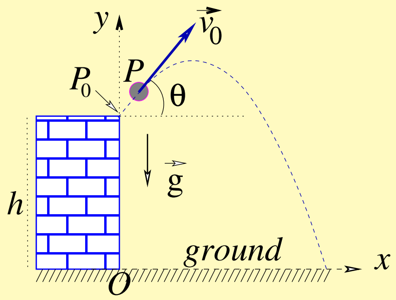
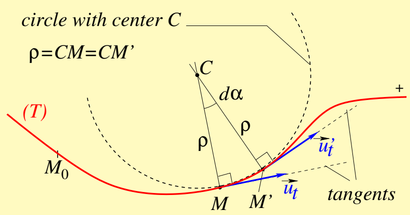
University of Batna 2 – Mostefa Ben Boulaïd

Faculty de Mathematics
and Informatics
Department of Mathematics
and Informatics - Common Base

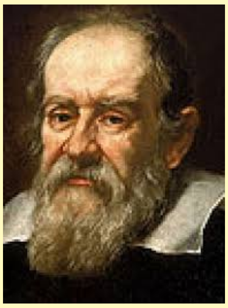
53, Road of Constantine, Fesdis,
Batna 05078, Algeria.



The equation of the trajectory of P is : $y = -\frac{g}{2v_0^2\cos^2\theta}x^2 + \tan\theta\, x + h$. The speed of the ball as it hits the ground is $v = \sqrt{v_0^2 + 2gh}$.
In terms of $\dot{x}$, $\dot{y}$, $\ddot{x}$, and $\ddot{y}$, the radius of curvature of M (right figure) reads : $\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{|\dot{x}\ddot{y}-\dot{x}\ddot{y}|}$.

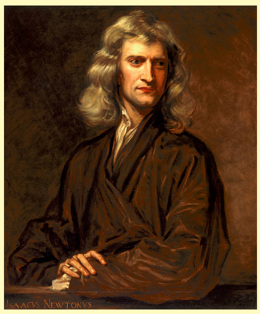


Course of Phy



Galileo (left portrait) writes : "Aristotle declares that a 100-pound iron ball has already descended 100 cubits when a 1-pound ball has traveled only one cubit. I affirm that the two balls arrive together."

In 1687, Newton (right portrait) published the mathematical principles of natural philosophy (*Philosophiae naturalis principia mathematica*). In it, he described his discoveries on universal gravitation and the three famous laws, known as Newton's Laws. These laws describe the physical phenomena of inertia and the forces exerted on objects.



Above, the blue figure (left) shows the trajectory of a small ball P launched from the top of a building (height h) with a velocity $vecv_0$ making an angle θ with the horizontal. The expressions to its right give the equation of the ball's trajectory and its velocity when it hits the ground. The figure on the far right defines the radius of curvature at a point on the trajectory. When M' tends towards M (trajectory (T) in red), the normals to the tangents at M and M' meet at a point C called the center of curvature. The lengths CM and CM' are then equal to a quantity ρ called radius of curvature. *A circle with center C and radius ρ will necessarily pass through M and M'. As for the two portrait photos below, they are of Galileo (left) and Newton (right), two great scientists who left their mark on the history of science and contributed greatly to mechanics.*

Prof. M. A. Belkhir, année 2023-2024

Table des matières

Chapitre 4

Work and Energy

4.1 Learning objectives

- Understand the concept of work and how to calculate the work done by a force.
- Understand the concept of the net work done on an object and how that relates to a change in speed of the object.
- Understand the concept of kinetic energy and where it comes from.
- Understand the concept of power.

4.2 Introduction

In everyday language, the word 'work' is used to designate any muscular or intellectual effort. So, for example, a teacher does work when he teaches, a student does work when he reads his lesson, a farmer does work when he tills his field, a bus-driver does work when he drives his bus, and an architect does work when he designs a building, etc. The word 'energy', on the other hand, conveys the idea of strength and firmness in action, or vitality. For example, protest with energy (force), or feel full of energy (vitality).

In physics, work and energy are fundamental concepts; they have a more specific meaning. We will see that work is related to the transfer of energy, and energy represents the capacity to do work.

In this chapter, we'll develop all these concepts, we'll talk about the work done by a force on an object, and then introduce the notions of kinetic energy, potential energy and mechanical energy. So, energy comes in various forms. In addition to kinetic, potential, and mechanical energies, energy can also have the form of :
- Thermal (Heat) Energy : The internal energy of an object due to the motion of its atoms and molecules. It is related to the temperature of the object.
- Chemical Energy : The energy stored in the bonds between atoms and molecules. It can be released through chemical reactions.
- Electrical Energy : The energy associated with the flow of electric charge.
- Nuclear Energy : The energy stored in the nucleus of an atom. It can be released through nuclear reactions.
- Electromagnetic (Radiant) Energy : The energy carried by electromagnetic waves, such as light.

We will talk about electrical energy in the next semster in Physics 2. Energy is subject to the principle of conservation of energy, which states that the total energy of an isolated system remains constant over time. Energy is neither created nor destroyed, only transferred or converted from one form to another. This principle is a fundamental concept in understanding and solving problems in physics.

