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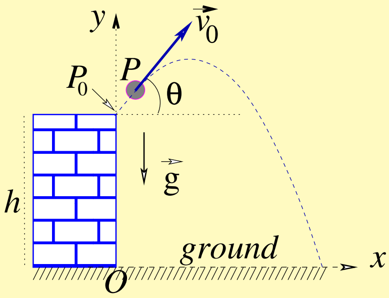
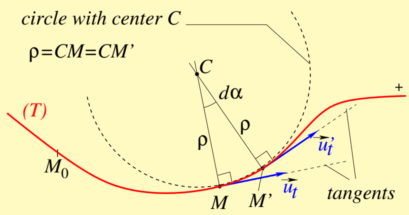


The equation of the trajectory of P is : $y = -\frac{g}{2v_0^2\cos^2\theta}x^2 + \tan\theta\, x + h$. The speed of the ball as it hits the ground is $v = \sqrt{v_0^2 + 2gh}$.
In terms of $\dot{x}$, $\dot{y}$, $\ddot{x}$, and $\ddot{y}$, the radius of curvature of M (right figure) reads : $\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{|\dot{x}\ddot{y}-\ddot{x}\dot{y}|}$.



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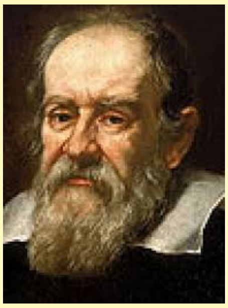
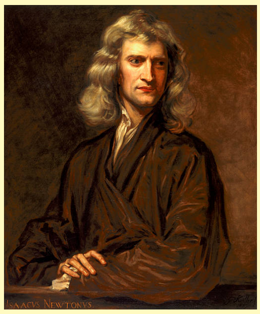


Galileo (left portrait) writes : "Aristotle declares that a 100-pound iron ball has already descended 100 cubits when a 1-pound ball has traveled only one cubit. I affirm that the two balls arrive together."

In 1687, Newton (right portrait) published the mathematical principles of natural philosophy (*Philosophiae naturalis principia mathematica*). In it, he described his discoveries on universal gravitation and the three famous laws, known as Newton's Laws. These laws describe the physical phenomena of inertia and the forces exerted on objects.



Above, the blue figure (left) shows the trajectory of a small ball P launched from the top of a building (height h) with a velocity vecv_0 making an angle θ with the horizontal. The expressions to its right give the equation of the ball's trajectory and its velocity when it hits the ground. The figure on the far right defines the radius of curvature at a point on the trajectory. When M' tends towards M (trajectory (T) in red), the normals to the tangents at M and M' meet at a point C called the center of curvature. The lengths CM and CM' are then equal to a quantity ρ called radius of curvature. A circle with center C and radius ρ will necessarily pass through M and M'. As for the two portrait photos below, they are of Galileo (left) and Newton (right), two great scientists who left their mark on the history of science and contributed greatly to mechanics.

Prof. M. A. Belkhir, année 2023-2024

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Chapitre 1

Vectors, SI unit system, and math reminders

Vectors are an important part of physics, and we'll be using them a lot in this course. That's why this chapter is devoted almost entirely to them. But before we really get into vectors, we'll introduce the International System of units (SI system), talk briefly about dimensional analysis and the homogeneity of equations in physics. We'll end this chapter with a few useful mathematical reminders.

1.1 SI units

To measure a physical quantity, it is necessary to use a unit of measurement (or standard). There are different systems of units based on different choices of the fundamental set of units, but today the most widely used system of units is the International System of Units (abbreviated to SI), in which there are seven base units and derived units. The base units are shown in the table below.

Physical quantity		**DIMENSION**	**Unit of measurement**	
Name of the quantity	Symbol	of the quantity	SI unit of measurement	Unit symbol
Length	$\ell, x, y, \ldots$	L	meter	m
Mass	M, m	M	kilogram	kg
Time	t	T	second	s
Electric Current	I, i	I	ampere	A
Thermodynamic temperature	T, θ	Θ	Kelvin	K
Luminous intensity	I_v, I	J	Candela	cd
Amount of substance (of matter)	n	N	mole	mol

Table 1.1 – Base Units of SI system

The symbols Θ and θ in the table are pronounced *thêta* and represent the eighth letter (upper and lower case) of the Greek alphabet. Other Greek letters will be encountered throughout the course. Greek alphabet is given at section 1.6.

Derived units : units formed from a combination of several base units are called *derived units*. For example, meters per second (m/s) for speed or kilograms per cubic meter ($\mathrm{kg/m^3}$) for density, etc.

Some derived units are given a special name ; for example, the $\mathrm{kg \cdot m/s^2}$ is the unit of force derived from Newton's law $F = ma$, and is called the newton (N) : 1 N= $1\,\mathrm{kg \cdot m/s^2}$.

Radian and steradian : Note that the radian (unit of plane angle ; symbol rad) and the steradian (unit of solid angle ; symbol sr) are considered derived units.
They are dimensionless, as they are defined by a ratio of two lengths for the plane angle and two areas for the solid angle. The use or non-use of these two units in other derived units is as required.

Examples
1- Use → Angular speed (the rate of change of angular displacement in radians per unit of time) of a point M moving on a circle is expressed in $\mathrm{rad} \cdot \mathrm{s}^{-1}$ to distinguish it from frequency, measured in s^{-1}, which gives the number of revolutions (cycles) per second.

2- Non-use → On a circle of radius R, the length of the arc intercepted by the central angle α is $L = \alpha R$. The unit of L is of course m and not rad·m.

Non-SI units : non-SI units are also widely used : °C, km/h, eV, inch, etc. You need to know how to convert non-SI units to SI units and vice versa. For example, convert 72 km/h to m/s :

$$72\,\frac{\mathrm{km}}{\mathrm{h}} = \frac{72 \times 1000}{3600}\frac{\mathrm{m}}{\mathrm{s}} = 20\,\mathrm{m/s}$$

1.2 Dimensional analysis and homogeneity of equations in physics

One can't add (or subtract) two quantities that have different dimensions : writing Length + width makes sense, but Length + surface or 2 kg + 5 m/s make no sense ! Only quantities of same dimension can be added or subtracted. On the other hand, it's perfectly possible to multiply or divide two physical quantities of the same dimension or of different dimensions : $v = L/t \to \mathrm{m/s}$, $S = L \times l \to \ \mathrm{m} \times \mathrm{m} = \mathrm{m}^2$, $P = m\,\mathrm{g} \to \mathrm{kg} \cdot \mathrm{m}/\,\mathrm{s}^2$.

notation : The dimension of a quantity X is indicated by the same letter in square brackets : $[X]$.

