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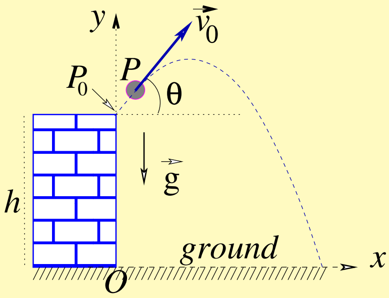
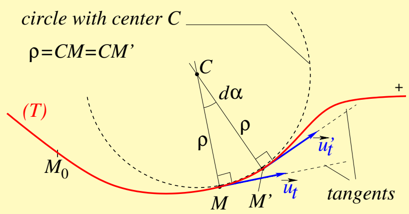


The equation of the trajectory of P is : $y = -\frac{g}{2v_0^2\cos^2\theta}x^2 + \tan\theta\, x + h$. The speed of the ball as it hits the ground is $v = \sqrt{v_0^2 + 2gh}$.
In terms of $\dot{x}$, $\dot{y}$, $\ddot{x}$, and $\ddot{y}$, the radius of curvature of M (right figure) reads : $\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{|\dot{x}\ddot{y}-\dot{x}\ddot{y}|}$.



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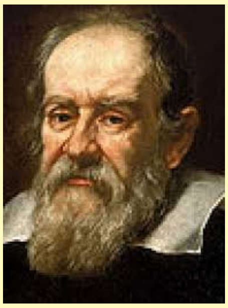
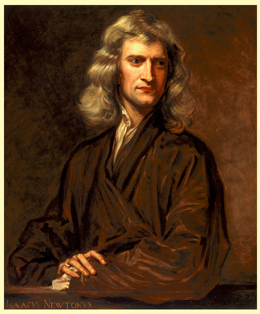


Galileo (left portrait) writes : "Aristotle declares that a 100-pound iron ball has already descended 100 cubits when a 1-pound ball has traveled only one cubit. I affirm that the two balls arrive together."

In 1687, Newton (right portrait) published the mathematical principles of natural philosophy (*Philosophiae naturalis principia mathematica*). In it, he described his discoveries on universal gravitation and the three famous laws, known as Newton's Laws. These laws describe the physical phenomena of inertia and the forces exerted on objects.



Above, the blue figure (left) shows the trajectory of a small ball P launched from the top of a building (height h) with a velocity vecv_0 making an angle θ with the horizontal. The expressions to its right give the equation of the ball's trajectory and its velocity when it hits the ground. The figure on the far right defines the radius of curvature at a point on the trajectory. When M' tends towards M (trajectory (T) in red), the normals to the tangents at M and M' meet at a point C called the center of curvature. The lengths CM and CM' are then equal to a quantity ρ called radius of curvature. A circle with center C and radius ρ will necessarily pass through M and M'. As for the two portrait photos below, they are of Galileo (left) and Newton (right), two great scientists who left their mark on the history of science and contributed greatly to mechanics.

Prof. M. A. Belkhir, année 2023-2024

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Chapitre 2

Kinematics of the point

2.1 Physics, mechanics and kinematics

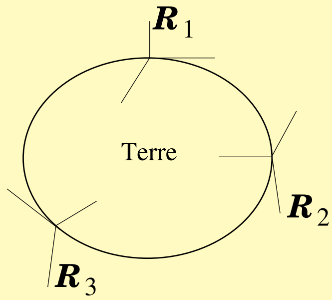
The word *physics* comes from ancient Greek ($\eta\ \varphi\upsilon\sigma\iota\kappa\eta$: pronounced ê physikê) and literally means knowledge of nature. The word 'nature' refers to the natural world around us. Physics is therefore the science that studies matter and its properties, from the most fundamental particles to the largest systems and the entire universe. To do this, physics relies on experiments, measurements and analyses ; it uses postulates, principles, concepts, models and mathematical theories to identify the laws that govern the phenomena we see around us. This is how Kepler's laws, Newton's laws, Coulomb's laws, etc. came into being.
Mechanics is the part of physics concerned with the study of the motion and equilibrium of bodies. It comprises three parts : kinematics, dynamics and statics.

Kinematics, to which we devote this chapter, studies the motion of objects in space and time, disregarding the causes (forces) that cause this motion.

2.2 Some notions

What is meant by the word "point" in point mechanics ? When you look at a moving object from a fairly long distance, a car for instance, the motion is the same for the front, the rear, the wheels, the passengers, etc. You can't see details such as the vibrations of the bodywork or the rotation of the wheels. Here, all such information is disregarded, with the result that the motion is the same for all points of the car. Therefore, to study the motion of the car, we simply need to track the motion of one of its points. The car as a whole can therefore be considered a single point. For the purposes of this course, this is how we picture a material object in motion.
We also use the words *particle* or *mobile* to designate a material point.

Reference frame : a reference frame is a solid relative to which the position or motion of an object can be located. An object cannot be referenced to the "void". Various coordinate systems can be attached to this solid. That's why a reference frame is, by extension, an arbitrary set of axes with reference to which the position or motion of a body is described or physical laws are formulated. When studying the motion of objects on or near the Earth's surface, the Earth is the most appropriate solid of reference. Various reference systems can be attached to it : R_1, R_2, R_3, ...Such reference frame can be designated by the set of axes and the origin, for instance, $Oxyz$ (i.e. $\mathcal{R} = \mathcal{O}$§†‡). The origin O is, of course, a point of the solid of refecrence.



Assuming a direct $Oxyz$ coordinate system, with a base vector set $\{\vec{i}, \vec{j}, \vec{k}\}$, the position of a particule P

relative to $Oxyz$ is :
$\overrightarrow{OP} = x_p\vec{i} + y_p\vec{j} + z_p\vec{k}$.
Finally, some motions are described in other frames of reference : the geocentric frame of reference for the motion of a satellite around the Earth, Kepler's heliocentric frame of reference for the motion of the planets around the Sun.

2.3 Position vector, velocity and acceleration

2.3.1 Time

Any phenomenon that recurs at fixed time intervals can be used as a clock. For example, the day corresponds to a complete rotation of the Earth around its axis, and the year corresponds to a complete rotation of the Earth around the sun. When an object moves, its position (x,y,z) changes over time, denoted by a real t. At each instant t, the position is $(x(t), y(t), z(t))$. *Instant* or *date*, are two synonyms often used in place of the word time.

2.3.2 Position vector

Relative to $Oxyz$, the (vector) position of M is $\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$. Sometimes we write $\overrightarrow{OM} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ to recall that x, y and z are generally functions of time.

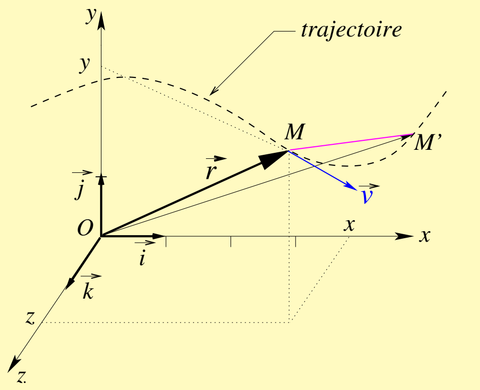


FIGURE 2.1 – *Position, vitesse et accélération*

Trajectory : The trajectory is the set of positions occupied by M over time. The relations giving $x = x(t)$, $y = y(t)$ and $z = z(t)$ constitute the parametric equation of the trajectory. The Cartesian equation is obtained by eliminating t between x, y, and z.

