University of Batna 2

Institute of Industrial Hygiene and Safety

Module : Math 1 (L1)

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Chapter 5 : Limited Development

Introduction

The limited development $LD_n(x_0)$ is useful in many areas of mathematics and physics, including solving differential equations, performing integrations, evaluating limits and analyzing local behavior of a function and its polynomial approximation.

Contents

1 Limited Development

Limited development

Let $f : I \to \mathbb{R}$ be a function, $x_0 \in I$ and $n \in \mathbb{N}$. It is said that f admits a limited development of order n, in a neighborhood of $x = x_0$, that we note $LD_n(x_0)$, if there exist real numbers $a_0, a_1, \ldots, a_n$ such that, when $x \to x_0$, it can be written as

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + (x - x_0)^n \epsilon(x)$$

where, $\epsilon : I \to \mathbb{R}$ is the function such that $\lim_{x \to x_0} \epsilon(x) = 0$

- $(x - x_0)^n \epsilon(x)$ is the remainder of order n.

Example: Let $f(x) = \dfrac{1}{1-x}, x \neq 1$. f admits $LD_n(0)$, indeed :
Since $1 - x^{n+1} = (1-x)(1 + x + \cdots + x^n)$, we have

$$\frac{1}{1-x} - \frac{x^{n+1}}{1-x} = \frac{1-x^{n+1}}{1-x} = \frac{(1-x)(1+x+\cdots+x^n)}{1-x} = 1 + x + \cdots + x^n$$

where

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \frac{x^{n+1}}{1-x} = 1 + x + \cdots + x^n + x^n \frac{x}{1-x}$$

Therefore the function $f(x) = \dfrac{1}{1-x}, x \neq 1$ admits a limited development of order n at $x_0 = 0$, with $\epsilon(x) = \dfrac{x}{1-x}$, where, $\lim_{x \to 0} \epsilon(x) = 0$.

1.1 Taylor's Formula

Taylor's Formula

Let x_0 be any real number and let $f : I \to \mathbb{R}$ be a function that can be differentiated at least n times at the point x_0. The Taylor's Formula for **f** of order **n** about the point $\mathbf{x_0}$ is defined by

$$f(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n)$$

where $o((x - x_0)^n)$ is called the Young remainder of order n.

- $o((x - x_0)^n) = (x - x_0)^n \epsilon(x)$
- $f^{(n)}(x_0)$ refers to the n^{th} derivative of the function f evaluated at x_0.

- $n!$ is the factorial of n where $n! = n \times (n-1) \times \cdots \times 3 \times 2 \times 1$.
- $P_n(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ is the Taylor polynomial in the variable x with $n+1$ terms.

Remark 1.

If $x_0 = 0$, Taylor formula with Young remainder is known as Maclaurin's formula:

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!}(x) + \frac{f^{(2)}(0)}{2!}(x)^2 + \cdots + \frac{f^{(n)}(0)}{n!}(x)^n + o(x^n)$$

Example 1: Find the Taylor formula of $f(x) = e^x$ of order $n = 3$ about the point $x_0 = 0$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}(x)^n$
0	e^x	1	1
1	e^x	1	x
2	e^x	1	$\frac{x^2}{2}$
3	e^x	1	$\frac{x^3}{6}$

Summing the last column we find that: $f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$

Example 2 The Taylor formula of $f(x) = \frac{1}{1-x}, x \neq 1$ of order n at $x_0 = 0$ is

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + o(x^n)$$

We notice that in this case the Taylor's formula is exactly the limited development.

Remark 2.

- Taylor-Young's formula of f of order n at x_0 is $LD_n(x_0)$, where $a_n = \frac{f^{(n)}(x_0)}{n!}$.
- The $LD_n(x_0)$ of f is given by the Taylor-Young's formula of order n at x_0, if f is differentiated at least n times at the point x_0.

1.2 Properties of Limited Development

- If f admits a $LD_n(x_0)$, then $\lim_{x\to x_0} f(x)$ exists, finite and is equal to a_0.
  This criterion is generally used to demonstrate that a function does not admit $LD_n(x_0)$.
  Example : The function $\ln(x)$ does not admit $LD_n(0)$, because $\lim_{x\to 0}\ln(x) = -\infty$.

- A function does not necessarily have an $LD_n(x_0)$, but if it exists, then it is unique.

- **Parity** **Even function** The $LD_n(x_0)$ of an even function has a main part that contains only monomials of even degree. That is to say the coefficients $a_{2k+1} = 0$.

  Odd function The $LD_n(x_0)$ of an odd function has a main part that contains only monomials of odd degree. That is to say the coefficients $a_{2k} = 0$.

- The $LD_n(x_0)$ of a polynomial of degree n is itself.

