = \exp(\alpha \ln(x)).$$

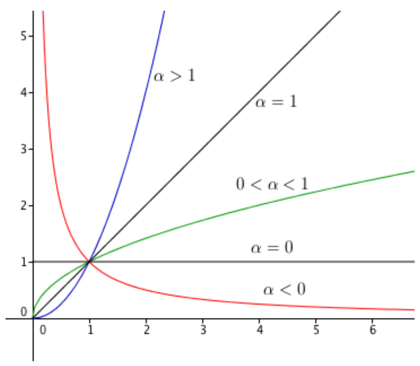
Operating properties :

For a and b two reals, $x > 0$ and $y > 0$, we have :

$$x^a y^a = (xy)^a \qquad x^a x^b = x^{a+b}$$
$$\left(x^a\right)^b = x^{ab} \qquad x^0 = 1$$

The power function checks the following properties :

- The function $x \longmapsto x^\alpha$ is continuous and differentiable on $\mathbb{R}_+^*$ and for all $x > 0, (x^\alpha)' = \alpha x^{\alpha-1}$
- The function $x \longmapsto x^\alpha$ is increasing if $\alpha > 0$, decreasing if $\alpha < 0$, constant if $\alpha = 0$.

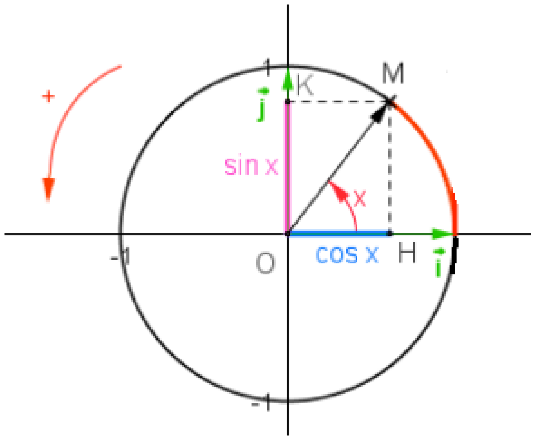


Graph of function x^α

2 Trigonometric Functions and their Inverses

In the plane provided with an orthonormal coordinate system $(O,\vec{i},\vec{j})$ and oriented in the direct direction, we consider a trigonometric circle with center O.

For any real number x, let us designate by the point M on the trigonometric circle such that the angle between the axis of abscissa Ox and $\overrightarrow{OM}$ is equal to x and by H and K its respective projections on the abscissa axis and to the y-axis passing through M.



Trigonometric functions are functions that relate an angle in a circle to the ratio of the sides in a right triangle.

2.1 Cosine and arccosine function

The cosine function

The *cosine* function is the function defined by $\cos:\mathbb{R}\longrightarrow[-1,1]$ which, to any real x, associates the real $\cos(x)$, where $\cos(x)$ designates the abscissa of the point M $\left(\cos(x)=\overline{OH}\right)$.

The function *cosine* is differentiable on $\mathbb{R}$ and, for all $x\in\mathbb{R}$, we have

$$\cos'(x)=-\sin(x)$$

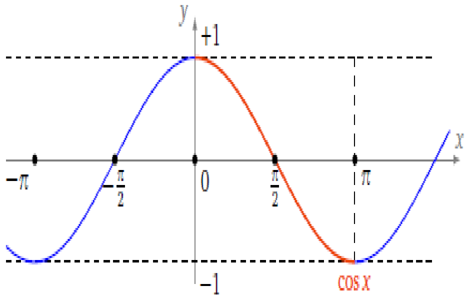
Properties

1. $\forall x\in\mathbb{R},\ -1\leqslant\cos(x)\leqslant 1$
2. $\cos(x)=\cos(x+2k\pi)$ where $k\in\mathbb{Z}$.
3. $\forall x\in\mathbb{R},\ \cos(-x)=\cos(x)$
4. $\cos(x)=\cos(a)\Longleftrightarrow x=a+2k\pi$ or $x=-a+2k\pi$ with $k\in\mathbb{Z}$.