2.3.3 Velocity

Average velocity :
Consider the motion of a particle. Let M be the particle's position at date t and M' its position at date t', $(t' > t)$. The average velocity between M and M' is given by :

$$\vec{v}_{aver} = \frac{\overrightarrow{OM'} - \overrightarrow{OM}}{t' - t} = \frac{\overrightarrow{MM'}}{t' - t} = \frac{\Delta\overrightarrow{OM}}{\Delta t} \tag{2.1}$$

The vector $\overrightarrow{MM'}$ (Figure 2.1) is the displacement vector of the particle from M to M'. The direction of $\vec{v}_{aver}$ is that of $\overrightarrow{MM'}$. Velocity is a vector quantity. A mobile that is in M at time t, moves and then returns to M at time t' has an average velocity of zero (because the displacement vector between t and t' is zero : $\overrightarrow{MM'} = \overrightarrow{MM} = \vec{0}$). Velocity is a vector quantity that gives the rate at which an object covers a displacement.

A plane flies from Algiers to Paris and then returns to Algiers. The distance Algiers to Paris is 1400 km. The total duration of the round trip is 4 hours. In physics, its average velocity is zero because it has returned to the starting point (final position = starting position). However, the average *speed* is given by the total distance covered, divided by the total duration of the journey. That is $(1400\,\text{km} + 1400\,\text{km})/(4\,\text{hours}) = 700$ km/hour. Speed is a scalar quantity that gives the rate at which an object covers a distance.

Instantaneous velocity :

Average velocity is easy to conceive, and is given by the displacement vector from M to M' divided by the time $t' - t$ taken to cover the displacement.

Instantaneous velocity is the velocity at an instant t. It is obtained from $\vec{v}_{aver}$ by taking a time t' that is infinitely close to t :

$$\overrightarrow{v} = \lim_{t' \to t} \frac{\overrightarrow{OM'} - \overrightarrow{OM}}{t' - t} = \frac{d\overrightarrow{OM}}{dt}, \tag{2.2}$$

The velocity $\overrightarrow{v}$ is given by the derivative of the position $\overrightarrow{OM}$ with respect to time t.[1]

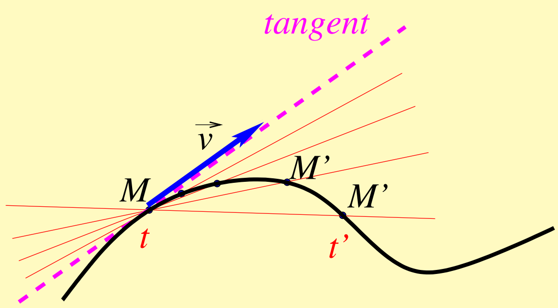


FIGURE 2.2 – Velocity is tangent to the trajectory and oriented in the direction of motion

Direction and sense of $\overrightarrow{v}$:
Equation (2.2) shows that $\vec{v}$ has the same direction and sense as the vector $\overrightarrow{OM'} - \overrightarrow{OM} = \overrightarrow{MM'}$. As M' approaches M (see figure 2.2), the straight line (MM') becomes closer and closer to the tangent at M to the trajectory, (the dotted magenta line on the figure) until it is superimposed on it when M' is infinitely close to M. *The velocity is therefore tangent to the trajectory at every instant, and is oriented in the direction of* $\overrightarrow{MM'}$*, i.e. in the direction of motion.*

Velocity components in Cartesian coordinates :
$\overrightarrow{v} = d\overrightarrow{OM}/dt = d(x\,\vec{i} + y\,\vec{j} + z\,\vec{k})/dt$. Therefore,

$$\overrightarrow{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \tag{2.3}$$

dx/dt is sometimes written more concisely $\dot{x}$, pronounced 'x-dot'. With this notation, the equation (2.3) is written :

$$\overrightarrow{v} = \dot{x}\,\vec{i} + \dot{y}\,\vec{j} + \dot{z}\,\vec{k} \tag{2.4}$$

The modulus of velocity is then :

$$\|\overrightarrow{v}\| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{2.5}$$

Velocity measures the rate of change of position with respect to time. If position is measured in meters (m), velocity will be in meters per second m/s (also noted $\mathrm{m{\cdot}s^{-1}}$ or $\mathrm{m\,s^{-1}}$).

2.3.4 The acceleration vector

A body undergoes acceleration when, in the course of time, its velocity changes modulus (e.g. a car moving in straight road and braking) or direction (e.g. a car taking a bend).

Acceleration gives the rate of change of velocity with respect to time. Average acceleration :

$$\overrightarrow{a}_{moy} = \frac{\overrightarrow{v'} - \overrightarrow{v}}{t' - t} = \frac{\Delta\vec{v}}{\Delta t} \tag{2.6}$$

1. If $\overrightarrow{OM}$ is denoted by $\vec{r}$, then we can write $\vec{v}_{aver} = \Delta\vec{r}/\Delta t$, and $\vec{v} = d\vec{r}/dt$, etc.

Instantaneous acceleration :

$$\overrightarrow{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \tag{2.7}$$

and since $\vec{v} = d\overrightarrow{r}/dt$, we also :

$$\overrightarrow{a} = \frac{d^2\overrightarrow{r}}{dt^2} \tag{2.8}$$

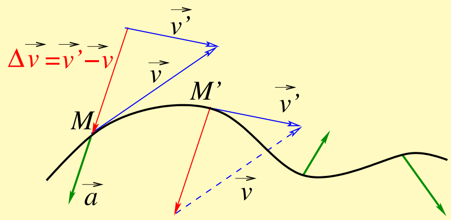


FIGURE 2.3 – Acceleration is always oriented towards the inside of the concavity of the trajectory

Orientation of $\vec{a}$ with respect to the trajectory : As shown in equation (2.6) or (2.7), the acceleration vector $\vec{a}$ has the direction and sense of $\Delta\vec{v}$. As $\Delta\vec{v}$, it is always oriented towards the inside of the concavity of the trajectory. Acceleration is measured in m/s^2, also denoted m$\cdot s^{-2}$ or ms^{-2}.

In Cartesian coordinates, the equation (2.8) can be broken down into :

$$a_x = \frac{d^2x}{dt^2} \equiv \ddot{x} \tag{2.9}$$

$$a_y = \frac{d^2y}{dt^2} \equiv \ddot{y} \tag{2.10}$$

$$a_y = \frac{d^2z}{dt^2} \equiv \ddot{z}, \tag{2.11}$$

the notation $\ddot{x}$, meaning second derivative with respect to time, reads "x two-dots".

Note : If we have the data for a motion at time t_0 (position $\vec{r}_0$, velocity $\vec{v}_0$), we can go back to the velocity $\vec{v}$ and then to the position $\vec{r}$ at time t using :

$$\vec{a} = \frac{d\vec{v}}{dt} \Longrightarrow d\vec{v} = \vec{a}dt \Longrightarrow \int_{\vec{v}_0}^{\vec{v}} d\vec{v} = \int_{t_0}^{t} \vec{a}dt$$

and then,

$$\vec{v} = \frac{d\vec{r}}{dt} \Longrightarrow d\vec{r} = \vec{v}dt \Longrightarrow \int_{\vec{r}_0}^{\vec{r}} d\vec{r} = \int_{t_0}^{t} \vec{v}dt$$

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2.4 Other coordinate systems

To describe certain types of motion, other coordinate systems are more convenient than Cartesian coordinates. Polar coordinates, cylindrical coordinates and spherical coordinates are widely used in physics. We'll introduce them in the following pages.

2.4.1 Polar coordinates

When the motion of a material point M takes place in a plane, we can locate M as a function of the distance $\rho = OM$ and the angle $\theta = (\overrightarrow{i}, \overrightarrow{OM})$. ρ and θ define the polar coordinates, which are related to the Cartesian coordinates by (see figure 2.4) :

$$x = \rho\cos\theta \tag{2.12}$$

$$y = \rho\sin\theta \tag{2.13}$$

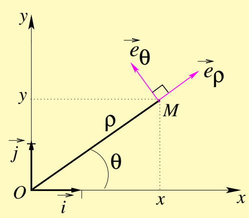


FIGURE 2.4 – Polar coordinates

The equation of the circle in Cartesian coordinates is $x^2 + y^2 = R^2$.
In polar coordinates, it is simplified to $\rho = R$.