The principle of homogeneity : A physical equation or relationship is homogeneous only if its two members and each additive term making up each member have the same dimension. The equation $a + b = c$ is homogeneous only and only if a, b et c ont même dimension :

$$a + b = c \iff [a] = [b] = [c]$$

Dimensional analysis is a powerful tool for detecting an error : *an inhomogeneous equation is necessarily false.*

If A, B and C are any physical quantities and α, β and γ are numbers, the following rules apply :
1) $[ABC] = [A][B][C]$
2) $[A/B] = [A]/[B]$
3) Consider the equality $A + A = 2A$. According to the principle of homogeneity, A and 2A have the same dimension : $[2A] = [A]$ or $[2][A] = [A]$ or $[2] = [A]/[A] = 1$. To say that a quantity is dimensionless is the same as saying that its dimension is 1. Numbers are dimensionless : $[2] = [1570] = [6.236] = [-2023] = 1$.
4) The equation $A^\alpha + B^\beta = C^\gamma$ is homogeneous $\iff [A]^\alpha = [B]^\beta = [C]^\gamma$.
5) If $A^\alpha B^\beta = C^\gamma$ then $[A]^\alpha [B]^\beta = [C]^\gamma$.

Take, for example, the equation $x = at^2/2$, where x is the space traveled in time t by a particle that starts at rest with acceleration a. To see if the equation is homogeneous, we calculate the dimension of each member. We have :

$$[x] = L$$

$$[at^2/2] = [a][t^2] = [a][t]^2 = (L/T^2)T^2 = L$$

so $[x] = [at^2/2]$, we conclude that the equation is homogeneous.

1.3 Vectors

1.3.1 Scalar quantities, vector quantities

Some physical quantities are completely defined using a *number*, followed or not by a unit of measurement. These are called *scalars*. Mass (a 2 kg block), time (a 1h30mn class session), energy (an electrical energy consumption of 30 kwh per day), temperature (a temperature of 25 °C), the density of a body (ratio of its density to that of water (iron has a density of 7.86 (without unit)), etc. are all scalar magnitudes or simply scalars. However, there are other physical quantities that cannot be completely defined by a simple number. For example, if I ask you to move 5 m forwards, it's not the same as saying backwards, right or left. To make the desired move, it's important to specify the direction and the sense of movement, in addition to the magnitude (the 5 m value). Displacement is not a scalar, it's a ***vector***. Unlike a scalar, a vector is defined by three characteristics : a ***direction***, a ***sense***, and a value called ***magnitude***. Speed, acceleration, force, electric field, etc., are also vectors.

Remarque : In textbooks from Anglo-Saxon countries, only two characteristics (length and direction) are used to define a vector. The word **direction** includes in its meaning the orientation of the vector. For example, a plane flying in an East-West direction, a projectile launched at 30° from the $+x$ axis, and so on. For our purposes, we'll adopt the same definition when the direction of the vector is implicit, otherwise, if the need arises, we'll specify the direction.

1.3.2 Graphical representation and notation of a vector

A vector is represented graphically by a line segment running from one **origin** to one **end**. For example, in the figure below, A is the tail (origin, initial point) and B is the tip (head, endpoint, terminal point) of the vector $\overrightarrow{AB}$.

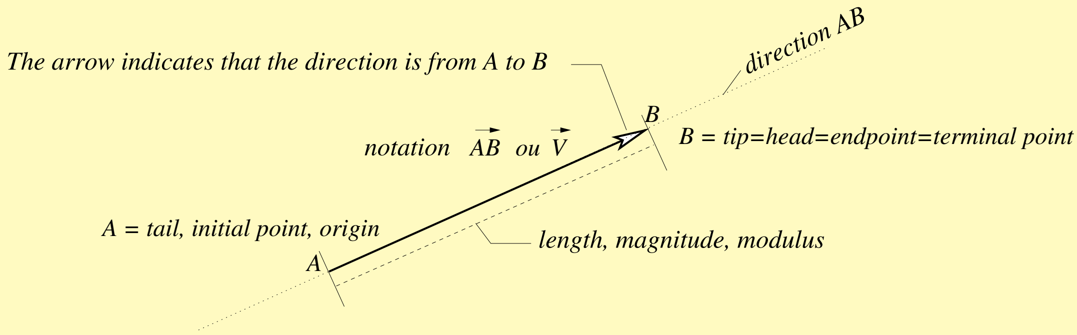

FIGURE 1.1 – Graphical representation and notation of a vector

- This is denoted by $\overrightarrow{AB}$ or by a simple letter ($\overrightarrow{V}$ for instance). Instead of putting an arrow above it, vectors are sometimes denoted with letters in **bold face** : **AB**, **V**.
- The modulus of a $\overrightarrow{V}$ vector is written $\| \overrightarrow{V} \|$, $| \overrightarrow{V} |$ or simply V.
- synonyms : modulus , length, magnitude, value,
intensity (of a force, electric field, etc.) norm of a vector (especially in mathematics)

Notice : the vector $\overrightarrow{BA}$ (figure below) has B as origin, A as head, and BA as direction. What distinguishes $\overrightarrow{BA}$ from $\overrightarrow{AB}$ is that $\overrightarrow{BA}$ points in the opposite direction.

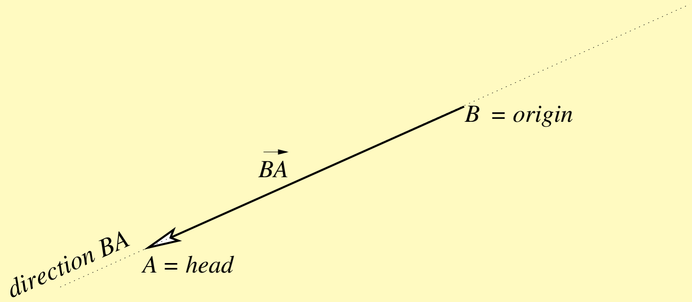

FIGURE 1.2 – Vector $\overrightarrow{BA}$ versus $\overrightarrow{AB}$

1.3.3 Addition, Commutativity, Associativity

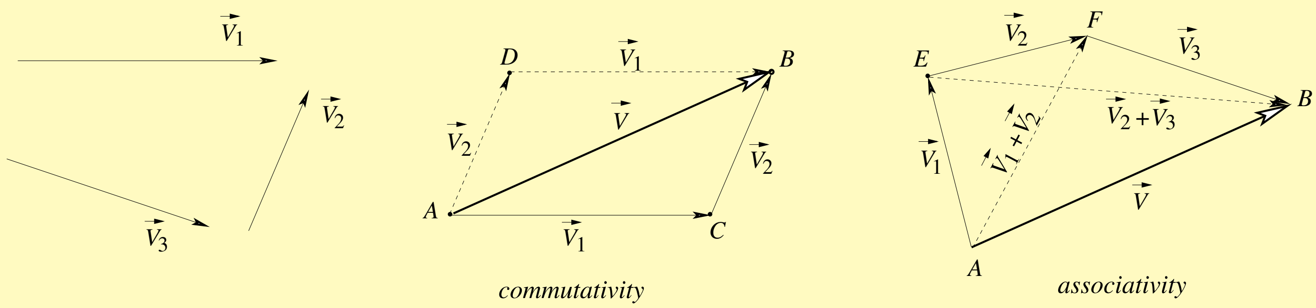

FIGURE 1.3 – Vector addition : graphical method

1. Parallelogram law of addition of two vectors : Bring the two vectors so that their tails meet at a common point. Draw one line parallel to one vector and a second line parallel to the other vector in a way to complete a parallelogram. Then the diagonal of the parallelogram originating from the common point will be the resultant (sum) vector $\overrightarrow{R}$: $\overrightarrow{R} = \overrightarrow{V_1} + \overrightarrow{V_2}$.