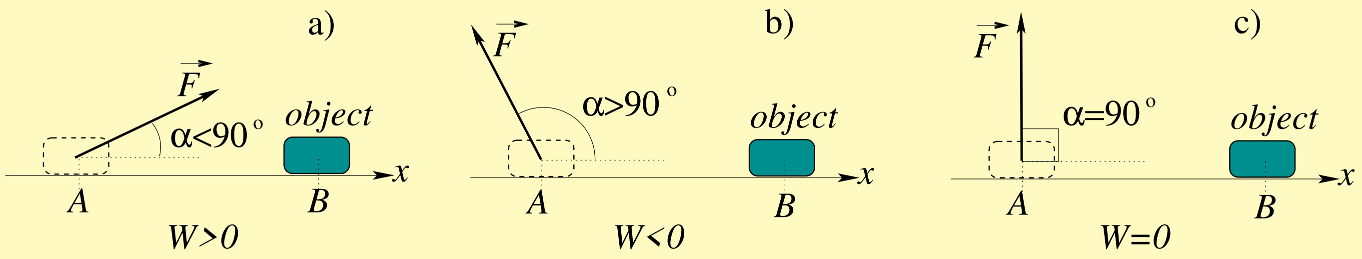
4.3 Work done by a force on an object

4.3.1 Work done by a constant force on an object moving in a straight line

Consider an object moving in a straight line while an external *constant* force $\vec{F}$ (modulus, direction and sense do not vary) acts on it. The work, noted W, that $vecF$ performs on the object, is defined to be the scalar product of $vecF$ dot the vector displacement, say $\overrightarrow{AB}$, through which the force acts. In equation form :

$$W = \vec{F} \cdot \overrightarrow{AB} \tag{4.1}$$

If α is the angle between $\vec{F}$ and $\overrightarrow{AB}$, the equation becomes :



$$W = F\,AB\,\cos\alpha \tag{4.2}$$

$*$ W is a scalar quantity. (4.2) shows that the SI unit of work is the newton meter. This unit is called the joule (symbol J) : $1\,\mathrm{J} = 1\,\mathrm{N}\cdot\mathrm{m}$. Therefore, a joule corresponds to the work done on (i.e. the energy transferred to) an object by a force of 1 N acting on the object during a displacement of this object by 1 m in the direction of the force.

$*$ Equation (4.1) shows that work can be zero, positive or negative. Its sign is that of $\cos\alpha$ because F and AB, being moduli, are positive.

- When $0 < \alpha < \pi/2$ and $F \neq 0$ and $AB \neq 0$, as in figure a), the cosine is positive and, consequently, the work is positive. In this case, the force is said to be doing motive work.

- For $\pi/2 < \alpha < \pi$, and $F \neq 0$ and $AB \neq 0$, as in figure b), the work is negative and the force is said to be doing resistive work.

- When $\alpha = \pi/2$, and $F \neq 0$ and $AB \neq 0$, as in figure c), the cosine is 0 and the work is zero. A force acting on an object perpendicularly to the displacement does no work.

$*$ Of course, no work is done for zero force ($F = 0$), and no work is done for zero displacement ($AB = 0$).

$*$ When we talk about the work of $\vec{F}$ along several displacements AB, BC, ..., we adopt a notation that avoids confusion, for example : W_{AB}, W_{BC}, ...
When there are several forces $\vec{F}_1$, $\vec{F}_2$ ..., we write $W_{AB}(\vec{F}_1)$, $W_{AB}(\vec{F}_2)$, ...

Questions :

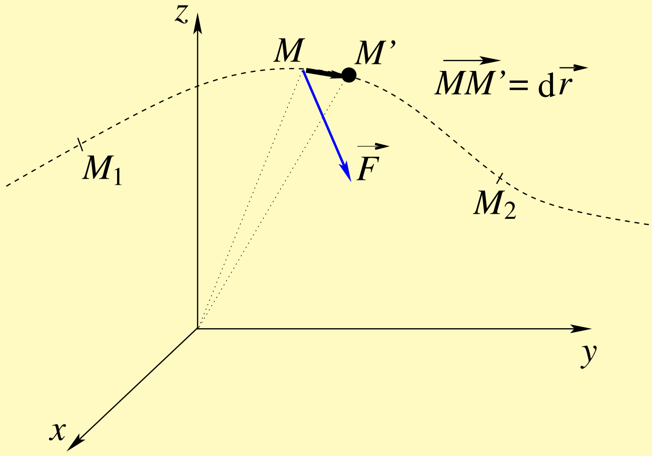
1) You exert a push on a wall. What is the value of the work done on the wall ? The work on the wall is zero because the wall makes no displacement.

2) When you walk with a bottle of water in your hand, do you do any work on the bottle ? No, because the force you exert on the bottle is vertical, whereas its displacement is horizontal.
3) What is the work done by the force of static friction on an object ? The work is zero because there is no displacement.
4) What is the work of the kinetic frictional force $\vec{F}_k$ on a 2 kg block descending at constant velocity a distance of 4 m along a plane inclined at 30° to the horizontal ?

Let's denote by $\vec{d}$ the displacement. Therefore, the work done by $\vec{F}_k$ is $W = \vec{F}_k \cdot \vec{d}$, $\vec{d}$ that gives $-F_k \times d$ because $\vec{F}_k$ is directed opposite to $\vec{d}$. Constant velocity means that the acceleration is zero, in other words, the sum of applied external forces is zero : $\vec{F}_k = -\vec{P}_{\parallel}$, $\vec{P}_{\parallel}$ being the parallel component (to the plane) of the block weight. So, $\vec{F}_k$ has same magnitude as $\vec{P}_{\parallel}$, i.e., $F_k = m\,g \sin 30$ and the work done by $\vec{F}_k$ reads : $W = -m\,g \sin 30 \times d = 2 \times 9.8 \times (1/2) \times 4 = -39.2$ J.

4.3.2 Work done by a variable force on an object moving along a non-rectilinear trajectory

For an elementary (i.e. infinitesimal) displacement متناهي الصغر) $d\vec{r} = \overrightarrow{MM'}$, we can assume that the motion is rectilinear and that the force $\vec{F}$ is constant.