To write vectors in the polar coordinate system, we need to define a basis. The polar basis $(\vec{e}_\rho, \vec{e}_\theta)$ is defined as follows :

$$\vec{e}_\rho = \overrightarrow{OM}/\rho, \text{ it is a unit vector along } \overrightarrow{OM} \text{ and of the same direction.} \quad (2.14)$$

$$\vec{e}_\theta = \text{ unit vector directly perpendicular to } \vec{e}_\rho, \text{ it is at } +\pi/2 \text{ from } \vec{e}_\rho. \quad (2.15)$$

In $(\vec{i}, \vec{j})$ basis, we have :

$$\vec{e}_\rho = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}, \quad (2.16)$$

$$\vec{e}_\theta = \cos(\theta + \pi/2)\,\vec{i} + \sin(\theta + \pi/2)\,\vec{j} = -\sin\theta\vec{i} + \cos\theta\vec{j}. \quad (2.17)$$

This is a local basis, attached to the point M and therefore moving with M. Let's express position, velocity and acceleration in this basis.

$$\overrightarrow{OM} = \rho\vec{e}_\rho \quad (2.18)$$

$$\overrightarrow{v} = \frac{d(\rho\vec{e}_\rho)}{dt} = \frac{d\rho}{dt}\,\vec{e}_\rho + \rho\,\frac{d\vec{e}_\rho}{dt} \quad (2.19)$$

Let's calculate $d\vec{e}_\rho/dt$:
We have $de_\rho/d\theta = -\sin\theta\vec{i} + \cos\theta\vec{j}$, which is none other than $\vec{e}_\theta$ (see equation (2.17)), so $d\vec{e}_\rho/d\theta = \vec{e}_\theta$. One more step and we're done with the derivative of $\vec{e}_\rho$.

$$\frac{d\vec{e}_\rho}{dt} = \frac{d\vec{e}_\rho}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\,\vec{e}_\theta \quad (2.20)$$

Velocity in polar coordinnates is finally :

$$\overrightarrow{v} = \dot{\rho}\,\vec{e}_\rho + \rho\dot{\theta}\,\vec{e}_\theta \quad (2.21)$$

Its modulus is : $\|\overrightarrow{v}\| = \sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2}$.
Now, let's calculate the acceleration :

$$\overrightarrow{a} = \frac{d(\dot{\rho}\,\vec{e}_\rho + \rho\dot{\theta}\,\vec{e}_\theta)}{dt} = \ddot{\rho}\,\vec{e}_\rho + \dot{\rho}\dot{\theta}\,\vec{e}_\theta + (\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\,\vec{e}_\theta + \rho\dot{\theta}^2(-\vec{e}_\rho) = (\ddot{\rho} - \rho\dot{\theta}^2)\,\vec{e}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\,\vec{e}_\theta \quad (2.22)$$

where we have taken into account : $\frac{d\vec{e}_\rho}{dt} = \dot{\theta}\vec{e}_\theta$ et $\frac{d\vec{e}_\theta}{dt} = \dot{\theta}(-\vec{e}_\rho)$. Modulus of $\overrightarrow{a}$ is : $\|\overrightarrow{a}\| = \sqrt{(\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})^2}$.

2.4.2 Cylindrical coordinates

The motion of M takes place in three-dimensional space represented by the orthonormal axis system $Oxyz$. Let m be the orthogonal projection of M on the xy plane.
The cylindrical coordinates are obtained by adding the z coordinate to the two polar coordinates ρ and θ already defined above. So, these three coordinates are $\rho = OM$, $\theta = (\overrightarrow{Ox}, \overrightarrow{Om})$, $z = \overline{mM}$ and are related to the Cartesian coordinates by :

$$x = \rho\cos\theta \quad (2.23)$$

$$y = \rho\sin\theta \quad (2.24)$$

$$z = z \quad (2.25)$$

Why are they called cylindrical coordinates ? Because when θ and z vary while fixing ρ, the point M describes a cylindrical surface.

Now, we attach to M the local basis $(\vec{e}_\rho, \vec{e}_\theta, \vec{k})$ such that $\vec{e}_\rho = \overrightarrow{Om}/Om$, $\vec{e}_\theta$ = unit vector directly perpendicular to $\vec{e}_\rho$ and $\vec{k}$ is the unit vector completing a direct orthonormal basis.

Let's express position, velocity and acceleration in this base. We won't repeat the calculations, since we've already done them in polar coordinates. We'll just add the z component.

$$\overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM} = \rho\vec{e}_\rho + z\vec{k} \tag{2.26}$$
$$\overrightarrow{v} = \dot\rho\,\vec{e}_\rho + \rho\dot\theta\,\vec{e}_\theta + \dot z\,\vec{k} \tag{2.27}$$
$$\overrightarrow{a} = (\ddot\rho - \rho\dot\theta^2)\,\vec{e}_\rho + (2\dot\rho\dot\theta + \rho\ddot\theta)\,\vec{e}_\theta + \ddot z\,\vec{k} \tag{2.28}$$

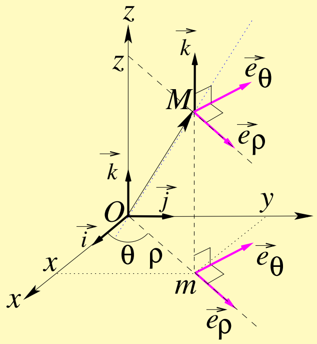


FIGURE 2.6 – Coordonnées cylindriques

2.4.3 Spherical coordinates

The spherical coordinates of M are r, θ and φ defined by : $r = OM$, $\theta = (\overrightarrow{Oz}, \overrightarrow{OM})$ et $\phi = (\overrightarrow{Ox}, \overrightarrow{Om})$, m étant, comme dans la sous-section précédente, la projection orthogonale de M sur le plan xy. θ varie entre 0 et π alors que φ va de 0 à 2π.
They are related to the Cartesian coordinates by :

$$x = r\sin\theta\cos\varphi \tag{2.29}$$
$$y = r\sin\theta\sin\varphi \tag{2.30}$$
$$z = r\cos\theta \tag{2.31}$$

When θ and φ vary while keeping r constant, the point M moves on a spherical surface of radius r, hence the name spherical coordinates.
We define a local basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ attached to M as follows :
The vector $\vec{e}_r$ is such that $\overrightarrow{OM} = r\,\vec{e}_r$, $\vec{e}_r$ is therefore the unit vector having the direction of $\overrightarrow{OM}$.
e_{theta} is the unit vector tangent at M to the (half)circle described by M when $theta$ varies, r and $varphi$ being kept constant.
$\vec{e}_\varphi$ is the unit vector that completes a direct orthonormal basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$:

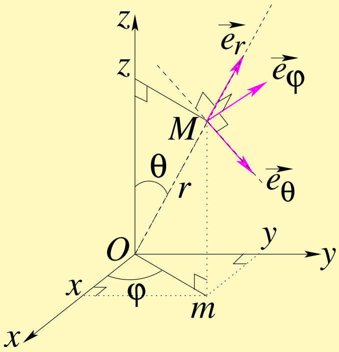


FIGURE 2.7 – Coordonnées sphériques

$$\vec{e}_\varphi = \vec{e}_r \times \vec{e}_\theta \tag{2.32}$$

It turns out that it is the vector tangent at M to the circle described by M when ϕ varies, r and θ held constant.