   Triangular Law of Vector Addition : Bring the head of vector $\overrightarrow{V_1}$ to the tail of vector $\overrightarrow{V_2}$. The resultant $\overrightarrow{R} = \overrightarrow{V_1} + \overrightarrow{V_2}$ is obtained by joining the tail of $\overrightarrow{V_1}$ to the head of $\overrightarrow{V_2}$. Triangular method is more practical when adding more than two vectors.
2. In the middle figure above, it's easy to see that
   $\overrightarrow{V}_1 + \overrightarrow{V_2} = \overrightarrow{V_2} + \overrightarrow{V_1}$; vector addition is commutative.
3. In the right figure above, we see that $(\overrightarrow{V_1} + \overrightarrow{V_2}) + \overrightarrow{V_3} = \overrightarrow{V_1} + (\overrightarrow{V_2} + \overrightarrow{V_3})$; vector addition is associative.
4. Null vector : If we don't move from A, or if we start from A and return to A, the displacement is *null*. This corresponds to the zero vector $\overrightarrow{0}$ (i.e. $\overrightarrow{AA} = \overrightarrow{0}$).
5. The vector $\overrightarrow{0}$ is a neutral element for vector addition : $\overrightarrow{V} + \overrightarrow{0} = \overrightarrow{0} + \overrightarrow{V} = \overrightarrow{V}$. Of course $\|\vec{0}\| = 0$.
6. Opposite of a vector : the opposite of a vector $\overrightarrow{V}$ is the vector $\overrightarrow{V'}$ such that the sum $\overrightarrow{V} + \overrightarrow{V'}$ gives $\vec{0}$, hence $\overrightarrow{V'} = -\overrightarrow{V}$.

7. Subtraction of two vectors : $\overrightarrow{V_1} - \overrightarrow{V_2} = \overrightarrow{V_1} + (-\overrightarrow{V_2})$, subtracting a vector is equivalent to adding its opposite.
8. Multiplication by a scalar, α and β two reals : $\alpha\overrightarrow{V} = \overrightarrow{V'}$, $\overrightarrow{V'}$ has the same direction as direction as $\overrightarrow{V}$, in the same direction if $\alpha > 0$, in the opposite direction if $\alpha < 0$.
9. Length of $\alpha\overrightarrow{V}$ is $|\alpha| \, \|\overrightarrow{V}\|$.
10. For $\alpha = 0$ we have, of course, $0\overrightarrow{V} = \vec{0}$.
11. $(\alpha + \beta)\overrightarrow{V} = \alpha\overrightarrow{V} + \beta\overrightarrow{V}$
12. $\alpha(\beta\overrightarrow{V}) = \beta(\alpha\overrightarrow{V}) = \alpha\beta\overrightarrow{V}$
13. $\alpha(\overrightarrow{V_1} + \overrightarrow{V_2}) = \alpha\overrightarrow{V_1} + \alpha\overrightarrow{V_2}$
14. **Unit vector** : A unit vector is a vector whose modulus is 1. A unit vector along any non-zero vector $\overrightarrow{V}$ is given by $\vec{u} = \overrightarrow{V}/V$. Note that a unit vector has no dimension (as it is a ratio of 2 quantities of the same dimension). The basic unit vectors of an orthonormal $Oxyz$ axis system are often denoted : $\vec{i}$, $\vec{j}$ and $\vec{k}$. Another notation used in English-speaking countries is $\hat{i}$, $\hat{j}$ and $\hat{k}$ ou $\hat{x}$, $\hat{y}$ and $\hat{z}$. $\hat{x}$ is pronounced 'x-hat'. But we may also come across other triplets such as ($\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$) ou ($\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$), etc.

1.3.4 Components of a vector : expression of a vector with respect to a system of axes

Any vector $\vec{V}$ can always be written as the sum of two vectors. Obviously, there are infinitely many pairs of vectors whose sum gives $\overrightarrow{V}$. But if we choose any two directions 1 and 2, then there will be a single pair of vectors $\overrightarrow{V_1}$ (along direction 1) and $\overrightarrow{V_2}$ (along direction 2) that satisfy $\overrightarrow{V} = \overrightarrow{V_1} + \overrightarrow{V_2}$. $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ are called (vector) components of $\vec{V}$ along directions 1 and 2 respectively.

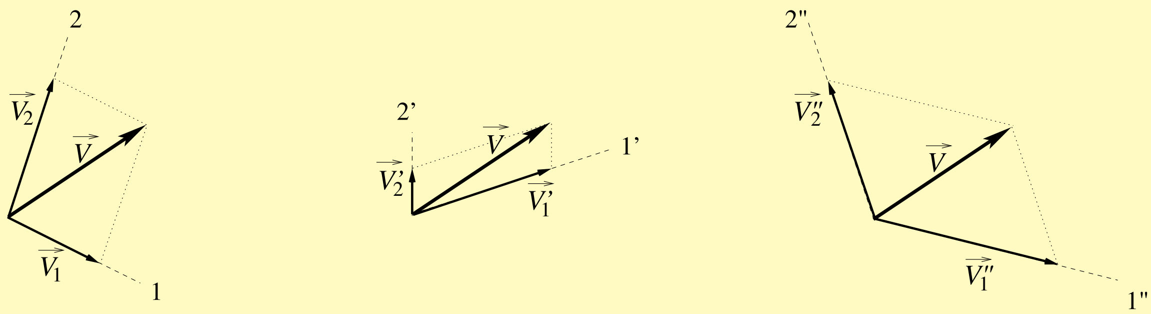

FIGURE 1.4 – Any vector could be resolved into an infinite number of pairs of components. But there is only one pair possible for each couple of directions.