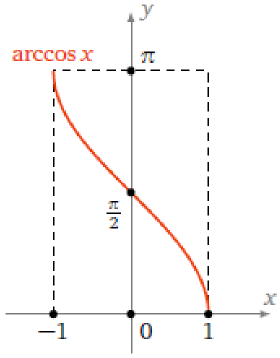
The arccosine function

The function cos is continuous and strictly decreasing from $[0,\pi]$ to $[-1,1]$, therefore achieves a bijection. Thus, it admits an inverse function defined as

$$\arccos:[-1,1]\longrightarrow[0,\pi].$$



Graph of the function cos



Graph of the function arccos

1) The function arccos is continuous and strictly decreasing on $[-1,1]$.

2) The function arccos is differentiable on $]-1,1[$ with

$$\forall x \in]-1,1[, \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}$$

3) The function arccos is not differentiable in ± 1, its representative curve admitting at these points a vertical tangent.

Properties:

1. $\forall x \in [-1,1], \cos(\arccos(x)) = x.$

2. $\forall x \in [0,\pi], \arccos(\cos(x)) = x.$

3. $\forall x \in]-1,1[,\ \sin(\arccos(x)) = \sqrt{1-x^2}.$

Example 1.

$$\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

because $\frac{\pi}{3}$ *is the only arc between* 0 *and* π *whose cosine is* $\frac{1}{2}$.

2.2 Sine and arcsine function

The Sine function

The function *sine* is the function defined by $\sin : \mathbb{R} \longrightarrow [-1,1]$ which, to any real x, associates the real $\sin(x)$, where $\sin(x)$ designates the ordinate from point M $\left(\sin(x) = \overline{OK}\right)$.

The function *sine* is differentiable on $\mathbb{R}$ and, for all $x \in \mathbb{R}$, we have

$$\sin'(x) = \cos(x)$$

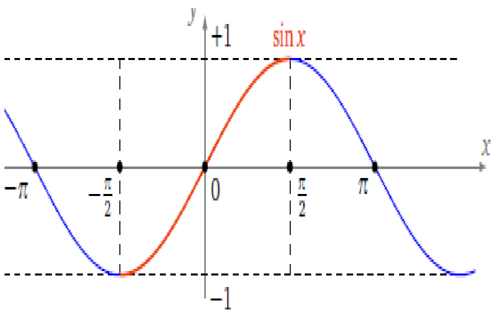
Properties

1. $\forall x \in \mathbb{R}$, $-1 \leqslant \sin(x) \leqslant 1$
2. $\sin(x) = \sin(x + 2k\pi)$ where $k \in \mathbb{Z}$.
3. $\forall x \in \mathbb{R}$, $\sin(-x) = -\sin(x)$
4. $\sin(x) = \sin(a) \Longleftrightarrow x = a + 2k\pi$ or $x = \pi - a + 2k\pi$ with $k \in \mathbb{Z}$.

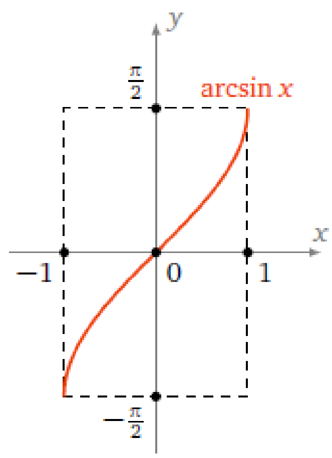
The arcsine function

The function sin is continuous and strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so performs a bijection of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ on $[-1, 1]$. Therefore, it admits an inverse function defined as

$$\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



Graph of function sin



Graph of function arcsin

1) The *arcsin* function is odd and continues on $[-1, 1]$.

2) The function *arcsin* is differentiable on $]-1, 1[$ and

$$\forall x \in]-1, 1[, \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}.$$

3) The function *arcsin* is not differentiable in ± 1, its representative curve admitting at these points a vertical tangent.