Let's express $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\varphi$ in $(\vec{i}, \vec{j}, \vec{k})$ basis
$\vec{e}_r = \overrightarrow{OM}/r = (x\vec{i} + y\vec{j} + z\vec{k})/r = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}$.
Calculation of $\vec{e}_\varphi$: it is given by $\vec{e}_r \times \vec{e}_\theta$. But, as we don't know e_θ, we can't use this formula. However, by noticing that $\vec{e}_r$ and $\vec{e}_\theta$ lie in $(\overrightarrow{Oz}, \overrightarrow{Om})$ plane and by considering the unit vector, let(s call it $\vec{u}$, of $\overrightarrow{Om}$, the vector $\vec{e}_\varphi$ can be written as $e_\varphi = \vec{k} \times \vec{u}$. Then
$\vec{e}_\varphi = \vec{k} \times (\cos\varphi\vec{i} + \sin\varphi\vec{j}) = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}$

Calculation of $\vec{e}_\theta$: $\vec{e}_\theta = \vec{e}_\varphi \times \vec{e}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\varphi & \cos\varphi & 0 \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{vmatrix} = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k}$.

These expressions are summarized in the following equations :

$$\vec{e}_r = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k} \tag{2.33}$$

$$\vec{e}_\theta = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k} \tag{2.34}$$

$$\vec{e}_\varphi = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j} \tag{2.35}$$

Let's express $\vec{v}$ and $\vec{a}$ in ($\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\varphi$) basis :
Velocity is obtained by deriving $\overrightarrow{OM}$ with respect to time :
$\vec{v} = d\overrightarrow{OM}/dt = d(r\,\vec{e}_r)/dt = \dot{r}\,\vec{e}_r + r d\vec{e}_r/dt$. To calculate $d\vec{e}_r/dt$, we first express the differential $d\vec{e}_r$:
$d\vec{e}_r = \frac{\partial\vec{e}_r}{\partial\theta}\,d\theta + \frac{\partial\vec{e}_r}{\partial\varphi}\,d\varphi \Longrightarrow \frac{d\vec{e}_r}{dt} = \frac{\partial\vec{e}_r}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\vec{e}_r}{\partial\varphi}\frac{d\varphi}{dt} = \dot{\theta}\frac{\partial\vec{e}_r}{\partial\theta} + \dot{\varphi}\frac{\partial\vec{e}_r}{\partial\varphi}$
$\frac{\partial\vec{e}_r}{\partial\theta} = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k} = \vec{e}_\theta$.
$\frac{\partial\vec{e}_r}{\partial\varphi} = -\sin\theta\sin\varphi\,\vec{i} + \sin\theta\cos\varphi\,\vec{j} = \sin\theta\vec{e}_\varphi$, il vient

$$d\vec{e}_r/dt = \dot{\theta}\,\vec{e}_\theta + \dot{\varphi}\sin\theta\,\vec{e}_\varphi. \tag{2.36}$$

Finally, we obtain :

$$\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta + r\dot{\varphi}\sin\theta\,\vec{e}_\varphi \tag{2.37}$$

Acceleration is obtained by deriving $\vec{v}$ with respect to time :
$\vec{a} = d\vec{v}/dt = d(\dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta + r\dot{\varphi}\sin\theta\,\vec{e}_\varphi)/dt$,
which gives :

$$\vec{a} = \ddot{r}\,\vec{e}_r + \dot{r}\,d\vec{e}_r/dt + (\dot{r}\dot{\theta} + r\ddot{\theta})\,\vec{e}_\theta + r\dot{\theta}\,d\vec{e}_\theta/dt + (\dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta + r\dot{\varphi}\cos\theta)\,\vec{e}_\varphi + r\dot{\varphi}\sin\theta d\vec{e}_\varphi/dt \tag{2.38}$$

$\frac{d\vec{e}_\theta}{dt} = \frac{\partial\vec{e}_\theta}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\vec{e}_\theta}{\partial\varphi}\frac{d\varphi}{dt} = \dot{\theta}\frac{\partial\vec{e}_\theta}{\partial\theta} + \dot{\varphi}\frac{\partial\vec{e}_\theta}{\partial\varphi} = \dot{\theta}(-\sin\theta\cos\varphi\,\vec{i} - \sin\theta\sin\varphi\,\vec{j} - \cos\theta\,\vec{k}) + \dot{\varphi}\cos\theta(-\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j})$

$$d\vec{e}_\theta/dt = -\dot{\theta}\,\vec{e}_\theta + \dot{\varphi}\cos\theta\,\vec{e}_\varphi. \tag{2.39}$$

$\frac{d\vec{e}_\varphi}{dt} = \frac{\partial\vec{e}_\varphi}{\partial\varphi}\frac{d\varphi}{dt} = \dot{\varphi}\frac{\partial\vec{e}_\varphi}{\partial\varphi} = -\dot{\varphi}(\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j})$, here the symbol (operator) ∂ can be denoted as d (right d), since $\vec{e}_\varphi$ depends on a single variable, φ in this case. Remember that we're trying to express $\vec{a}$ in ($\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\varphi$) basis. So we'll need to express $\frac{d\vec{e}_\varphi}{dt}$ in this base. To do this, we know that $\frac{d\vec{e}_\varphi}{dt}$ is perpendicular to $\vec{e}_\varphi$ (we can also see that the scalar product $\frac{d\vec{e}_\varphi}{dt}\cdot\vec{e}_\varphi$ is zero). So $\frac{d\vec{e}_\varphi}{dt}$ is in the plane of the two vectors $\vec{e}_r$ and $\vec{e}_\theta$) and can therefore be decomposed according to these two vectors. It's easy to see that : $(\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}) = \sin\theta\,\vec{e}_r + \cos\theta\,\vec{e}_\theta$, and

$$d\vec{e}_\varphi/dt = -\dot{\varphi}\sin\theta\,\vec{e}_r - \dot{\varphi}\cos\theta\,\vec{e}_\theta \tag{2.40}$$

Substituting (2.36), (2.39) and (2.40) into (2.38), we obtain :

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\right)\,\vec{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta\right)\,\vec{e}_\theta + \left(2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta\right)\,\vec{e}_\varphi \tag{2.41}$$

2.5 Order of magnitude of some examples of speed and acceleration

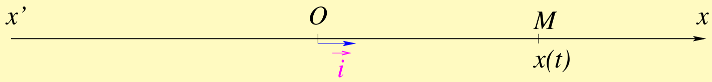
The table below gives a few orders of magnitude for speed and acceleration :

Speed				**Acceleration**		
Example	**m/s**	**km/h**		**Example**	**m/s²**	in terms of g
Escargot	0,0013	0,00468		Acceleration d'un train	0.25	$\approx$ g/40
Walking	1,3	4,5		Starting acceleration of a car	3.2	$\approx$ g/3
Galloping horse	15	55		Braking a car	8	$\approx$ 0.8 g
A car on the freeway	30	108		Gravitational acceleration g	9.8	1 g
Airliner takeoff	69	250		Rocket acceleration	90	$\approx$ 9 g
High-speed train	90	320		Acceleration supported by man	100	$\approx$ 10 g
Aircraft cruise speed	280	1000		Car hitting a wall at 100 km/h	982	$\approx$ 100 g
Speed of sound in air	340	1225		Kicked soccer ball	2946	$\approx$ 300 g
Rocket speed	7800	28000		Washing machine drum rotation	4944	$\approx$ 500 g
Earth around the Sun	30000	108000		Ball fired from a pistol	304 000	$\approx$ 31000 g
Speed of light	3×10^8	$1{,}08\times10^9$		Proton acceleration at the LHC [!]	2.1×10^{13}	$\approx 2.1\times10^{12}$ g

[!] The LHC (Large Hadron Collider) is the world's largest and most powerful particle gas pedal. Commissioned in 2008, it is located in the border region between France and Switzerland, between the north-western outskirts of Geneva and the Pays de Gex. It consists of a 27-kilometer ring of superconducting magnets and accelerator structures that increase the energy of the particles flowing through it.