1.3 Limited Development of usual Functions

Below, we show some very famous limited development of common function of order n, at $x = 0$ using Maclaurin's formula :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n)$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n)$$
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n)$$
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots + (-1)^{n-1}\frac{1\times 3\times 5\times\cdots\times(2n-3)}{2^n n!}x^n + o(x^n)$$
$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + o(x^n)$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$
$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

Remark 3.

We will often work at $x_0 = 0$, based on changes of variables:

1. If $x_0 \in \mathbb{R}^*$, we put $t = x - x_0$, and then $t \to 0$ when $x \to x_0$.
2. If $x_0 \to \infty$, we put $t = \frac{1}{x}$, and then $t \to 0$ when $x \to \infty$.

Example Find $LD_3(\frac{\pi}{4})$ for the function $x \mapsto \sin(x)$
We put $t = x - \frac{\pi}{4}$, then $t \to 0$ when $x \to \frac{\pi}{4}$. Thus, $x = t + \frac{\pi}{4}$

$$\begin{aligned} f(x) = \sin(x) = \sin\left(t+\frac{\pi}{4}\right) &= \sin(t)\cos\left(\frac{\pi}{4}\right) + \cos(t)\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\sin(t) + \frac{\sqrt{2}}{2}\cos(t) \\ &= \frac{\sqrt{2}}{2}\left(t - \frac{t^3}{6} + o(t^3)\right) + \frac{\sqrt{2}}{2}\left(1 - \frac{t^2}{2} + o(t^3)\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t - \frac{\sqrt{2}}{4}t^2 - \frac{\sqrt{2}}{12}t^3 + o(t^3) \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3 + o\left(\left(x-\frac{\pi}{4}\right)^3\right) \end{aligned}$$

1.4 Operation on Limited Development

Sum If f admits a $LD_n(0)$: $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + o(x^n)$
and g admits a $LD_n(0)$: $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + o(x^n)$
Then $f+g$ admits a $LD_n(0)$, which is given by the sum of the two limited development :

$$(f+g)(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_n+b_n)x^n + o(x^n)$$

Example : Find the $LD_4(0)$ of $\ln(1+x) + e^x$:
As

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)$$

Hence : $\ln(1+x) + e^x = 1 + 2x + \frac{x^3}{2} - \frac{5x^4}{24} + o(x^4)$

Product If f admits a $LD_n(0)$: $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + o(x^n)$
and g admits a $LD_n(0)$: $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + o(x^n)$
Then fg admits a $LD_n(0)$, obtained by keeping only the monomials of degree n or less in the product: $(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)(b_0 + b_1x + b_2x^2 + \cdots + b_nx^n)$

Example : Find $LD_3(0)$ of $x \mapsto \cos(x)\sin(x)$:
We have

$$\cos(x) = 1 - \frac{x^2}{2} + o\left(x^3\right)$$

$$\sin(x) = x - \frac{x^3}{6} + o\left(x^3\right)$$

Then, we developing the product, only considering terms of order 3 or less :

$$\begin{aligned}\cos(x)\sin(x) &= \left(1-\frac{x^2}{2}+o\left(x^3\right)\right)\left(x-\frac{x^3}{6}+o\left(x^3\right)\right)\\ &= x-\frac{2x^3}{3}+o\left(x^3\right)\end{aligned}$$

Quotient If f admits a $LD_n(0)$: $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n+x^n\epsilon(x)$
and g admits a $LD_n(0)$: $g(x)=b_0+b_1x+b_2x^2+\cdots+b_nx^n+x^n\epsilon(x)$, with $b_0\neq 0$
Then $\frac{f}{g}$ admits a $LD_n(0)$, obtained by the division according to the increasing degrees to order n of the polynomial $(a_0+a_1x+a_2x^2+\cdots+a_nx^n)$ by the polynomial $(b_0+b_1x+b_2x^2+\cdots+b_nx^n)$

Example : Let us compute $LD_5(0)$ for $x\mapsto\tan(x)=\dfrac{\sin(x)}{\cos(x)}$
We have

$$\sin(x)=x-\frac{x^3}{6}+\frac{x^5}{120}+o\left(x^5\right)$$

$$\cos(x)=1-\frac{x^2}{2}+\frac{x^4}{24}+o\left(x^5\right)$$

Thus,

$$\tan(x)=\frac{\sin(x)}{\cos(x)}=\frac{x-\frac{x^3}{6}+\frac{x^5}{120}+o\left(x^5\right)}{1-\frac{x^2}{2}+\frac{x^4}{24}+o\left(x^5\right)}$$