Components with respect to an orthonormal axis system In one dimension, we need one direction. In figure 1.5 (a) we have : $\vec{V} = -3\vec{i}$, which means that the (scalar) component of $\vec{V}$ that goes with $\vec{i}$ is -3. *Here, the component is simply designated by the algebraic value of the vector component.*

In two or three dimensions, two or three directions are needed. Of particular importance is the case in which these directions are mutually perpendicular. It's then more convenient to work relative to a system of axes chosen along these directions.

Figure (b) shows : $\vec{V} = 5\vec{i} + 1.5\vec{j}$, which means that the component of $\vec{V}$ along x is 5 and that along y is 1.5. Figure (c) shows : $\vec{V} = \vec{V_1} + \vec{V_2} + \vec{V_3} = 2.5\,\vec{i} + 4\,\vec{j} + 3\,\vec{k}$, which means that the components of $\vec{V}$ are $+2.5$ along x, $+4$ along y and $+3$ along z. V_{xy} and V_{xz} are the orthogonal projections of $\vec{V}$ on the xy and yz

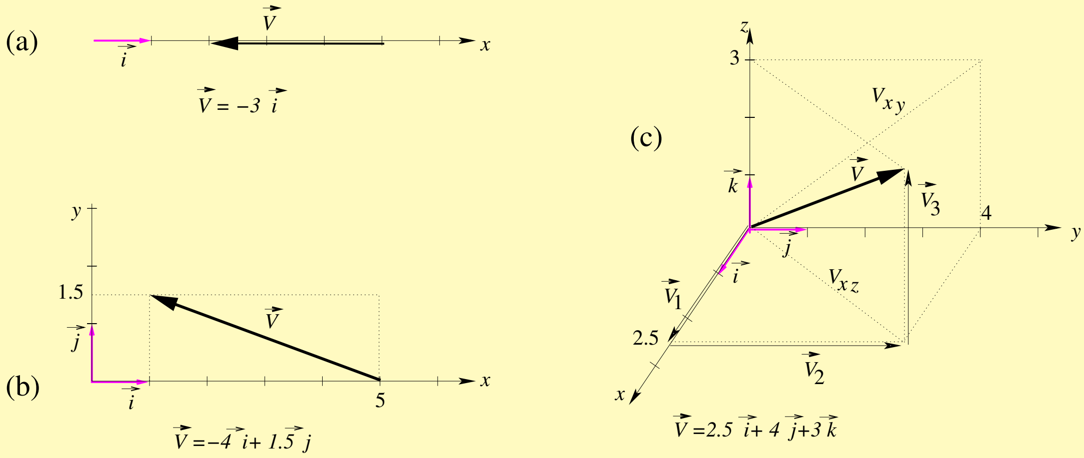

FIGURE 1.5 – 1-, 2-, 3-dimension vector

planes respectively.

Generally speaking, in an orthonormal basis $(\vec{\imath},\vec{\jmath},\vec{k})$, a vector $\vec{a}$ is written as : $\vec{a} = a_x\,\vec{\imath} + a_y\,\vec{\jmath} + a_z\,\vec{k}$. It's easy to check (Pythagorean theorem) that the modulus of $\vec{a}$ is given by :

$$\boxed{a^2 = a_x^2 + a_y^2 + a_z^2 \text{ ou encore } a = \sqrt{a_x^2 + a_y^2 + a_z^2}} \tag{1.1}$$

If $\vec{b} = b_x\,\vec{\imath} + b_y\,\vec{\jmath} + b_z\,\vec{k}$, then the equality $\vec{a} = \vec{b}$ is equivalent to $\boxed{a_x = b_x, \quad a_y = b_y, \quad a_z = b_z}$.

1.3.5 Addition of vectors : analytical method

Let be $\vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}$ et $\vec{b} = b_x\vec{\imath} + b_y\vec{\jmath} + b_z\vec{k}$. In terms of the components, the sum $\vec{c} = \vec{a} + \vec{b}$ reads :

$$\vec{c} = (a_x + b_x)\vec{\imath} + (a_y + b_y)\vec{\jmath} + (a_z + b_z)\vec{k}, \tag{1.2}$$

which gives : $\vec{c} = c_x\vec{\imath} + c_y\vec{\jmath} + c_z\vec{k}$ with

$$\boxed{c_x = a_x + b_x, \qquad c_y = a_y + b_y, \qquad c_z = a_z + b_z}. \tag{1.3}$$

1.4 Multiplication of vectors

We saw previously that vectors can be added or substracted. In this section, we'll see that two vectors can be multiplied by dot product and cross product. Let us understand more about each of these two multiplications.

1.4.1 Scalar product or dot product

Consider two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ making an angle θ between them. [1]. Their scalar product, denoted $\overrightarrow{a} \cdot \overrightarrow{b}$ and reads a scalar b or a dot b or a point b, is defined by :

1. Between $\overrightarrow{a}$ and $\overrightarrow{b}$ there are two angles : angle β (called reentrant angle) greater than 180°, and angle θ (called salient angle) less than 180°. It is the salient angle that enters the definition of the scalar product

$$\overrightarrow{a} \cdot \overrightarrow{b} = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta \quad (1.4)$$

When there is no ambiguity, simple notation is preferred :

$$\overrightarrow{a} \cdot \overrightarrow{b} = a\, b\, \cos\theta \quad (1.5)$$

Scalar product of two vectors = modulus of one times modulus of the other multiplied by the cosine of the angle between them.

The scalar product, as its name suggests, is **a scalar**. Its sign is that of $\cos\theta$ since a and b, being moduli, are positive. Consequently, the scalar product is positive if $\cos\theta > 0$ i.e. $0 \leq \theta < \pi/2$; negative if $\cos\theta < 0$ i.e. $\pi/2 \leq \theta < \pi$, null if $a = 0$ or $b = 0$ or $\cos\theta = 0$ i.e. $\theta = \pi/2$. This last case means that if the scalar product of two (non-zero) vectors is zero, then they are **perpendicular**.

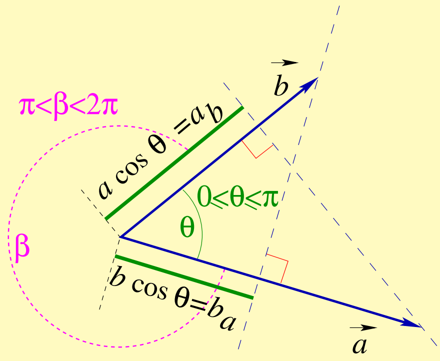

FIGURE 1.6 – Scalar product of two vectors

Properties :

1- $ab\,\cos\theta = ba\cos\theta \implies \overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{b} \cdot \overrightarrow{a} \to \to$ Scalar product is commutative.