2.6 Rectilinear motion (also called straight-line motion)

The motion of a material point is said to be rectilinear when its trajectory is a straight line. Such motion is simply described by taking the axis $x'Ox$ bearing the line along which the motion takes place, and provided with the unit vector $\vec{i}$. With respect to $x'Ox$, we have :



$$\overrightarrow{OM} = x\,\vec{i} \tag{2.42}$$
$$\vec{v} = v\,\vec{i} \tag{2.43}$$
$$\vec{a} = a\,\vec{i} \tag{2.44}$$

Given the expressions (2.42), (2.43) and (2.44), velocity and acceleration can be written as :

$$v\,\vec{i} = \frac{d(x\vec{i})}{dt} = \frac{dx}{dt}\,\vec{i} \Longrightarrow v = \frac{dx}{dt} \tag{2.45}$$
$$a\,\vec{i} = \frac{d(v\vec{i})}{dt} = \frac{dv}{dt}\,\vec{i} \Longrightarrow a = \frac{dv}{dt} \tag{2.46}$$

Note that in the previous relations, we were able to take $\vec{i}$ out of the derivative because it's a constant vector (it doesn't depend on t), allowing us to simplify it in the following. This is why, when doing one-dimensional kinematics, vector notation is not necessary for position, velocity and acceleration, which are simply denoted by x, v and a. Once the axis orientation is fixed, their vector character is contained in their sign.
A positive (negative) x means that M is on the positive (negative) side of the axis. A positive (negative) v means that M is moving in the positive (negative) direction of the axis. The sign of a will be interpreted according to that of v. If a has the same sign as v (i.e. both are negative or positive), then velocity increases and motion is said to be accelerated. If a and v have opposite signs (i.e. one negative and the other positive),

then the modulus of velocity decreases and motion is said to be decelerated (or retarded). We'll come back to this point a little later in the section 2.7. *In conclusion, knowledge of x, v and a completely determines the motion of M.*

Calculate v and x from a.
If we know the acceleration a of a motion between instant t_0 (position x_0, velocity v_0) and instant t (position x, velocity v) we can find the velocity and position of M using :
$a = dv/dt \Longrightarrow dv = adt \Longrightarrow \int_{v_0}^{v} dv = \int_{t_0}^{t} adt$ et $v = dx/dt \Longrightarrow dx = vdt \Longrightarrow \int_{x_0}^{x} dx = \int_{t_0}^{t} vdt$, ce qui conduit à :

$$v = v_0 + \int_{t_0}^{t} adt \tag{2.47}$$

$$x = x_0 + \int_{t_0}^{t} vdt \tag{2.48}$$

Uniform rectilinear motion

A rectilinear motion is said to be uniform when the velocity is constant, i.e., does not vary with time. If v is constant, then $dv/dt = 0 \Longleftrightarrow a = 0$. Saying that the velocity of a moving body does not vary with time means that it does not accelerate, or that its acceleration is zero. For uniform rectilinear motion (for $a = 0$), the equations (2.47) and (2.48) give :

$$v = v_0 \tag{2.49}$$

$$x = x_0 + v_0(t - t_0) \tag{2.50}$$

In uniform rectilinear motion, the speed of the moving body is constant throughout the duration of the motion, and is equal to the speed the body had at t_0. If $t_0 = 0$, the equation (2.50) becomes : $x = x_0 + v_0 t$.

Case of constant acceleration : uniformly varied rectilinear motion A constant acceleration of a m/s^2 means that the modulus of velocity increases or decreases by a m/s every second. This is called uniformly varied rectilinear motion (accelerated or retarded). For a constant value of a, integration of the equations (2.47) and (2.48) gives :

$$v = v_0 + a(t - t_0) \tag{2.51}$$

$$x = x_0 + \int_{t_0}^{t} vdt = x_0 + \int_{0}^{t-t_0} (v_0 + a(t - t_0))d(t - t_0) \rightarrow x = x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2 \tag{2.52}$$

If $t_0 = 0$, the relationships (2.51) and (2.52) become :

$$v = v_0 + at \tag{2.53}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \tag{2.54}$$

By eliminating t from the preceding equations, we obtain this important equation in kinematics : $v = v_0 + at \Longrightarrow t = \frac{v-v_0}{a} \Longrightarrow x - x_0 = v_0\frac{v-v_0}{a} + \frac{1}{2}a\frac{(v-v_0)^2}{a^2} = \frac{v^2-v_0^2}{2a}$, that is :

$$\boxed{v^2 - v_0^2 = 2a(x - x_0)} \tag{2.55}$$

Free fall : An example of uniformly varied rectilinear motion is vertical free fall. An object is said to be in free fall when it falls towards the Earth, subject only to the Earth's gravitational acceleration $\vec{g}$ (the only force acting on it is its weight). The $\vec{g}$ vector is directed at the center of the Earth, and its modulus near the

Earth's surface is 9.81 m/s^2.
In air, the fall can be considered "free" if friction and Archimedes' buoyancy can be neglected. The latter is negligible when the density of the object is greater than that of the air.
Let's take the example of a ball M released without initial velocity ($v_0 = 0$) from a certain height h. The motion is vertical, and we choose to locate it relative to a vertical axis y pointing upwards, with origin O and unit vector $vecj$.
With respect to the y axis, the acceleration, velocity and position of the ball are :

$$\vec{\mathrm{g}} = -\mathrm{g}\vec{j} \tag{2.56}$$

$$\vec{v} = v\vec{j} = \int -\mathrm{g}\vec{j}dt \rightarrow \quad v = -\mathrm{g}t + C_1 = -\mathrm{g}t \text{ car } C_1 = v_0 = 0. \tag{2.57}$$

$$\overrightarrow{OM} = y\vec{j} \rightarrow y = \int -\mathrm{g}tdt = -\frac{1}{2}\mathrm{g}t^2 + C_2 = -\frac{1}{2}\mathrm{g}t^2 + h \text{ car } C_2 = y(t=0) = h. \tag{2.58}$$

The position is $y = \int -\mathrm{g}tdt = -\frac{1}{2}\mathrm{g}t^2 + C_2 = -\frac{1}{2}\mathrm{g}t^2 + h$ car $C_2 = y(t=0) = h$.
Solve the same problem by choosing a downward axis.

2.7 Accelerated motion, decelarated motion

The motion of a moving body is accelerated if the modulus v of the velocity increases with time, which implies that v^2 also increases with time and mathematically this is expressed by :

$$\frac{dv^2}{dt} > 0 \tag{2.59}$$

Knowing that (see equation (1.6)) $v^2 = \vec{v}^2$, the previous derivative is written $d\vec{v}^2/dt = 2\,(d\vec{v}/dt)\cdot\vec{v} = 2\,\vec{a}\cdot\vec{v}$ and the equation (2.59) leads to :

$$\boxed{\vec{a}\cdot\vec{v} > 0} \tag{2.60}$$

The inequation (2.60) means that $\vec{a}$ and $\vec{v}$ make an angle of less than 90°. So, for a motion to be accelerated, the acceleration vector must be within 90° from the velocity vector.
In a decelerated motion, the modulus of velocity decreases with time and a calculation similar to the previous one leads to :

$$\boxed{\vec{a}\cdot\vec{v} < 0} \tag{2.61}$$

To decelerate a motion, the acceleration vector must form an obtuse angle (between 90° and 180°) with the velocity vector.