In the same way as in equation (4.1), we can express the elementary work [1] of $\vec{F}$ from M to M' as :

$$\delta W = \vec{F} \cdot d\vec{r} \tag{4.3}$$

The work done by $vecF$ for a displacement from M_1 to M_2 is obtained by integration :

$$W = \int_{M_1}^{M_2} \delta W = \int_{M_1}^{M_2} \vec{F} \cdot d\vec{r} \tag{4.4}$$

Note : When n forces $\vec{F}_1$, $\vec{F}_2$,...,$\vec{F}_n$ are acting on the object, their resultant is written $\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n$ and the previous equation becomes

$$W = \int_{M_1}^{M_2} (\vec{F}_1 + \vec{F}_2 + \cdot + \vec{F}_n) \cdot d\vec{r} = W_1 + W_2 + \ldots + W_n \tag{4.5}$$

In other words, the net (i.e. the total) work done on the object by all these forces can also be written as the algebraic sum of the individual works.
The previous equation can also be written as :

$$W = \int_{M_1}^{M_2} \vec{R} \cdot d\vec{r} \tag{4.6}$$

1. In mathematics, an exact differential form of a function f is written df. In physics, the work done by a force $\vec{F}$ along an elementary displacement is noted δW rather than dW because, in general, δW is not an exact differential form. This means that the integral $\int_{M_1}^{M_2} \delta W$ depends upon the path taken to move from M_1 to M_2. The elementary work is an exact differential form if the force $\vec{F}$ is conservative (see in section 4.6.1). In this case we can use the notation dW for the elementary work.

where $\vec{R} = \vec{F}_1 + \vec{F}_2 + \cdot + \vec{F}_n$ is the resultant of all of the forces applied to the object. That is, the net work done on the object is also given by the work of the net force force applied to it.

4.4 Kinectic energy and kinectic energy theorem

We know that Newton's second law for an object (a material point) [2] of constant mass m and velocity $vecv$ is written as :

$$\vec{F} = m\frac{d\vec{v}}{dt}, \tag{4.7}$$

where $\vec{F}$ represents the sum of the forces applied to m. By multiplying both members scalarly by $\vec{v}$, we obtain :

$$\vec{F} \cdot \vec{v} = m\frac{d\vec{v}}{dt} \cdot \vec{v}. \tag{4.8}$$

Note that

$$\frac{d\vec{v}}{dt} \cdot \vec{v} = \frac{d}{dt}\left(\frac{1}{2}\vec{v} \cdot \vec{v}\right),$$

and since $\vec{v} \cdot \vec{v} = v^2$ and m is constant, we finally have :

$$m\frac{d\vec{v}}{dt} \cdot \vec{v} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right),$$

and then,

$$\vec{F} \cdot \vec{v} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right), \tag{4.9}$$

or

$$\vec{F} \cdot \vec{v}dt = d\left(\frac{1}{2}mv^2\right). \tag{4.10}$$

The first member of equation (4.10) represents the elementary work of $\vec{F}$ for an elementary displacement $\vec{v}dt = d\vec{r}$, i.e., $\vec{F} \cdot \vec{v}dt = \vec{F} \cdot d\vec{r} = \delta W$ (see equation (4.3)). We therefore have :

$$\delta W = d\left(\frac{1}{2}mv^2\right) \tag{4.11}$$

The quantity $\frac{1}{2}mv^2$ has the same dimension as W, i.e. the dimension of an energy. It is an energy that results from the fact that there is motion (velocity $\neq 0$). It's called kinetic energy and is often referred to as E_c :

$$E_c = \frac{1}{2}mv^2 \tag{4.12}$$

Equation (4.11) can be rewritten as follows :

$$\delta W = dE_c \tag{4.13}$$

When m goes from M_1 (velocity v_1, kinetic energy E_{c_1}) to M_2 (velocity v_2, kinetic energy E_{c_2}), we have by integration :

$$\int_{M_1}^{M_2} dW = \int_{E_{c_1}}^{E_{c_2}} dE_c \tag{4.14}$$

or,

$$W = \Delta E_c \text{ ou encore } W = E_{c_2} - E_{c_1} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \tag{4.15}$$

2. Here, we recall that objects are considered as "point materials". In reality, objects' size is not point-like but they are qualified as such because we limit ourselves to movements where all the points of the object perform the same motion.

This last equation constitutes the kinetic energy theorem, which states that :

The work done by the sum of all the forces acting on an object moving from M_1 to M_2 is equal to the variation in the object's kinetic energy between M_1 and M_2.

From the kinetic energy theorem, we deduce that :

1- When the modulus of $\vec{v}$ does not change, no work is done. If there are applied forces, they are necessarily perpendicular to the displacement. This is the case in uniform rectilinear motion or uniform circular motion.

2- If E_k decreases, work is negative (resistive work). If E_k increases, work is positive (motive work).

Example : A girl pushes a toy car (سيارة لعبة مصغرة) towards her little brother on level ground. She exerts a horizontal force of 5 N on the car, initially at rest, over a distance of 1 m.

a) What is the value of the work supplied to the car ? $W = 5\,\mathrm{N} \times 1\,\mathrm{m} = 5\,\mathrm{J}$

b) What is the kinetic energy at the end of $1\,\mathrm{m}$? $E_{c_2} - E_{c_i} = W \implies E_{c_2} = 5\,\mathrm{J}$

c) If the car has a mass of $0.1\,\mathrm{kg}$, what will its speed be after $1\,\mathrm{m}$? $mv^2_{c_1m}/2 = E_{c_2} \implies v = \sqrt{2E_{c_2}/m} = 10\,\mathrm{m/\,s}$.

d) The girl lets go of the car after $1\,\mathrm{m}$. The car continues to roll along the ground and reaches the brother, who stops it over a distance of $0.25\,\mathrm{m}$ by exerting a constant force F' in the opposite direction. Évaluate F' neglecting friction.