Properties :

1. $\forall x \in [-1, 1], \quad \sin(\arcsin(x)) = x$
2. $\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad \arcsin(\sin(x)) = x$
3. $\forall x \in]-1, 1[, \quad \cos(\arcsin(x)) = \sqrt{1-x^2}$.

2.3 Tangent and arctangent function

The tangent function

We call the tangent function the function defined on $\mathbb{R}$-$\left\{\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\right\}$ by:

$$\tan(x)=\frac{\sin(x)}{\cos(x)}.$$

The function tan is differentiable on $\mathbb{R}$-$\left\{\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\right\}$ and, for all $x\in\mathbb{R}$-$\left\{\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\right\}$, we have

$$\tan'(x)=1+\tan^2(x).$$

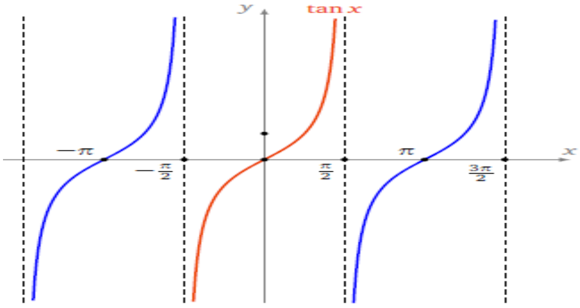
Properties

1. $\forall x\in\mathbb{R}$-$\left\{\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\right\}$, $\tan(-x)=-\tan(x)$
2. $\forall x\in\mathbb{R}$-$\left\{\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\right\}$, $1+\tan^2(x)=\dfrac{1}{\cos^2(x)}$.

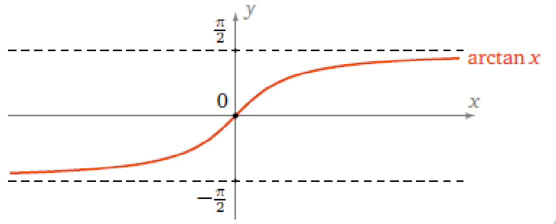
The arctangent function

The function tan is continuous and strictly increasing on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, so performs a bijection of $\left]\frac{-\pi}{2},\frac{\pi}{2}\right[$ on $\mathbb{R}$. Therefore, it admits an inverse function defined as

$$\arctan:\mathbb{R}\to\left]\frac{-\pi}{2},\frac{\pi}{2}\right[$$



Graph of function tan



Graph of function arctan

1) The function arctan is odd, continuous and strictly increasing on $\mathbb{R}$.

2) The function arctan is differentiable on $\mathbb{R}$ and

$$\forall x\in\mathbb{R},\quad \arctan'(x)=\frac{1}{1+x^2}.$$

Properties :

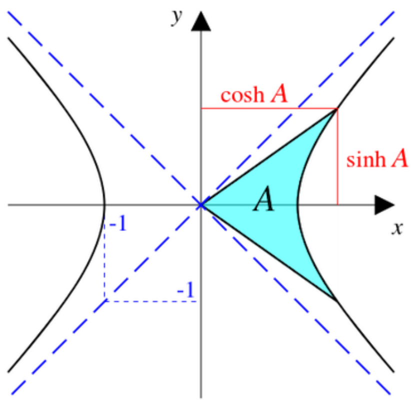
1. $\forall x\in\mathbb{R},\quad \tan(\arctan(x))=x$
2. $\forall x\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[,\quad \arctan(\tan(x))=x$
3. $\forall x\in\mathbb{R},\quad \cos^2(\arctan(x))=\dfrac{1}{1+x^2}$.

Remark :

$$\arctan(x)\neq\frac{\arcsin(x)}{\arccos(x)}$$

3 Hyperbolic Functions and their Inverses

This representation shows how an area relates to hyperbola. Let $(\cosh A, \sinh A)$ be a point on the unit hyperbola of eqaution $x^2 - y^2 = 1$. A line segment from the origin to the unit hyperbola sweeps out an area. like the unit circle, it satisfies the property that the area of the sector is precisely half the corresponding angle.