2- $\overrightarrow{a} \cdot \overrightarrow{a} = \|\vec{a}\|\ \|\vec{a}\|\ \cos 0$, or, since $\cos 0 = 1$,

$$\boxed{\overrightarrow{a} \cdot \overrightarrow{a} = \|\vec{a}\|^2 \text{ ou bien } \overrightarrow{a}^2 = a^2} \quad (1.6)$$

The square of the vector is equal to the square of the modulus. This is an important result to remember.

3- $\alpha\vec{a} \cdot \beta\overrightarrow{b} = \beta\vec{a} \cdot \alpha\overrightarrow{b} = \alpha\beta\,\vec{a} \cdot \overrightarrow{b}$

4- $\overrightarrow{a} \cdot (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{a} \cdot \overrightarrow{c} \to$ the scalar product is distributive with respect to addition.

1.4.1.1 Equivalent definitions of the scalar product

By posing $b\cos\theta = \bar{b}_a$ = projection of $\overrightarrow{b}$ on $\overrightarrow{a}$, equation (1.5) can be rewritten as :
2nd definition

2) $\overrightarrow{a} \cdot \overrightarrow{b} = a\,\underbrace{b\cos\theta}_{\bar{b}_a} = a\,\bar{b}_a$ (= modulus of $\overrightarrow{a}$ times *algebraic* projection [2] of $\overrightarrow{b}$ on $\overrightarrow{a}$. 3rd definition

3) $\overrightarrow{a} \cdot \overrightarrow{b} = b\,\underbrace{a\cos\theta}_{\bar{a}_b} = b\,\bar{a}_b$ (= modulus of $\overrightarrow{b}$ times *algebraic* projection of $\overrightarrow{a}$ on $\overrightarrow{b}$.

1.4.1.2 Scalar product as a function of components

In an orthonormal axis system $Oxyz$ with the base $(\vec{i}, \vec{j}, \vec{k})$: $\overrightarrow{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ et $\overrightarrow{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$.

$$\begin{aligned}
\overrightarrow{a} \cdot \overrightarrow{b} &= (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) \cdot (b_x\vec{i} + b_y\vec{j} + b_z\vec{k}) && (1.7)\\
&= a_xb_x\vec{i}\cdot\vec{i} + a_xb_y\vec{i}\cdot\vec{j} + a_xb_z\vec{i}\cdot\vec{k} + a_yb_x\vec{j}\cdot\vec{i} + a_yb_y\vec{j}\cdot\vec{j} + a_yb_z\vec{j}\cdot\vec{k} && (1.8)\\
&+ a_zb_x\vec{k}\cdot\vec{i} + a_zb_y\vec{k}\cdot\vec{y} + a_zb_z\vec{k}\cdot\vec{k} && (1.9)
\end{aligned}$$

2. The word algebraic here means that the projection can take on negative values. This is what happens when the angle θ is between $\pi/2$ and π .

From the definition (1.5) and the fact that $\vec{i}, \vec{j}, \vec{k}$ are unitary, we have : $\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$, and $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{i} = \vec{i} \cdot \vec{k} = \vec{k} \cdot \vec{i} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{j} = 0$. It follows :

$$\overrightarrow{a} \cdot \overrightarrow{b} = a_x b_x + a_y b_y + a_z b_z \tag{1.10}$$

Applying (1.6) and (**??**) we get : $a^2 = a_x^2 + a_y^2 + a_z^2$, we find, of course, the result (1.1).

1.4.2 Vector product or cross product

The vector product of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ **is a vector**
- of modulus $ab \sin\theta$, θ being the salient angle (the smallest, $0 < \theta < \pi$) between the two vectors,

- of bluedirection perpendicular to the plane of the two vectors, and
- its underline is such that it forms a direct trihedron with $veCa$ and $veCb$ (see below).
The vector product
- is written as $veCa \times \overrightarrow{b}$.
- reads a vector b or a cross b.

Defining the unit vector $\vec{n}$ carried by $\overrightarrow{a} \times \overrightarrow{b}$ and of the same direction, the vector product takes the expression :

$$\overrightarrow{a} \times \overrightarrow{b} = ab \sin\theta \, \vec{n}. \tag{1.11}$$

There are several rules for determining the direction of $\overrightarrow{a} \times \overrightarrow{b}$. The figure below shows the two most commonly used rules.

1) Right hand thumb rule : The fingers of the right hand (except the thumb) are curved in the direction from the first vector to the second. The raised thumb indicates the direction of their vector product.
2) The screw-cap rule (for a bottle of water, for example) : Imagine that the two vectors rest on the cap as shown in figure 1.7. If $\overrightarrow{a}$ towards $\overrightarrow{b}$ goes in the sense the opening, then the cap comes out and $\overrightarrow{a} \times \overrightarrow{b}$ is directed towards the outside of the bottle. If $\overrightarrow{a}$ towards $\overrightarrow{b}$ goes in the closing direction, then the cap moves inwards and $\overrightarrow{a} \times \overrightarrow{b}$ is directed towards the inside of the bottle.

Properties
i) $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}$ because the direction of $\vec{n}$ is reversed when we turn from $\overrightarrow{b}$ to $\overrightarrow{a}$. The vector product is *anticommutative*.
ii) The modulus $ab \sin\theta$ of the vector product can be interpreted geometrically as the surface of the parallelogram formed by the two vectors.
iii) The vector product of two collinear vectors (parallel or antiparallel) is zero because the sine cancels out for $\theta = 0$ or π. In particular, $\overrightarrow{a} \times \overrightarrow{a} = \vec{0}$.
iv) $\alpha \overrightarrow{a} \times \beta \overrightarrow{b} = \alpha\beta (\overrightarrow{a} \times \overrightarrow{b})$
v) $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{c}$
vi) $(\overrightarrow{a} + \overrightarrow{b}) \times \overrightarrow{c} = \overrightarrow{a} \times \overrightarrow{c} + \overrightarrow{b} \times \overrightarrow{c}$