Let's apply the kinetic energy theorem in the $0.25\,\mathrm{m}$ phase : $W(F') = E_{k_f} - E_{k_i}$, $W(F') = -F' \times 0.25$, $E_{cf} = 0$, $E_{k_i} = 5\,\mathrm{J} \implies -0.25F' = -5 \implies F' = 20\,\mathrm{N}$. The negative sign ($W = -5\,\mathrm{J}$) indicates that it's, in a way, the car that is doing the work on the child.

This exercise teaches us that when positive work is done on an object, it gives it kinetic energy. This energy is then available for further work.

4.5 Power

How quickly work is done is measured by power. Power is the rate at which work is done on an object.

Average power :

$$P_{av} = \frac{\text{work done by a force on an object between time } t_1 \text{ and time } t_2}{t_2 - t_1} = \frac{\Delta W}{\Delta t}$$

Instantaneous Power :

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

If power is constant, then $P = P_m$ which yiels :

$$\Delta W = P\Delta t$$

In SI unit system, power is expressed in watts (symbol W), $1\,\mathrm{W} = 1\,\mathrm{J/\,s}$. A watt is the power required to do one joule of work in one second.

We can also write the power supplied to an object as a function of the force $\vec{F}$ acting on it. Since $dW = \vec{F} \cdot d\vec{r}$, we have $dW/dt = \vec{F} \cdot d\vec{r}/dt = \vec{F} \cdot \vec{v}$ and then :

$$P = \vec{F} \cdot \vec{v}$$

Example 4.5.1 : A boy with a mass of $51\,\mathrm{kg}$ climbs, at constant speed, a vertical rope 6 m long in 10 s. a) How much work does the boy do ? b) How much power does the boy produce during the climb ? Take g = 9.8 m/s^2.

Answer : a) To maintain a constant climbing speed, the boy just needs to exert a force to compensate for his weight. The work he does is therefore given by : $W = m\,gd = 51\,\text{kg} \times 9.8\,\text{m}/s^2 \times 6\,\text{m} = 2998.8\,\text{J}$, b) The power produced by the boy during the climb is $P = W/t = 2998.8\,\text{J}/10\,\text{s} = 299.88\,\text{W}$.

Example 4.5.2 : If a car engine can do work on the car (mass $m = 1000$ kg) with a power of P, what will be the speed of the car at some time t if the car starts from rest ? How long does it take the car to accelerate from 0 to 100 km/ h if its power is $P = 90$ hp ? hp is the symbol for *horsepower* a British unit for power, 1 hp = 745,7 W. Horsepower is one of the main measures that quantifies an engine's power output. The power of a Clio 4 gt-line car is typically 90 hp. The "Wonder of the Seas" cruise ship (362 m long and 65 m wide) can deliver a power of 90000 hp, the equivalent of 1000 Clio 4 gt-line !

Solution : The total work done on the car is $W = P \times t$. Then, using the Work-Energy Theorem between $t = 0\,\text{s}$ and $t = t\,\text{s}$, we can find the speed of the car at some time t : $W = \frac{1}{2}mv_t^2 - \frac{1}{2}mv_0^2$ or $Pt = \frac{1}{2}mv_t^2$. Therefore, $v_t = \sqrt{2Pt/m}$. The car reaches 100 km/ s at $t = mv_t^2/2P$.

4.6 Law of Conservation of Mechanical Energy

When several forces act on a particle, the kinetic energy theorem (equation (4.15)) is written :

$$W_1 + W_2 + \ldots + W_n = \Delta E_c \tag{4.16}$$

Among the applied forces, we'll distinguish two types : *nonconservative* and *non conservative* forces. Conservative forces lead us to introduce the concept of potential energy, which in turn enables us to express the conservation of mechanical energy.

4.6.1 Conservative force and nonconservative force

If we throw an object of mass m vertically upwards from a point O with a velocity v_0, the object will rise until it reaches the point where its velocity cancels out (the summit), then fall back down instantaneously. Its velocity, now directed downwards, increases and it arrives at O (starting position) with velocity $-v_0$. This is, in fact, true in a more general way : if, during its ascent, the object is located at a position P with velocity v, then, when it descends again, it will arrive at P with velocity $-v$. In other words, the object returns to the same point with the same kinetic energy $E_k = mv^2/2$. Even if the kinetic energy varies over time (decreasing on the outward journey and increasing on the return), it is *conserved* in the sense that the object returns to the initial point with the same kinetic energy.

In the previous reasoning, we implicitly ignored air resistance and assumed that the only force acting on the object once launched is gravitational force (its weight). Because it conserves kinetic energy, gravitational force is referred to as a conservative force. Any force that behaves like the gravitational force, i.e. any force that conserves kinetic energy over a round trip, is a *conservative force*.

If, on the other hand, an object acted upon by one or more forces returns to its initial position with more or less kinetic energy, then kinetic energy is not conserved over a round trip. In this case, at least one of the forces applied to the object is a nonconservative force. In the example of the object launched upwards, the kinetic energy on return would not be the same if the air resistance is not negligible. The force of air resistance, which is a form of frictional force, is a nonconservative force.

To sum up, a force acting on an object is said to be a conservative force if the object's kinetic energy returns to its initial value after any round trip. The force is nonconservative if the object's kinetic energy does not return to its initial value (changes) after any round trip. .