The hyperbolic functions are functions taht take an area as their argument instead of an angle.

3.1 hyperbolic cosine function and its inverse

Hyperbolic Cosine Function

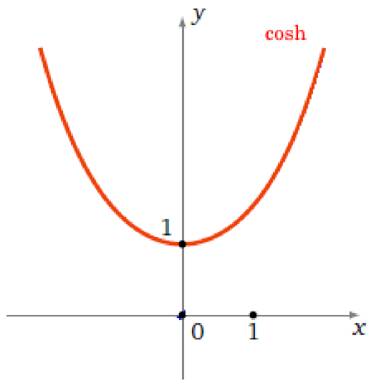
Definition 3.1

Given a real x, we call hyperbolic cosine of x, denoted $\cosh(x)$*, the function defined as*

$$\begin{aligned} \cosh : \mathbb{R} &\to [1, +\infty[\\ x &\mapsto \frac{e^x + e^{-x}}{2} \end{aligned}$$

Properties of hyperbolic cosine :

- The function cosh is continuous, strictly decreasing on $\mathbb{R}_-$ and strictly increasing on $\mathbb{R}_+$
- The function cosh is differentiable on $\mathbb{R}$ and
$$\cosh'(x) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$
- The function cosh is even.
- $\forall x \in \mathbb{R}, \cosh(x) \geqslant 1.$



Graph of function cosh

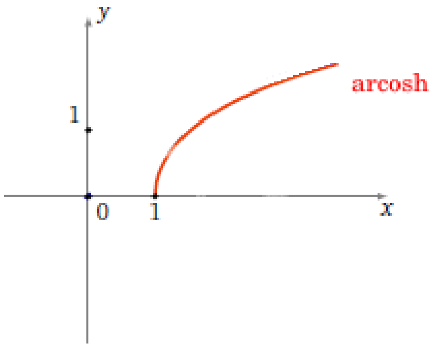
Inverse hyperbolic cosine function

The restriction of the function cosh to the interval $[0, +\infty[$ is continuous and strictly increasing. It establishes a bijection of $[0, +\infty[$ on $[1, +\infty[$. Therefore, It admits an inverse function denoted as, $\text{arcosh} : [1, +\infty[\to [0, +\infty[$

Properties of inverse hyperbolic cosine :

- The function arcosh is continuous and strictly increasing on $[1,+\infty[$.
- The arcosh function is neither even nor odd.
- The function arcosh is differentiable on $]1,+\infty[$ and

$$\forall x \in]1,+\infty[, \quad \operatorname{arcosh}'(x) = \frac{1}{\sqrt{x^2-1}}$$



Graph of function arcosh

Properties :

1. $\forall x \in [1,+\infty[, \quad \cosh(\operatorname{arcosh}(x)) = x$
2. $\forall x \in [0,+\infty[, \quad \operatorname{arcosh}(\cosh(x)) = x$
3. $\forall x \in]1,+\infty[, \quad \sinh(\operatorname{arcosh}(x)) = \sqrt{x^2-1}.$

3.2 Hyperbolic sine function and its inverse

Hyperbolic Sine Function

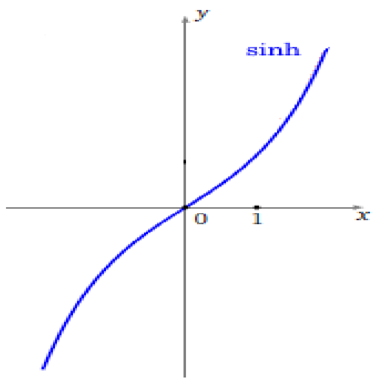
Definition 3.2

Given a real x*, we call hyperbolic sine of* x*, denoted* $\sinh(x)$*, the function defines as*

$$\sinh : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \frac{e^x - e^{-x}}{2}$$

Properties of hyperbolic sine :

- The function sinh is continuous and strictly increasing on $\mathbb{R}$.
- The function sinh is differentiable on $\mathbb{R}$ and
$$\sinh'(x) = \frac{e^x + e^{-x}}{2} = \cosh(x).$$
- The function sinh is odd.