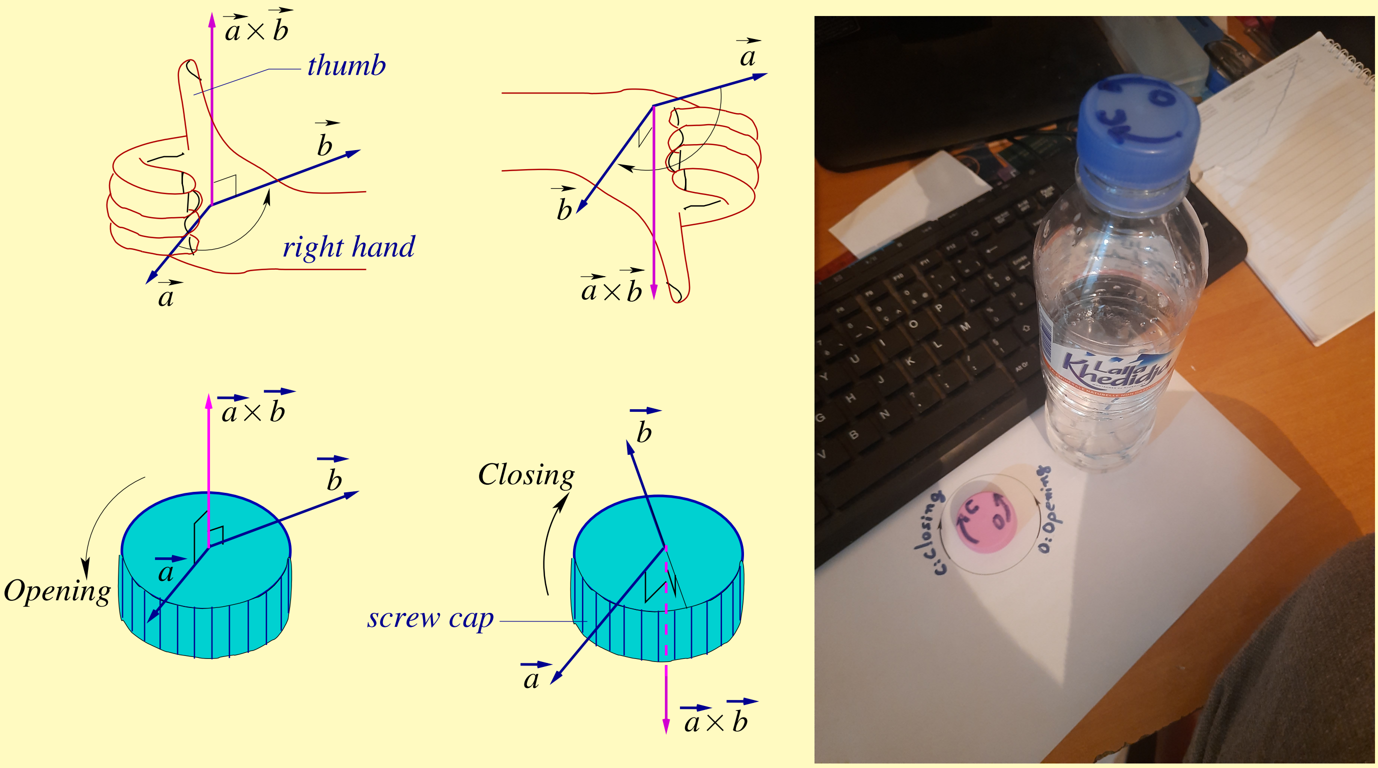

FIGURE 1.7 – Sense of the vector product : right hand and screw cap rules

1.4.2.1 Analytical expression of the vector product

We have : $\overrightarrow{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$, and $\overrightarrow{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$.

$$\begin{aligned}
\overrightarrow{a} \times \overrightarrow{b} &= (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) \times (b_x\vec{i} + b_y\vec{j} + b_z\vec{k}) && (1.12)\\
&= a_xb_x\,\vec{i}\times\vec{i} + a_xb_y\,\vec{i}\times\vec{j} + a_xb_z\,\vec{i}\times\vec{k} && (1.13)\\
&+ a_yb_x\,\vec{j}\times\vec{i} + a_yb_y\,\vec{j}\times\vec{j} + a_yb_z\,\vec{j}\times\vec{k} && (1.14)\\
&+ a_zb_x\,\vec{k}\times\vec{i} + a_zb_y\,\vec{k}\times\vec{j} + a_zb_z\,\vec{k}\times\vec{k} && (1.15)
\end{aligned}$$

But, according to the definition of the vector product, we have : [3].

$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = \vec{0} \tag{1.16}$$

$$\vec{i}\times\vec{j} = \vec{k},\quad \vec{i}\times\vec{k} = -\vec{j},\quad \vec{j}\times\vec{i} = -\vec{k},\quad \vec{j}\times\vec{k} = \vec{i},\quad \vec{k}\times\vec{i} = \vec{j},\quad \vec{k}\times\vec{j} = -\vec{i}. \tag{1.17}$$

It follows :

$$\begin{aligned}
\overrightarrow{a} \times \overrightarrow{b} &= \vec{0} + a_xb_y\,\vec{k} - a_xb_z\vec{j} - a_yb_x\,\vec{k} + \vec{0} + a_yb_z\,\vec{i} + a_zb_x\,\vec{j} - a_zb_y\,\vec{i} + \vec{0} && (1.18)\\
&= (a_yb_z - a_zb_y)\,\vec{i} + (a_zb_x - a_xb_z)\,\vec{j} + (a_xb_y - a_yb_x)\vec{k} && (1.19)
\end{aligned}$$

Using determinants, we can rewrite this expression in a more concise and practical way :

3. To find the relationships (1.17) without calculation, we can use this mnemonic : we write the list $\vec{i},\vec{j},\vec{k},\vec{i},\vec{j},\vec{k}$ in this order and the vector product of two consecutive vectors is then given by the next vector on the list with a + sign if we go from left to right and − if we go from right to left. For example $\vec{i}\times\vec{j} = \vec{k}$ but $\vec{k}\times\vec{j} = -\vec{i}$.

$$\overrightarrow{a} \times \overrightarrow{b} = + \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k}, \tag{1.20}$$

The equation (1.20) can be put in the form of a 3×3 determinant :

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1.21}$$

Note the alternating signs +,−,+ in the equation (1.20), which we shouldn't forget to apply when using the equation (1.21).

1.4.3 Scalar triple product or mixed product

The mixed product of any three vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ is the scalar defined by :

$$veCacdot(\overrightarrow{b} \times veCc).$$

The mixed product, also called scalar triple product, can be positive, negative or zero. It cancels out if :
- at least one of the three vectors is zero,
- two of the three vectors are proportional (parallel). In fact (we won't demonstrate this here), the mixed product is invariant to circular permutation of the three vectors, i.e. :

$$\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) = \overrightarrow{b} \cdot (\overrightarrow{c} \times \overrightarrow{a}) = \overrightarrow{c} \cdot (\overrightarrow{a} \times \overrightarrow{b}).$$

One of the above vector products gives 0 if two of the three vectors are parallel (or antiparallel),
- the three vectors are coplanar. Indeed, if the three vectors lie in the same plane, then $(\overrightarrow{b} \times \overrightarrow{c})$ is a vector perpendicular to the plane, and therefore to $\overrightarrow{a}$ (since $\overrightarrow{a}$ lies in the same plane), and consequently the scalar product $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c})$ is zero.

This property is very useful for demonstrating that three vectors with common origin lie in the same plane.