2.8 Two dimensional motion with constant acceleration

Motion with constant acceleration in the one-dimensional case, i.e. uniformly varied rectilinear motion, has already been described in detail. When motion takes place in two- or three-dimensional space with constant acceleration, the equations (2.53), (2.54) and (2.55) generalize and can be written in vector form :

$$\vec{v} = \vec{v}_0 + \vec{a}t \tag{2.62}$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \tag{2.63}$$

$$\vec{v}^2 - \vec{v}_0^2 = 2\vec{a}(\vec{r} - \vec{r}_0) \tag{2.64}$$

2.8.1 Case of projectile motion :

At $t = 0$ the projectile is launched from O with an initial velocity $\vec{v}_0$ making an angle θ_0 with the positive direction of the x axis. If O designates the origin of the x and y axes, at $t = 0$ we have $x_0 = 0$ and $y_0 = 0$.

The components of the initial velocity are :

$$v_{0x} = v_0 \cos\theta \qquad (2.65)$$
$$v_{0x} = v_0 \sin\theta \qquad (2.66)$$

The acceleration is $\vec{a} = \vec{g} = -g\vec{j}$, so its components are :

$$a_{0x} = 0 \qquad (2.67)$$
$$a_{0y} = -g \qquad (2.68)$$

Acceleration is zero along x, so the v_x component of velocity will retain its initial value throughout the entire movement, leading to :

$$v_x = v_0 \cos\theta_0 \qquad (2.69)$$
$$x = v_0 \cos\theta_0\, t \qquad (2.70)$$

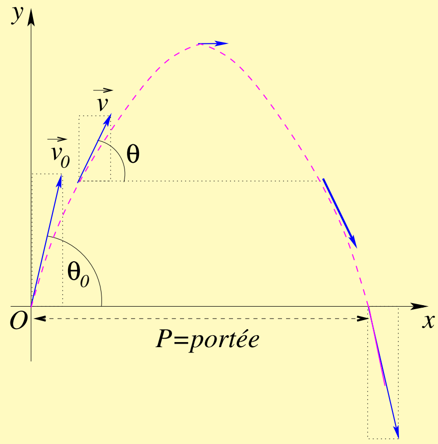


FIGURE 2.8 – Mouvement d'un projectile

Suivant l'axe y on a :

$$v_y = v_0 \sin\theta_0 - gt \qquad (2.71)$$
$$y = v_0 \sin\theta_0\, t - \frac{1}{2}gt^2 \qquad (2.72)$$

The angle θ which $\vec{v}$ makes with the x axis is given at each instant by :

$$\tan\theta = \frac{v_y}{v_x} = \frac{v_0 \sin\theta_0 - gt}{v_0 \cos\theta_0} \qquad (2.73)$$

2.8.2 Trajectory equation

To find the shape of the trajectory, we express y as a function of x. To do this, we eliminate the time t between x and y. From the equation (2.70) we derive $t = \frac{x}{v_0 \cos\theta_0}$, and by substitution in the equation (2.72), we obtain :

$$y = (\tan\theta_0)\, x - \left(\frac{g}{2v_0^2 \cos^2\theta_0}\right) x^2 \qquad (2.74)$$

The resulting equation has the form of the equation of a parabola $y = c + bx + ax^2$. The trajectory of a projectile is therefore parabolic. The distance between the starting point (here $y = 0$) and the point where the projectile returns to the same horizontal level ($y = 0$) is given by :

$$\tan\theta_0\, x - \frac{g}{2v_0^2 \cos^2\theta_0} x^2 = 0 \text{ ou } x\left(\tan\theta_0 - \frac{g}{2v_0^2 \cos^2\theta_0} x\right) = 0 \qquad (2.75)$$

This equation has two solutions :

$$x_1 = 0 \text{ et } x_2 = \frac{2v_0^2 \cos^2\theta_0}{g} \tan\theta_0 = \frac{2v_0^2 \cos\theta_0 \sin\theta_0}{g} = \frac{v_0^2 \sin 2\theta_0}{g} \qquad (2.76)$$

The first corresponds to the starting point, and is of no interest to us. The second corresponds to the distance between the launch point and the point where the projectile touches the horizontal axis x on landing. This distance, noted P, is called the horizontal range :

$$P = \frac{v_0^2 \sin 2\theta_0}{g} \qquad (2.77)$$

Note that P takes on a maximum value when $\sin 2\theta_0 = 1$, i.e. when $2\theta_0 = 90^\circ$. In other words, maximum range is obtained when the launch angle is 45°.

2.9 Curvilinear motion

Curvilinear motion is motion where the trajectory is not a straight line but a curved line. A special case of curvilinear motion is circular motion, which we'll discuss in section 2.12.

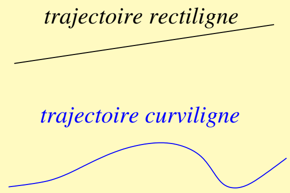


FIGURE 2.9 – Trajectoire curviligne

2.9.1 Curvilinear abscissa

Let M_0 be a point on the trajectory chosen as origin (reference point). At any instant t, the moving body is at M. To express the measure of the length of the path from M_0 to M, i.e. the arc $\overset{\frown}{M_0M}$, we introduce the curvilinear abscissa s :

$$s = \overset{\frown}{M_0M} \tag{2.78}$$

The path can be oriented either to the left or to the right of M_0. In this course, we choose to orient it in the direction of motion.

Relationship between $\vec{v}$ and s :

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\overrightarrow{OM}}{ds}\frac{ds}{dt} \tag{2.79}$$

Tha derivative $\frac{d\overrightarrow{OM}}{ds}$ can be written, by definition, as :

$$\frac{d\overrightarrow{OM}}{ds} = \lim_{s'\to s} \frac{\overrightarrow{OM'} - \overrightarrow{OM}}{s'-s} = \lim_{M'\to M} \frac{\overrightarrow{MM'}}{\overset{\frown}{MM'}} \tag{2.80}$$

As M' approaches M (i.e. becomes infinitely close to it),

- the direction of the vector $\overrightarrow{MM'}$ is that of the tangent at M to the trajectory, and it is also that of the velocity $\vec{v}$ since the vector $\vec{v}$ is, by definition, always tangent to the trajectory.

- the arc $\overset{\frown}{MM'}$ merges with the segment (the chord) $[MM']$, i.e. $\overset{\frown}{MM'}$ and $\overrightarrow{MM'}$ have the same length, which leads to :

$$\frac{\|d\overrightarrow{OM}\|}{ds} = 1. \tag{2.81}$$

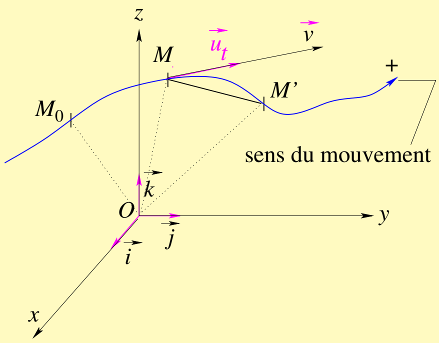


FIGURE 2.10 – Abscisse curviligne

From the above we deduce that $d\overrightarrow{OM}/ds$ is a unit vector carried by the tangent at M to the trajectory and oriented in the direction of $\overrightarrow{MM'}$. It therefore has the same direction and sense as the velocity vector $\vec{v}$. Designating it $\vec{u}_t$, we can write :

$$\vec{v} = \frac{ds}{dt}\,\vec{u}_t. \tag{2.82}$$

Since the trajectory is oriented in the direction of motion, we have $ds > 0$ and the modulus of $\vec{v}$ is written as :

$$v = \frac{ds}{dt}. \tag{2.83}$$

Note : If the trajectory were oriented in the opposite direction, we'd have $ds < 0$ and in this case $v = -ds/dt$, with dt assumed to be always positive.