Graph of function sinh

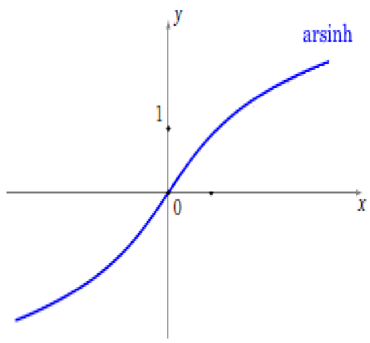
Inverse hyperbolic sine function

The function sinh is continuous and strictly increasing from $\mathbb{R}$ in $\mathbb{R}$. It establishes a bijection of $\mathbb{R}$ on $\mathbb{R}$. Therefore, it admits an inverse function denoted $\operatorname{arsinh} : \mathbb{R} \to \mathbb{R}$

Properties of inverse hyperbolic sine :

- The function arsinh is continuous and strictly increasing on $\mathbb{R}$.
- The function arsinh is odd.
- The function arsinh is differentiable on $\mathbb{R}$ and

$$\forall x \in \mathbb{R}, \quad \operatorname{arsinh}'(x) = \frac{1}{\sqrt{1+x^2}}$$



Graph of function arsinh

Properties :

1. $\forall x \in \mathbb{R}, \quad \sinh(\operatorname{arsinh}(x)) = x$
2. $\forall x \in \mathbb{R}, \quad \operatorname{arsinh}(\sinh(x)) = x$
3. $\forall x \in \mathbb{R}, \quad \cosh(\operatorname{arsinh}(x)) = \sqrt{1+x^2}$.

3.3 Hyperbolic tangent function and its inverse

Hyperbolic tangent function

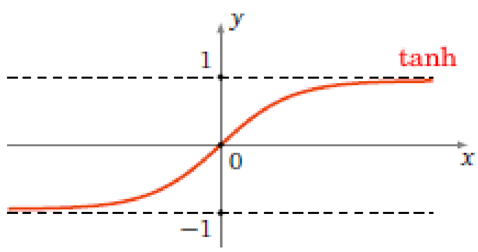
Definition 3.3

Given a real x, we call the hyperbolic tangent of x, denoted $\tanh(x)$, *the function defined as*

$$\begin{aligned} \tanh : \mathbb{R} &\to]-1,1[\\ x &\mapsto \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \end{aligned}$$

Properties of hyperbolic tangent :

- The function tanh is continuous and strictly increasing on $\mathbb{R}$.
- The function tanh is differentiable on $\mathbb{R}$ and $\tanh'(x) = 1 - \tanh^2(x) = \dfrac{1}{\cosh^2(x)}$.
- The function tanh is odd.



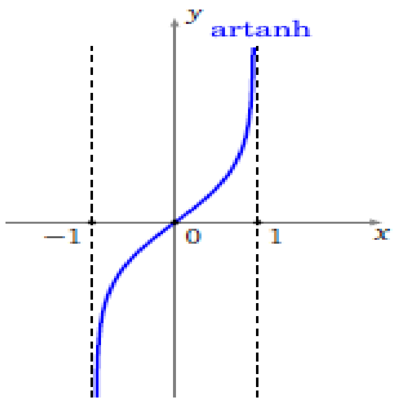
Graph of function tanh

Inverse hyperbolic tangent function

The function tanh is continuous and strictly increasing from $\mathbb{R}$ over $]-1,+1[$. It establishes a bijection of $\mathbb{R}$ on $]-1,+1[$. Therefore, it admits an inverse function denoted

$$\text{artanh} :]-1,+1[\to \mathbb{R}$$

Properties of inverse hyperbolic tangent :