Note that by virtue of the commutativity of the scalar product, the previous equality $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) = \overrightarrow{c} \cdot (\overrightarrow{a} \times \overrightarrow{b})$ can be rewritten as :

$$\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{c},$$

a result which shows that the mixed product does not change if we exchange the symbols $\cdot$ and $\times$. For this reason, it is sometimes referred to as $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ without having to specify where to place the $\cdot$ and $\times$.

In an orthonormal basis $(\vec{i},\vec{j},\vec{k}))$, the mixed product can be written (using (1.20)) :

$$(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{c} = \left(+ \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k} \right) \cdot \left(c_x\vec{i} + c_y\vec{j} + c_z\vec{k} \right) \tag{1.22}$$

$$= + \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} c_x - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} c_y + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} c_z \tag{1.23}$$

The latter expression is also that of a 3×3 determinant :

$$\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{1.24}$$

With respect to an orthonormal direct basis $(\vec{i},\vec{j},\vec{k})$), a positive mixed product $\overrightarrow{a}\cdot(\overrightarrow{b}\times\overrightarrow{c})$ means that the three vectors form a direct trihedron. If, on the other hand, $\overrightarrow{a}\cdot(\overrightarrow{b}\times\overrightarrow{c})$ is negative, then the trihedron $(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c})$ is indirect.
Note : The absolute value of a mixed product gives the volume of the parallelepiped formed by the three vectors.

1.4.4 Vector product of three vectors or double vector product

With three vectors $\overrightarrow{a}$, $\overrightarrow{b}$ et $\overrightarrow{c}$ we can form two double vector products : $\overrightarrow{a}\times(\overrightarrow{b}\times\overrightarrow{c})$ and $(\overrightarrow{a}\times\overrightarrow{b})\times\overrightarrow{c}$.
We admit that (but it can be demonstrated) :

$$\overrightarrow{a}\times(\overrightarrow{b}\times\overrightarrow{c}) = \overrightarrow{b}(\overrightarrow{a}\cdot\overrightarrow{c})-\overrightarrow{c}(\overrightarrow{a}\cdot\overrightarrow{b}) \tag{1.25}$$

$$(\overrightarrow{a}\times\overrightarrow{b})\times\overrightarrow{c} = \overrightarrow{b}(\overrightarrow{a}\cdot\overrightarrow{c})-\overrightarrow{a}(\overrightarrow{b}\cdot\overrightarrow{c}) \tag{1.26}$$

It's easy to see that with the change $\overrightarrow{c}=\overrightarrow{a'}$, $\overrightarrow{a}=\overrightarrow{b'}$ and $\overrightarrow{b}=\overrightarrow{c'}$, the formula (1.26) becomes, thanks to the anticommutativity of the vector product and the commutativity of the scalar product : $\overrightarrow{a'}\times(\overrightarrow{b'}\times\overrightarrow{c'})=\overrightarrow{b'}(\overrightarrow{a'}\cdot\overrightarrow{c'})-\overrightarrow{c'}(\overrightarrow{a'}\cdot\overrightarrow{b'})$. We recover the form (1.25).
You can quickly find these formulas using this mnemonic way : *we take out the middle vector, put a − sign in front of the other vector in the parenthesis and add the scalar products of the other two vectors*. Practise using this method to find the relationships (1.25) (1.26).

1.5 A few mathematical reminders

1.5.1 Trigonometry

The trigonometric circle is a circle with radius equal to 1, centered at the origin O of the x-axis and y-axis. The coordinates x and y of a point P on the circle simply correspond to the cosine and sine of the angle θ made by the vector $\overrightarrow{OP}$ with the x-axis : $x=\cos\theta$ and $y=\sin\theta$. This circle can help you locate the remarkable values below on $[0{,}2\pi]$.
$\cos 0=\cos 2\pi=1,\quad \cos\pm\pi/2=0,\quad \cos\pm\pi/3=1/2,\quad \cos\pm\pi/6=\sqrt{3}/2,\quad \cos\pm\pi=-1.$
$\sin 0=\sin\pm\pi=\sin 2\pi=0,$
$\sin\pm\pi/2=\pm 1\rightarrow \arcsin 1=\pi/2,$
$\sin\pm\pi/3=\pm\sqrt{3}/2\rightarrow \arcsin\sqrt{3}/2=\pi/3,$
$\sin\pm\pi/6=\pm 1/2\rightarrow \arcsin 1/2=\pi/6.$

$\arcsin 0=0$	$\arcsin 1/2=\pi/6$	$\arcsin\sqrt{2}/2=\pi/4$
$\arcsin\sqrt{3}/2=\pi/3$	$\arcsin 1=\pi/2$	
$\arccos 0=1$	$\arcsin 1/2=\pi/3$	$\arcsin\sqrt{2}/2=\pi/4$
$\arcsin\sqrt{3}/2=\pi/6$	$\arccos 1=0$	
$\arctan 0=0$	$\arctan\sqrt{3}/3=\pi/6$	$\arctan 1=\pi/4$
$\arctan\sqrt{3}=\pi/3$	$\arctan\infty=\pi/2$	

$\forall\,\alpha$ and β real, one has :
$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta,$
$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta,$
$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta,$
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$

From previous relations, one deduces that :
$\cos^2\alpha + \sin^2\alpha = 1$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$
$$= 2\cos^2\alpha - 1$$
$$= 1 - 2\sin^2\alpha$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$\forall k$ integer and $\forall\theta$ real, we have :
$\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$
$\tan\theta = \sin\theta/\cos\theta \to \tan(-\theta) = -\tan\theta$
$\tan\theta = \sin\theta/\cos\theta \to \tan(-\theta) = -\tan\theta$
$\tan(2x) = 2\tan x/(1-\tan^2 x)$

$\sin(\pi/2-\theta) = \cos\theta\,;\ \cos(\pi/2-\theta) = \sin\theta\,;$
$\sin(\pi/2+\theta) = \cos\theta\,;\ \cos(\pi/2+\theta) = -\sin\theta\,;$
$\sin(\pi-\theta) = \sin\theta\,;\ \cos(\pi-\theta) = -\cos\theta\,;$
$\sin(\pi+\theta) = -\sin\theta\,;\ \cos(\pi+\theta) = -\cos\theta.$