Second definition : Conservative and nonconservative forces can be defined in terms of the work performed

by the applied forces.
Let's take the example of the object thrown vertically upwards.
If we neglect air resistance, the only force acting on the object is its weight. On the outward journey, the work done by the weight on the object is $m\vec{g}\cdot\vec{d} = -m\,gd$ (negative because on the outward journey the weight $m\vec{g}$ and the displacement $\vec{d}$ are in opposite directions). On the return trip, the work done by the weight on the object is $m\vec{g}\cdot(-\vec{d}) = +m\,gd$ (positive because on the return trip the weight $m\vec{g}$ and the displacement $-\vec{d}$ are in the same direction). On the round trip, the work is therefore **null**.
If we take into account the force of air resistance, this force opposes the displacement and will do negative work on both the outward and return journey, so that over the whole of the outward and return journey the work cannot be zero (negative + negative $\neq 0$). Hence the second definition : A force is conservative if the work it does on an object in any round trip is zero. A force is non-conservative if the work it does on an object in any round trip is non-zero.
This second way of defining conservative and non-conservative forces is completely equivalent to the first definition. Indeed, if the variation ΔE_c of kinetic energy is zero in a closed path (outward and return), then, according to the kinetic energy theorem (equation (4.15)), the work is zero and all forces applied to the object must be conservative. If, on the other hand, $\Delta E_k \neq 0$ then at least one of the forces applied is nonconservative.

Third definition : Consider an object moving from A to B along path 1 and back from B to A along path 2 (see figure 4.1A).

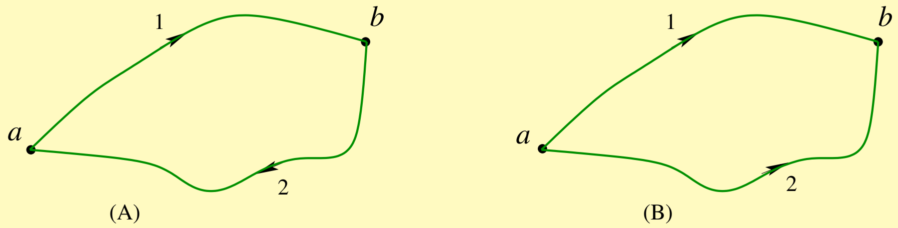


FIGURE 4.1 –

Several forces can act on the object. Let's consider them separately. If the force under consideration is conservative, then the work it does on a round trip is zero, i.e. $W_{AB,1} + W_{BA,2} = 0$ or

$$W_{AB,1} = -W_{BA,2} \tag{4.17}$$

The work done to go from A to B along path 1 is equal to minus the work done to go from B to A along path 2. But, if the object takes path 2 to get from A to B (see figure 4.1B), the work is simply given by minus the work done to get from B to A following path 2 :

$$W_{AB,2} = -W_{BA,2}, \tag{4.18}$$

and therefore

$$W_{AB,1} = W_{AB,2}. \tag{4.19}$$

Since points A to B and paths 1 and 2 are arbitrary, equation (4.19) tells us that the work done on a particle by a conservative force in moving from A to B is the same whatever the path taken to move from A to B. This brings us to the third definition of conservative and nonconservative forces : A force is conservative if the work it does on a particle moving between two points depends only on these two points and not on the path followed. A force is nonconservative if the work it does on a particle moving between two points depends on the path taken to make the displacement between the two points. We'll illustrate this last definition with the following example. Consider a mass m and calculate the work done by its weight (weight $m\vec{g}$) when it is made to move from A to B via path 1, then path 2 and then path 3, figure below.

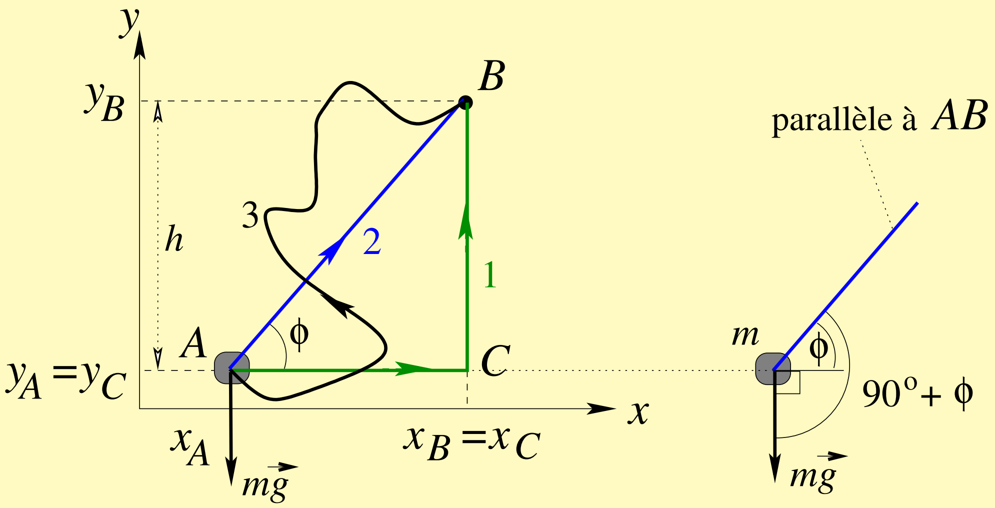


FIGURE 4.2 – Weight work from A to B is est le même whatever the path taken.

Path 1 : $W_1 = m\,\vec{g}\cdot\overrightarrow{AC} + m\,\vec{g}\cdot\overrightarrow{CB} = m\,g\times ac\times\underbrace{\cos 90°}_{0} + m\,g\times cb\times\underbrace{\cos 180°}_{-1} = -m\,gh$

Path 2 : $W_2 = m\,\vec{g}\cdot\overrightarrow{AB}$. The angle between $m\,\vec{g}$ and $\overrightarrow{AB}$ (see figure below) is $90° + \phi$, then $W_2 = m\,g\times AB\times\cos(90°+\phi) = m\,g\times AB\times(-\sin\phi) = -m\,g\times CB = -m\,gh$