Compute s from v : If v is known, we can compute s this way :

$$ds = vdt \Longrightarrow \int_{s_0}^{s} ds = \int_{t_0}^{t} vdt \Longrightarrow s = s_0 + \int_{t_0}^{t} vdt. \tag{2.84}$$

2.10 Tangential and normal components of acceleration

The velocity is written : $\vec{v} = v\vec{u}_t$, with $v = ds/dt$. By deriving $\vec{v}$, we obtain the expression for acceleration as a function of $\vec{u}_t$.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(v\vec{u}_t)}{dt} = \frac{dv}{dt}\vec{u}_t + v\frac{d\vec{u}_t}{dt} \tag{2.85}$$

Let's take a closer look at the vector $d\vec{u}_t/dt$. We can already see that :

$$\frac{d\vec{u}_t}{dt}\cdot\vec{u}_t = \frac{1}{2}\frac{d(\vec{u}_t\cdot\vec{u}_t)}{dt} = \frac{1}{2}\frac{d(|\vec{u}_t|^2)}{dt} = \frac{1}{2}\frac{d(1)}{dt} = 0, \tag{2.86}$$

which means that $d\vec{u}_t/dt$ is *normal* (perpendicular) to $d\vec{u}_t$. Now let's calculate $d\vec{u}_t/dt$:

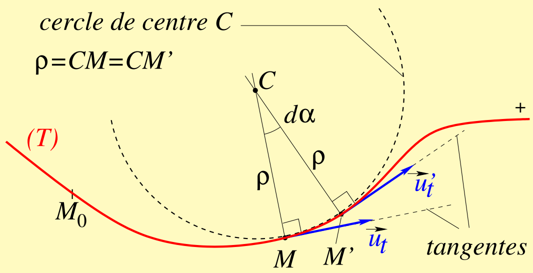


FIGURE 2.11 – Rayon de courbure

$$\frac{d\vec{u}_t}{dt} = \frac{d\vec{u}_t}{ds}\frac{ds}{dt} = v\frac{d\vec{u}_t}{ds}, \tag{2.87}$$

but,

$$\frac{d\vec{u}_t}{ds} = \lim_{M'\to M}\frac{\vec{u'}_t - \vec{u}_t}{\widehat{M_0M'} - \widehat{M_0M}}, \tag{2.88}$$

The normals to the trajectory at M and M' intersect at a point C. When M and M' are infinitely close, the arc $\widehat{MM'}$ on the trajectory merges with that of the circle of center C and radius $\rho = CM = CM'$. The quantity ρ is called the radius of curvature of the trajectory at point M. By definition, ρ is a positive quantity. Point C is the center of curvature.
Denoting $d\alpha$ by the angle $\widehat{MCM'}$, we have the relationship known in a circle :

$$ds = \rho d\alpha. \tag{2.89}$$

We then have

$$\frac{d\vec{u}_t}{ds} = \frac{d\vec{u}_t}{d\alpha}\frac{d\alpha}{ds} = \frac{1}{\rho}\frac{d\vec{u}_t}{d\alpha} \Longrightarrow \frac{d\vec{u}_t}{dt} = \frac{v}{\rho}\frac{d\vec{u}_t}{d\alpha} \tag{2.90}$$

The equation (2.85) then becomes :

$$\vec{a} = \frac{dv}{dt}\vec{u}_t + \frac{v^2}{\rho}\frac{d\vec{u}_t}{d\alpha} \tag{2.91}$$

Since v and ρ are positive quantities, the equation (2.90) tells us that $d\vec{u}_t/d\alpha$ is a vector of the same direction and sense as $d\vec{u}_t/dt$.

Let's find the modulus of $d\vec{u}_t/d\alpha$. In the figure opposite, we have : $\|d\vec{u}_t| = \|\vec{u}_t\|,d\alpha = d\alpha$ because $\|\vec{u}_t\| = 1$, donc $d\vec{u}_t/d\alpha$ is a unit vector.
We have already shown that it is normal to $\vec{u}_t$. All that remains is to specify its direction with respect to the trajectory. As shown in the adjacent figure, its direction is that of $vecu'_t - vecu_t$, which is always oriented towards the inside of the concavity of the trajectory. More precisely, when M and M' are infinitely close, this vector points towards the center of curvature C. It

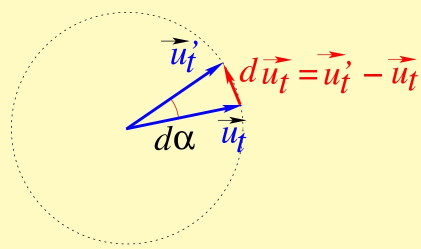


FIGURE 2.12 – Module de $d\vec{u}_t/d\alpha$

is denoted by $\vec{u}_n$ ($\frac{d\vec{u}_t}{d\alpha} = \vec{u}_n$)) and the equation (2.91) is finally written as :

$$\vec{a} = \frac{dv}{dt}\vec{u}_t + \frac{v^2}{\rho}\,\vec{u}_n \tag{2.92}$$

Acceleration can be decomposed in a tangential component $\vec{a}_t$ and a normal component $\vec{a}_n$.
$\vec{a}_t = dv/dt\vec{u}_t$, it is tangential to the trajectory and results from the variation in the modulus of velocity over time. $a_n = fracv^2\rho\,\vec{u}_n(= vd\vec{u}_t/dt)$, it is normal to the trajectory and results from the variation in direction of the velocity vector over time.

2.11 Expression of the radius of curvature ρ as a function of $\vec{v}$ and $\vec{a}$

On a :

$$\vec{v} \times \vec{a} = \vec{v} \times \left(\frac{dv}{dt}\vec{u}_t + \frac{v^2}{\rho}\,\vec{u}_n\right) = \frac{v^3}{\rho}(\vec{u}_t \times \vec{u}_n) \tag{2.93}$$

Since the vectors $\vec{u}_t$ and $\vec{u}_n$ are unitary, the same applies to $\vec{u}_t \times \vec{u}_n$. Turning to moduli, the equation (2.93) is as follows :

$$\|\vec{v} \times \vec{a}\| = \frac{v^3}{\rho} \Longrightarrow \rho = \frac{v^3}{\|\vec{v} \times \vec{a}\|} \tag{2.94}$$

This expression is valid in any coordinate system. In Cartesian coordinates, it reads :

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{3/2}}{\left[(\dot{y}\ddot{z} - \dot{z}\ddot{y})^2 + (\dot{z}\ddot{x} - \dot{x}\ddot{z})^2 + (\dot{x}\ddot{y} - \dot{y}\ddot{x})^2\right]^{1/2}} \tag{2.95}$$

If motion takes place in the xy plane, then $z = 0, \dot{z} = 0$ and $\ddot{z} = 0$ and the expression for ρ becomes :

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\left[(\dot{x}\ddot{y} - \dot{y}\ddot{x})^2\right]^{1/2}} = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|} \tag{2.96}$$

2.12 Circular motion

A motion is said to be circular when the trajectory is an *circle*. It can be considered as a special case of curvilinear motion where the radius of curvature is the radius R of the circle and the center of curvature is the center O of the circle.
If M_0 is a point of the circle chosen as origin, then

$$s = \overset{\frown}{M_0M} = R\theta, \quad \theta = \widehat{M_0OM} \tag{2.97}$$

$$\vec{v} = \frac{ds}{dt}\vec{u}_t = R\dot{\theta}\,\vec{u}_t \tag{2.98}$$

$$\vec{a} = R\ddot{\theta}\,\vec{u}_t + R\dot{\theta}^2\,\vec{u}_n \tag{2.99}$$

$\dot{\theta}$ gives the variation of angle θ as a function of time. It's called *angular velocity*, and its unit is radian per second (rad/s). $\ddot{\theta}$ is *angular acceleration*, its unit is radian per second squared (rad/ s^2).

Uniform circular motion Circular motion is said to be uniform when the angular velocity $\dot{\theta}$ does not change over time, i.e. when $\dot{\theta}$ is a constant (often noted ω).