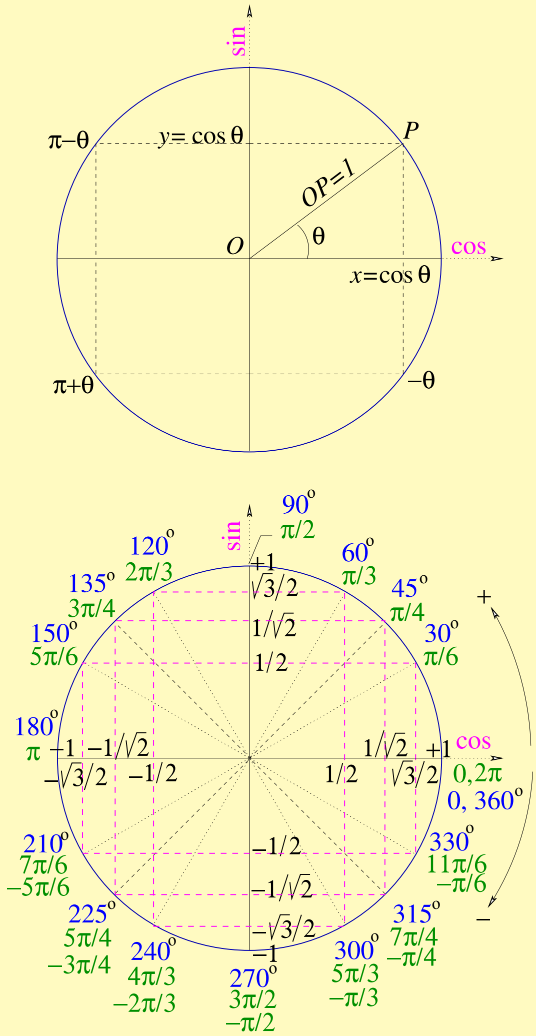

FIGURE 1.8 – Trigonometric circle

1.5.2 Derivatives and integrals with respect to x

The derivative at x of a scalar function $f(x)$ is by definition : $f'(x) = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}$ where Δx is an increase in x and Δf is the corresponding increase of f. The derivative function f' is also given by $\frac{df}{dx}$.
Some properties :
$d(f\pm g)/dx = df/dx \pm dg/dx$
$d(\alpha f)/dx = \alpha df/dx$
$d(fg)/dx = df/dx + dg/dx$
$d(f/g)/dx = (gdf/dx + fdg/dx)/g^2$
$(u^\alpha)' = \alpha u' u^{\alpha-1} \to \int x^\alpha dx = x^{\alpha+1}/(\alpha+1)$ + constant
$(\sqrt{u})' = u'/(2\sqrt{u})$
$(\sin x)' = \cos x \to \int \cos x dx = \sin x +$ constant
$(\cos x)' = -\sin x\ \int \sin x dx = -\cos x +$ constant
$(\tan x)' = 1+\tan^2 x = 1/\cos^2 x \to \int(1/\cos^2 x)dx = \tan x +$ constant

$(\arcsin x)' = 1/\sqrt{1-x^2} \to \int 1/\sqrt{1-x^2}dx = \arcsin x$ + constant
$(\arccos x)' = -1/\sqrt{1-x^2} \to \int 1/\sqrt{1-x^2}dx = -\arccos x$ + constant
$(\arctan x)' = 1/(1+x^2) \to \int 1/(1+x^2)dx = \arctan x$ + constant

$(\ln(x))' = 1/x \to \int 1/x dx = \ln(x) +$ constant
$(\exp(x))' = \exp(x) \to \int \exp(x)dx = \exp(x) +$ constant

1.5.3 Multi-variable scalar function

Example : $f(x,y,z) = x^3 + 2z/y + zx$
Partial derivative with respect to x (notation $\partial f/\partial x$) : during derivation, y and z are treated as constants,

$\partial f/\partial x = 3x^2 + z$, $\partial f/\partial y = -2z/y^2$ et $\partial f/\partial z = 2/y + x$.

Total differential : $df = \dfrac{\partial f}{\partial x}\,dx + \dfrac{\partial f}{\partial y}\,dy + \dfrac{\partial f}{\partial z}\,dz = (3x^2 + 2/y + z)\,dx + (-2x/y^2)\,dy + xdz.$

1.5.4 Derivation, differentiation and vector integration

$\vec{V}$ depends on a single variable t : the derivative is written as $\frac{d\vec{V}}{dt}$ and is calculated as for a scalar function. If in a reference frame $(Oxyz)$ $\vec{V} = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$ then in this frame
1) $\frac{d\vec{V}}{dt} = dV_x/dt\,\vec{i} + dV_y/dt\,\vec{j} + dV_z/dt\,\vec{k}$, and 2) $\int \vec{V}\,dt = \vec{i}\int V_x\,dt + \vec{j}\int V_y\,dt + \vec{k}\int V_z\,dt$.

Other properties :

3) $\frac{d(\alpha\vec{V})}{dt} = \frac{d\alpha}{dt}\vec{V} + \alpha\frac{d\vec{V}}{dt}$ 4) $\frac{d(\vec{V}_1+\vec{V}_2)}{dt} = \frac{d\vec{V}_1}{dt} + \frac{d\vec{V}_2}{dt}$ 5) $\frac{d(\vec{V}_1\cdot\vec{V}_2)}{dt} = \frac{d\vec{V}_1}{dt}\cdot\vec{V}_2 + \vec{V}_1\cdot\frac{d\vec{V}_2}{dt}$ 6) $\frac{d(\vec{V}_1\times\vec{V}_2)}{dt} = \frac{d\vec{V}_1}{dt}\times\vec{V}_2 + \vec{V}_1\times\frac{d\vec{V}_2}{dt}$
If $\vec{V}$ depends on several variables x,y,z, the differential is written as 7) $d\vec{V} = \frac{\partial\vec{V}}{\partial x}\,dx + \frac{\partial\vec{V}}{\partial y}\,dy + \frac{\partial\vec{V}}{\partial z}\,dz$

1.6 Greek alphabet letters

A number of Greek letters are widely used in mathematics and physics to name numbers, variables or functions. Some examples : Σ to denote a sum of multiple terms, the number π (or Archimedes' constant $\pi = 3.14159265...$), Δ for the Laplacian operator, ω for the speed of rotation, Ω for the electrical resistance unit, θ and ϕ for spherical coordinates, etc.

So it's important to know what they look like, and how to read and write them. They are summarized in the table below.

Uppercase	Lowercase	Letter name		Uppercase	Lowercase	Letter name
A	α	alpha		**N**	ν	nu
B	β	bêta		Ξ	ξ	ksi/xi
Γ	γ	gamma		**O**	**o**	omicron
Δ	δ	delta		Π	π et ϖ	pi
E	ϵ et ε	epsilon		**P**	ρ et ϱ	rhô
Z	ζ	zêta		Σ	σ et ς	sigma
H	η	êta		**T**	τ	tau
Θ	θ et ϑ	thêta		Υ	υ	upsilon
I	ι	iota		Φ	ϕ et φ	phi
K	κ et $\varkappa$	kappa		**X**	χ	khi/chi
Λ	λ	lambda		Ψ	ψ	psi
M	μ	mu		Ω	ω	omega

TABLE 1.2 – The letters of the Greek alphabet with their names and pronunciation in English. You'll notice that some of them (colored letters **Magenta**) have the same spelling as Latin letters and will be considered as such.