Path 3 : For a displacement $d\vec{r}$ on path 3, the elementary work done by the weight is : $dW = m\,\vec{g}\cdot d\vec{r}$. Let's decompose $d\vec{r}$ according to x (unit vector $\vec{i}$) and y (unit vector $\vec{j}$) : $d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$. So, $dW = m\,\vec{g}\cdot dx\vec{i} + m\,\vec{g}\cdot dy\vec{j}$. The first term is zero because $\vec{g}$ and $\vec{i}$ are perpendicular. Therefore $dW = m\,\vec{g}\cdot dy\vec{j} = -m\,g\vec{j}\cdot dy\vec{j} = -m\,gdy$ because $\vec{j}\cdot\vec{j} = 1$. The total work on the whole path 3 is obtained by integration from A to B, that is from y_A to y_B : $W_3 = \int_{y_A}^{y_B} -m\,gdy = -m\,g(y_B - y_A) = -m\,gh$

We find $-m\,gh$ for all three paths. The same result was obviously expected since, as already mentioned, gravitational force (weight) is a conservative force. Note that if the paths were descending as in the figure below (A higher than B), we would have found $+m\,gh$. This quantity can be written as : $-m\,g(-h)$ so that even here we can express the work as $W_3 = -m\,g(y_B - y_A)$.

4.6.2 Potentoial energy

When the particle moves from position 1 (velocity v_1) to position 2 (velocity v_2), the change in kinetic energy is :

$$E_{c2} - E_{c1} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \tag{4.20}$$

If we assume that weight is the only force acting on the particle, the work done on the particle is :

$$W_{12} = -m\,g(y_2 - y_1) \tag{4.21}$$

By introducing the quantity $E_p = m\,gy$, which depends only on the position y, the work W_{12} can be written as $-(E_{p2} - E_{p1})$, i.e., minus the variation in E_p. The kinetic energy theorem then gives :

$$E_{c2} - E_{c1} = -(E_{p2} - E_{p1}), \tag{4.22}$$

or more succinctly

$$\Delta E_c = -\Delta E_p \tag{4.23}$$

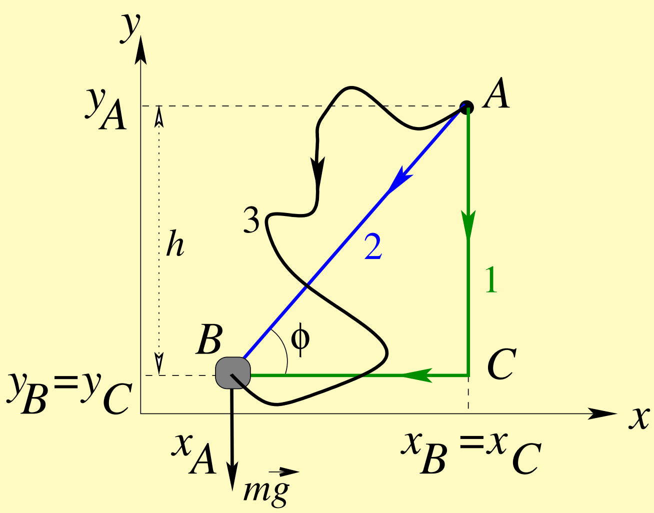


FIGURE 4.3 – The work of the weight is $+m\,gh$ if point A is at a height h above B.

The quantity E_p is called *potential energy.* Remember that the expression of E_p is closely linked to the fact that the force involved (in this case, weight) is a conservative force. In other words, potential energy only makes sense for a conservative force. Potential energy cannot be associated with a nonconservative force!
Just as the kinetic energy of a particle represents its capacity to do work due to its motion (i.e. its velocity), the potential energy of a particle represents its capacity to do work due to its position. Example : By lifting an object to a height h, I do an amount of +mgh work on it, which I transfer to it in the form of potential energy. If I let go of the object, it will be able to fall thanks to its potential energy.
The equation (4.23) tells us that if the kinetic energy E_k increases (decreases) by an amount, then the potential energy E_p decreases (increases) by the same amount.

4.6.3 Conservation of mechanical energy

The equation (4.23) results in $\Delta E_k + \Delta E_p = 0$ or $\Delta(E_k + E_p) = 0$, which also means that the sum $E_k + E_p$, called mechanical energy, does not vary over time :

$$E_c + E_p = \text{constant} \tag{4.24}$$

When the forces applied to a particle are conservative, its mechanical energy remains constant during motion.
Application exercise : An object of mass m, located at a height h from the ground, is dropped without initial velocity. Air resistance is neglected. What is its velocity when it reaches the ground?

1) First method : The relationship (2.55) seen in kinematics is written here $v^2 - 0^0 = 2\,gh \implies v^2 = 2\,gh$ and therefore $v = \sqrt{2\,gh}$.

2) Second method : Once released, the object is subject only to its weight. Since weight is a conservative force, we can apply the equation (4.24). This gives $0 + m\,gh = mv^2/2 + 0 \implies v = sqrt2\,gh$. Of course, the same result applies.

4.6.3.1 Alternative definition of a conservative force

Taking the differential of the two sides of the equation (4.24), we have : $d(E_c + E_p) = d(constante)$, since a differential of a constant is zero, then $dE_c + dE_p = 0$ or

$$dE_c = -dE_p. \tag{4.25}$$

This last equation is none other than the differential form of the equation (4.23). Remember that this result only applies for conservative forces. If F designates the sum of conservative forces, the kinetic energy theorem for an elementary displacement dr gives $dE_k = \vec{F} \cdot d\vec{r}$ and the equation (4.25) leads to :

$$\vec{F} \cdot d\vec{r} = -dE_p. \tag{4.26}$$

Considering E_p as a function of position $M(x,y,z)$, its total differential form is written (there are 3 variables) :

$$dE_p = \frac{\partial E_p}{\partial x}dx + \frac{\partial E_p}{\partial y}dy + \frac{\partial E_p}{\partial z}dz \tag{4.27}$$