In this case $\ddot{\theta} = d\omega/dt = 0$. Equations (2.98) et (2.99) become :

$$\vec{v} = R\omega\,\vec{u}_t \quad (2.100)$$

$$\vec{a} = R\omega^2\,\vec{u}_n = -\omega^2\overrightarrow{OM} \quad (2.101)$$

The linear velocity v has a constant modulus $v = R\omega$. The acceleration vector $\vec{a}$ has a constant modulus ($a = R\omega^2 = v^2/R$), constantly changing direction but pointing at all times to the center O of the circle ; acceleration is said to be *centripetal.*

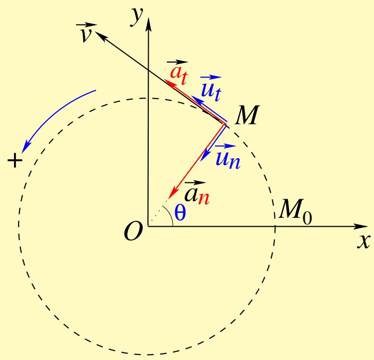


FIGURE 2.13 – Circular motion

Equation θ_0 as a function of time

$$\frac{d\theta}{dt} = \omega \Longrightarrow \int_{\theta_0}^{\theta} d\theta = \int_{t_0}^{t} \omega\,dt \Longrightarrow \theta = \theta_0 + \omega(t - t_0), \quad (2.102)$$

θ_0 is the value of θ at initial time t_0.

The motion repeats itself identically with each complete rotation. The duration T of a complete rotation is the *period* of the motion :

$$T = \frac{2\pi R}{v} = \frac{2\pi R}{R\omega} = \frac{2\pi}{\omega} \text{ (en secondes)} \quad (2.103)$$

We can also define the frequency f of the motion, which corresponds (here) to the number of complete rotations performed per second :

$$f = \frac{1}{T} \quad (2.104)$$

Frequency has the dimension of the inverse of time, T^{-1}. Its SI unit is the hertz (Hz), 1 Hz $= 1\,\mathrm{s}^{-1}$.

2.13 Movement with central acceleration

The motion of a moving body M is said to be centrally accelerated if its acceleration $\vec{a}$ is constantly directed towards a fixed point (center), noted O on the figure 2.14.

The vectors $\overrightarrow{OM}$ and $\vec{a}$ are, of course, collinear, which implies that :

$$\boxed{\overrightarrow{OM} \times \vec{a} = \vec{0}} \quad (2.105)$$

The equation (2.105) shows that a motion with central acceleration is a *plane motion*, i.e. it takes place in a plane. Indeed, if $\vec{v}$ is the velocity of M, then : $\overrightarrow{OM} \times \vec{a} = \vec{0} \iff \overrightarrow{OM} \times d\vec{v}/dt = \vec{0}$.

Knowing that $\overrightarrow{OM} \times d\vec{v}/dt = d(\overrightarrow{OM} \times d\vec{v})/dt - d\overrightarrow{OM}/dt \times \vec{v}$, il vient $d(\overrightarrow{OM} \times \vec{v})/dt - d\overrightarrow{OM}/dt \times \vec{v} = \vec{0}$, soit

$$\frac{d(\overrightarrow{OM} \times \vec{v})}{dt} = \vec{0}, \quad (2.106)$$

car $d\overrightarrow{OM}/dt \times \vec{v} = \vec{v} \times \vec{v} = \vec{0}$.

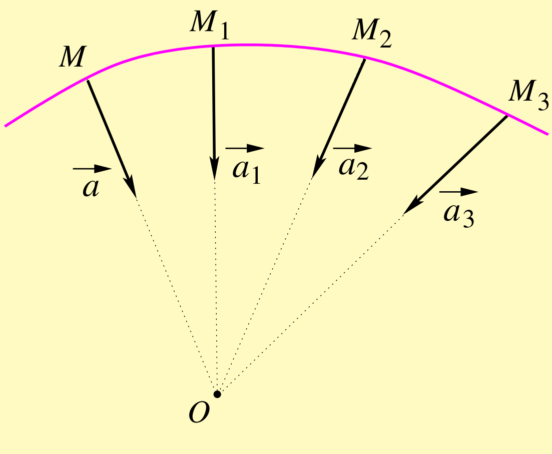


FIGURE 2.14 – Mouvement à accélération centrale

Equality to 0 of the derivative in the equation (2.106) implies that the vector $\overrightarrow{OM}/dt \times \vec{v}$ is constant :

$\overrightarrow{OM}/dt \times \vec{v} = \vec{C}'$, $\vec{C}'$ constant in modulus, direction and sense.

By definition of the vector product, the vector $\overrightarrow{OM}$ is perpendicular to $\vec{C}'$. So the point M moves in the plane perpendicular in O to $\vec{C}'$. *Centrally accelerated motions are planar motions.*

2.13.1 Areal speed

$\Delta A =$ variation in area swept by $\overrightarrow{OM}$ between t and $t + \Delta t$. Pour Δt assez petit, c'est-à-dire M' assez proche de M, we can confuse the arc $\overset{\frown}{MM'}$ with the cord MM'. The area ΔA is then just half of the area of the parallelogram built on $\overrightarrow{OM}$ and $\overrightarrow{MM'}$:

$$\Delta A = \frac{1}{2}\left\|\overrightarrow{OM} \times \overrightarrow{MM'}\right\| \implies \frac{\Delta A}{\Delta t} = \frac{1}{2}\left\|\overrightarrow{OM} \times \frac{\overrightarrow{MM'}}{\Delta t}\right\|$$

At the limit $\Delta t \to 0$, we'll have :

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2}\left\|\overrightarrow{OM} \times \lim_{\Delta t \to 0} \frac{\overrightarrow{MM'}}{\Delta t}\right\|,$$

then, since $\lim_{\Delta t \to 0} \frac{\overrightarrow{MM'}}{\Delta t} = \vec{v}$,

$$\boxed{\frac{dA}{dt} = \frac{1}{2}\left\|\overrightarrow{OM} \times \vec{v}\,\right\|} \tag{2.107}$$

The areal speed is a positive quantity and measures the area swept by the $\overrightarrow{OM}$ radius-vector in one second. Its SI unit is square meters per second ($\mathrm{m^2\,s^{-1}}$) and its dimension is square length per time ($\mathrm{L^2\ T^{-1}}$). The areal speed is the **magnitude** of the *areal velocity* $\frac{1}{2}(\overrightarrow{OM} \times \vec{v})$.

Note : For motion with central acceleration, the $\overrightarrow{OM} \times \vec{v}$ is constant and, consequently, $\frac{dA}{dt}$ is equal to a positive constant C. We therefore have :

$$\frac{dA}{dt} = C \tag{2.108}$$

The constant C is called the area constant because $\overrightarrow{OM}$ sweeps out equal areas over equal durations. The motion is said to follow the law of areas. The converse is also true. If $\frac{dA}{dt} = C$, then motion has central acceleration.

The law of areas applies to the motion of planets around the Sun : A radius vector joining any planet to the Sun sweeps out equal areas in equal intervals of time. This is known as the Kepler's second law.

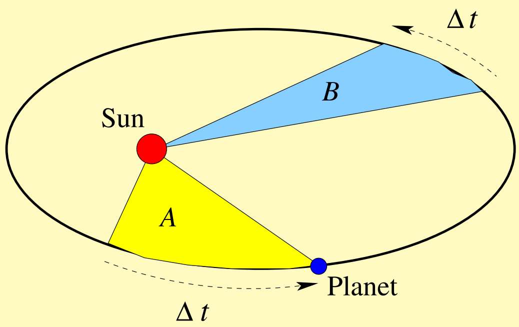


FIGURE 2.15 – For the same interval of time Δt, area A equals area B