Graph of function artanh

- The function artanh is continuous and strictly increasing on $]-1,1[$.
- The function artanh is odd.
- The function artanh is differentiable on $]-1,1[$ and

$$\forall x \in]-1,1[, \quad \text{artanh}'(x) = \frac{1}{1-x^2}$$

Properties :

1. $\forall x \in]-1,1[, \quad \tanh(\text{artanh}(x)) = x$
2. $\forall x \in \mathbb{R}, \quad \text{artanh}(\tanh(x)) = x$
3. $\forall x \in]-1,1[, \quad \cosh^2(\text{artanh}(x)) = \dfrac{1}{1-x^2}.$

Logarithmic expression of inverse hyperbolic function

1. $\text{arcosh}(x) = \ln(x + \sqrt{x^2-1}), \quad x \geqslant 1$
2. $\text{arsinh}(x) = \ln(x + \sqrt{x^2+1}), \quad x \in \mathbb{R}$
3. $\text{artanh}(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), \quad x \in]-1,1[$

Useful equalities

1. $\forall x \in \mathbb{R}, \quad \cosh(x) + \sinh(x) = e^x$
2. $\forall x \in \mathbb{R}, \cosh(x) - \sinh(x) = e^{-x}$
3. $\forall x \in \mathbb{R}, \quad \dfrac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$

Formulas: trigonometric and hyperbolic

Trigonometric properties: replace cos by cosh and sin by i-sinh.

$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$	$\cosh(a+b) = \cosh a \cdot \cosh b + \sinh a \cdot \sinh b$
$\sin(a+b) = \sin a \cdot \cos b + \sin b \cdot \cos a$	$\sinh(a+b) = \sinh a \cdot \cosh b + \sinh b \cdot \cosh a$
$\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \cdot \tan b}$	$\tanh(a+b) = \dfrac{\tanh a + \tanh b}{1 + \tanh a \cdot \tanh b}$
$\cos(a-b) = \cos a \cdot \cos b + \sin a \cdot \sin b$	$\cosh(a-b) = \cosh a \cdot \cosh b - \sinh a \cdot \sinh b$
$\sin(a-b) = \sin a \cdot \cos b - \sin b \cdot \cos a$	$\sinh(a-b) = \sinh a \cdot \cosh b - \sinh b \cdot \cosh a$
$\tan(a-b) = \dfrac{\tan a - \tan b}{1 + \tan a \cdot \tan b}$	$\tanh(a-b) = \dfrac{\tanh a - \tanh b}{1 - \tanh a \cdot \tanh b}$
$\cos a \cdot \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$	$\cosh a \cdot \cosh b = \frac{1}{2}[\cosh(a+b) + \cosh(a-b)]$
$\sin a \cdot \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$	$\sinh a \cdot \sinh b = \frac{1}{2}[\cosh(a+b) - \cosh(a-b)]$
$\sin a \cdot \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$	$\sinh a \cdot \cosh b = \frac{1}{2}[\sinh(a+b) + \sinh(a-b)]$
$\cos p + \cos q = 2\cos\dfrac{p+q}{2} \cdot \cos\dfrac{p-q}{2}$	$\cosh p + \cosh q = 2\cosh\dfrac{p+q}{2} \cdot \cosh\dfrac{p-q}{2}$
$\cos p - \cos q = -2\sin\dfrac{p+q}{2} \cdot \sin\dfrac{p-q}{2}$	$\cosh p - \cosh q = 2\sinh\dfrac{p+q}{2} \cdot \cosh\dfrac{p-q}{2}$
$\sin p + \sin q = 2\sin\dfrac{p+q}{2} \cdot \cos\dfrac{p-q}{2}$	$\sinh p + \sinh q = 2\sinh\dfrac{p+q}{2} \cdot \cosh\dfrac{p-q}{2}$
$\sin p - \sin q = 2\sin\dfrac{p-q}{2} \cdot \cos\dfrac{p+q}{2}$	$\sinh p - \sinh q = 2\sinh\dfrac{p-q}{2} \cdot \cosh\dfrac{p+q}{2}$