If M is referenced to a system of axes $Oxyz$ with an orthonormal basis $\vec{i}, \vec{j}, \vec{k}$, we have $\overrightarrow{OM} \equiv \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. So $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$. The equation (4.27) can be written as a scalar product :

$$dE_p = \left(\frac{\partial E_p}{\partial x}\vec{i} + \frac{\partial E_p}{\partial y}\vec{j} + \frac{\partial E_p}{\partial z}\vec{k}\right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k}), \tag{4.28}$$

The first factor in brackets represents the gradient of E_p. It is denoted $\overrightarrow{textrmgrad}\, E_p$ or $\overrightarrow{nabla} E_p$. The second factor is simply $d\vec{r}$. The total differential dE_p can therefore be written as :

$$dE_p = \overrightarrow{\mathrm{grad}}\, E_p \cdot d\vec{r}, \tag{4.29}$$

and equation (4.26) becomes

$$\vec{F} \cdot d\vec{r} = -\overrightarrow{\mathrm{grad}}\, E_p \cdot d\vec{r}, \tag{4.30}$$

or, since $d\vec{r}$ is arbitrary,

$$\vec{F} = -\overrightarrow{\mathrm{grad}} E_p. \tag{4.31}$$

4.7 Frictional forces and mechanical energy

The kinetic energy theorem is : $W = Delta E_c$ (see equation (4.15)). If, in addition to conservative forces, there are also non-conservative forces, we can decompose the work W into two terms, W_c and W_{nc}. We then have :

$$W_c + W_{nc} = \Delta E_c \tag{4.32}$$

If E_p is the potential energy associated with conservative forces, we have : $W_c = -\Delta E_p$. The equation (4.32) then becomes :

$$-\Delta E_p + W_{nc} = \Delta E_c \tag{4.33}$$

When nonconservative forces are reduced to frictional forces, then $W_{nc} = W_f$ and the equation (4.33) becomes :

$$-\Delta E_p + W_f = \Delta E_c \tag{4.34}$$

This equation can be rearranged as :

$$\Delta(E_c + E_p) = W_f, \tag{4.35}$$

which means that $(E_c + E_p)$ is not constant, i.e. frictional forces are non-conservative forces. When frictional forces act in addition to conservative forces, the change in mechanical energy $(E_c + E_p)$ is equal to the work of the frictional forces. The work W_f of the friction forces is negative, since friction opposes displacement. Equation (4.35) therefore reflects a loss of mechanical energy.

4.8 Résumé

- Travail d'une force constante d'un point A à un point B d'une trajectoire rectiligne : $W = \vec{F} \cdot \overrightarrow{AB}$

- Travail d'une force variable fait sur un déplacement élémentaire $d\vec{r}$ pris sur une trajectoire quelconque : $dW = \vec{F} \cdot d\vec{r}$

- Énergie cinétique d'une masse ponctuelle m animée d'une vitesse v : $E_c = \frac{1}{2}mv^2$

- Théorème de la variation d'énergie cinétique : $W = \Delta E_c$, plus explicitement, entre deux points 1 et 2, $W_{1\to 2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

- Forces conservatives et Forces non conservatives :

Comparaison entre forces conservatives et forces non conservatives		
	Forces conservatives	**Forces non conservatives**
1	Le travail effectué dépend du chemin suivi	Le travail effectué ne dépend pas du chemin suivi
2	Le travail effectué sur un circuit fermé est nul	Le travail effectué sur un circuit fermé n'est pas nul
3	Force conservative $\vec{F_c}$ et énergie potentielle E_p sont reliées par : $\vec{F_c} = -\overrightarrow{\text{grad}} E_p$	Pour force non conservative, une telle relation n'existe pas
4	L'énergie mécanique est conservée : $E_c + E_p = constante$	L'énergie mécanique n'est pas conservée : $E_c + E_p \neq constante$
5	Le travail effectué par une force conservative est entièrement récupérable	Le travail effectué par une force non conservative n'est pas entièrement récupérable
	Exemple : force d'attraction gravitationnelle	Exemple : force de frottement

- En présence de forces conservatives seulement, l'énergie mécanique se conserve : $\Delta(E_c + E_p) = 0 \iff \Delta E_c = -\Delta E_p$.

- En présence de forces conservatives et non conservatives, l'énergie mécanique ne se conserve pas, sa variation est égal au travail des forces non conservatives : $\Delta(E_c + E_p) = W_{nc}$. Lorsque les forces non conservatives se réduisent à des forces de frottement, alors $W_{nc} = W_f$.

4.9 Exercices d'application

Exercice 1 : Un bloc de masse $m = 2$ kg, initialement au repos sur une table horizontale, se met en mouvement sous l'action d'une force $F = 7$ N appliquée horizontalement. Le coefficient de frottement cinétique entre le bloc et la table est $\mu_c = 0.1$.
a) Calculer l'accélération du bloc.
Rép. : En plus de la force appliquée F, le bloc est soumis à son poids $m\,\mathrm{g}$, la réaction normale N de la table et à la force de frottement cinétique $F_r = \mu_c N$. Les forces $m\,\mathrm{g}$ et N se compensent, i.e. $N = m\,\mathrm{g}$. Le bloc se déplace dans le sens de F alors que F_r s'oppose à la direction de déplacement. Au total, le bloc subit la force nette $F - F_r$. D'après la deuxième loi de Newton, la force nette communique au bloc l'accélération $a = (F - F_r)/m = (7 - 0.1 \times 2 \times 9.8)/2 = (7 - 1.96)/2 = 2.52$ m/ s^2.

b) En déduire la distance parcourue après 10 s.
Rép. : Le mouvement est uniformément accéléré puisqu'on a une accélération constante. Sachant que le bloc a une vitesse initiale nulle, la distance parcourue après 10 s est donnée par : $d = at^2/2 = 2.52 \times 10^2/2 = 126$ m.

c) Calculer le travail effectué sur le bloc après 10 s par chacune des forces qui agissent sur lui.
Rép. : $W_F = Fd = 7 \times 126 = 882$ J.
$W_{F_r} = F_r \times d = -1.96 \times 126 = -247$ J.
$W_{m\,\mathrm{g}} = W_N$, car ces deux forces sont $\perp$ au déplacement.

d) Calculer le travail total sur le bloc après 10 s.
Le travail total est donné par la somme algébrique des travaux calculés en c).
$W_{\text{total}} = 882 - 247 + 0 + 0 = 635$ J.
Le travail W_{total} est équivalent au travail de la force nette agissant sur le bloc : $W_{\text{total}} = (F - F_r) \times d = 5.04 \times 126 = 635$ J.