Then, we developing the division according to the increasing degrees to order 5 :

$$\begin{array}{r|l} x-\frac{x^3}{6}+\frac{x^5}{120}+o(x^5) & 1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^5) \\ x-\frac{x^3}{2}+\frac{x^5}{24}+o(x^5) & \\ \hline \frac{x^3}{3}-\frac{x^5}{30}+o(x^5) & x+\frac{x^3}{3}+\frac{2x^5}{15} \\ \frac{x^3}{3}-\frac{x^5}{6}+o(x^5) & \\ \hline \frac{2x^5}{15}+o(x^5) & \\ \frac{2x^5}{15}+o(x^5) & \\ \hline o(x^5) & \end{array}$$

Therefore, $\tan(x)=x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+o(x^5)$

Composition If f admits a $LD_n(g(0))$:

$$f(x) = a_0 + a_1(x - g(0)) + a_2(x - g(0))^2 + \cdots + a_n(x - g(0))^n + (x - g(0))^n \epsilon(x)$$

and g admits a $LD_n(0)$: $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + x^n \epsilon(x)$
Then, $f \circ g$ admits a $LD_n(0)$, obtained by replacing the limited development of g in that of f and keeping only the monomials of degree n or less.
Example : Let us compute $LD_3(0)$ for $x \mapsto \sin\left(\frac{1}{1-x} - 1\right)$
Since,

$$\frac{1}{1-x} - 1 = -x + x^2 - x^3 + o(x^3)$$

$$\sin(x) = x - \frac{x^3}{6} + o(x^3)$$

Then, we compose, only considering terms of order 3 or less :

$$\begin{aligned}\sin\left(\frac{1}{1-x} - 1\right) &= -x + x^2 - x^3 - \frac{1}{6}(-x)^3 + o(x^3)\\ &= -x + x^2 - \frac{5x^3}{6} + o(x^3)\end{aligned}$$

Differentiability If $f : I \to \mathbb{R}$ admits a $LD_{n+1}(0)$ and f is differentiated at least $n+1$ times, then f' admits a $LD_n(0)$, obtained by deriving the limited development of f.
Example : compute $LD_3(0)$ for $x \mapsto \frac{1}{(1-x)^2}$
Since, $\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)'$ and $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + o(x^4)$
Derive the $LD_4(0)$ of $\frac{1}{1-x}$, we obtain $LD_3(0)$ for $\frac{1}{(1-x)^2}$:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + o(x^3)$$

2 Applications on Calculating Limits

Limited development is very useful in the case of an indeterminate form when computing a limit:

- Evaluate $\lim_{x \to 0} \frac{x - \sin(x)}{x^3}$
  We have $\lim_{x \to 0} \frac{x - \sin(x)}{x^3} = \frac{0}{0}$ (IF)

Since, $\sin(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$, we obtain

$$\begin{aligned}\lim_{x\to 0} \frac{x - \sin(x)}{x^3} &= \lim_{x\to 0} \frac{x - \left(x + \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right)}{x^3} \\ &= \lim_{x\to 0} -\frac{1}{6} - \frac{x^2}{120} + o(x^2) \\ &= -\frac{1}{6}\end{aligned}$$

○ Evaluate $\lim_{x\to +\infty} x^2 \left(e^{\frac{1}{x}} - e^{\frac{1}{1+x}}\right)$

We have $\lim_{x\to +\infty} x^2 \left(e^{\frac{1}{x}} - e^{\frac{1}{1+x}}\right) = 0.\infty$ (IF)

First, we put $t = \frac{1}{x}$, and then $t \to 0$ when $x \to +\infty$ and

$$x^2 \left(e^{\frac{1}{x}} - e^{\frac{1}{1+x}}\right) = \frac{1}{t^2}\left(e^t - e^{\frac{t}{1+t}}\right)$$

Since, $\frac{t}{1+t} = t - t^2 + o(t^2)$ and $e^t = 1 + t + \frac{t^2}{2} + o(t^2)$

We obtained,

$$e^{\frac{t}{1+t}} = 1 + t - t^2 + \frac{(t-t^2)^2}{2} + o(t^2) = 1 + t - \frac{1}{2}t^2 + o(t^2)$$

Hence,

$$\begin{aligned}\frac{1}{t^2}\left(e^t - e^{\frac{t}{1+t}}\right) &= \frac{1}{t^2}\left(\left(1 + t + \frac{t^2}{2}\right) - \left(1 + t - \frac{1}{2}t^2\right) + o(t^2)\right) \\ &= \frac{1}{t^2}\left(t^2 + o(t^2)\right) \\ &= 1 + o(1)\end{aligned}$$

Consequently,

$$\begin{aligned}\lim_{x\to +\infty} x^2 \left(e^{\frac{1}{x}} - e^{\frac{1}{1+x}}\right) &= \lim_{t\to 0} \frac{1}{t^2}\left(e^t - e^{\frac{t}{1+t}}\right) \\ &= \lim_{t\to 0}(1 + o(1)) \\ &= 1\end{aligned}$$