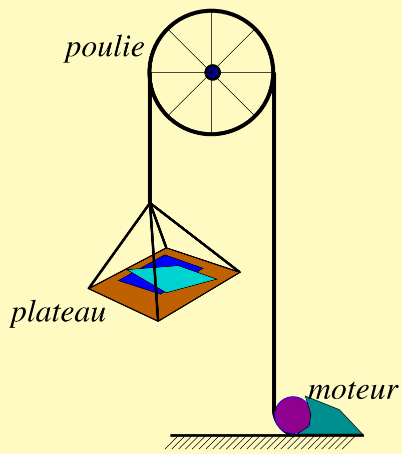
Exercice 2 : Une voiture de 1000 kg dispose d'une force de freinage maximum égale à 5000 N.

a) Exprimer la distance d'arrêt minimum en fonction de sa vitesse ? (R : 0,1 v^2 en unités S.I.).

b) Quelle est la puissance maximum développée par les freins si la vitesse initiale de la voiture est de 108 km/ h ? (R : 150 kW).

Solution : a) $\frac{1}{2} \times 1000 \times 0^2 - \frac{1}{2} \times 1000 \times v^2 = -5000 \times d \implies d = (-500/-5000)v^2 = 0.1v^2$
b)

Exercice 3 : Un monte-charge consiste en un plateau sur lequel on dépose la charge à monter ou à descendre. Ce plateau est suspendu à un câble passant sur treuil actionné par un moteur électrique.



a) Quel travail le monte-charge effectue-t-il lorsqu'il monte une charge de 60 kg, comprenant la masse du plateau et de ses accessoires, d'une hauteur de 3 m ?

b) Quelle doit être la puissance minimum du moteur pour effectuer ce travail en une minute ?

c) Sachant que la poulie a un rayon de 25 cm, quelle doit être la vitesse angulaire de la poulie pour effectuer ce travail en une minute ?

d) Lorsque la charge est arrivée au sommet de sa trajectoire, le câble casse et la charge retombe sur le sol, 3 m plus bas. Avec quelle vitesse touchera-t-elle le sol ?

e) Combien de temps mettra-t-elle pour arriver au sol ? On néglige la résistance de l'air.

Rép. : a) 1800 J ; b) 30 W ; c) 0,2 rad/s ; d) 7,8 m/ s ; e) 0,77 s.

δ (delta minuscule)

Une variation peut s'étudier comme résultant de l'accumulation successive de plusieurs petits apports. Chacun de ces apports n'est pas considéré comme une variation à proprement parler, mais comme une quantité élémentaire. On utilise la lettre grecque delta minuscule (δ) pour indiquer une telle petite quantité n'étant pas une variation. Cependant, la variation Δ, d ou ∂ d'une grandeur peut dépendre de cet apport δ.

Par exemple, considérons un compte bancaire sur lequel sont effectués plusieurs petits prélèvements P d'argent (petits par rapport au total de tout ce qui sera prélevé). Si P vaut 10 euros, on peut noter cette quantité $\delta P = 10$. Cette quantité est simplement une valeur numérique qui ne correspond pas à un écart entre deux sommes d'argent, à un gain ni à une perte. Le compte subit maintenant une variation de valeur $-\delta P$ (retrait de la quantité numérique 10 euros), si bien qu'on peut maintenant parler de la variation du montant total T du compte. Pour autant, le montant δP n'est pas, lui, une variation (un billet de 10 euros a la valeur qu'il a, c'est une quantité, pas une variation). Si on voyait les comptes (débité et destinataire) comme deux récipients reliés par un tuyau, on pourrait parler de la variation de niveau d'un des récipients. Mais on ne parlerait pas de variation de niveau d'eau dans le tuyau. L'eau y circule mais le tuyau est toujours plein. C'est aussi le cas de notre δP : c'est une quantité créée ou déplacée, pas une variation en tant que telle. L'introduction de la notation δ correspond donc essentiellement à un besoin de distinction sémantique entre variation et amplitude d'une variation.

On retrouve souvent cette distinction en physique. Par exemple, considérons le travail d'une force $\vec{F}$ sur un petit déplacement $d\vec{r}$: on note δW un travail élémentaire sur un court déplacement et on a la relation $\delta W = \vec{F} \cdot d\vec{r}$ — on fait généralement le bilan énergétique d'un système, dont on étudie par exemple l'évolution de l'énergie interne. Ce travail élémentaire étant « hors système », δW n'est pas la variation d'une grandeur mais un prélèvement ou un dépôt élémentaire d'énergie. C'est une manière de voir les choses, car on pourrait dans un autre contexte décider de s'intéresser à une fonction f représentant l'énergie totale apportée par le travail de cette force $\vec{F}$: on noterait alors df comme précédemment. De façon générale, pour le travail d'une force, on n'écrit pas ΔW mais W : on ne le voit pas comme la variation d'une grandeur mais comme une quantité d'énergie. Et δW est donc un des apports élémentaires en cours de route, une partie de la quantité totale.