Chapter's exercises with answers

Exercice 1

For problems 1 to 4, verify if the function y_i is a solution to the differential equation.

1. $y'' - y = 0; \quad y_1(x) = e^x, \quad y_2(x) = \cosh(x)$
2. $xy' - y = x^2; \quad y(x) = x^2 + 3x$
3. $y'' + 2y' - 3y = 0; \quad y_1(x) = e^x, \quad y_2(x) = e^{-3x}$
4. $x^2y'' + 5xy' + 4y = 0, \quad x > 0; \quad y_1(x) = x^{-2}, \quad y_2(x) = x^{-2}\ln(x)$

Correction

1

$$\begin{cases} y_1(x) = & e^x \\ y_1'(x) = & e^x \wedge y_1''(x) = e^x \end{cases} \Rightarrow y_1'' - y_1 = e^x - e^x = 0$$

So y_1 is a solution to the differential equation

$$\begin{cases} y_2(x) = & \cosh(x) \\ y_2'(x) = & \sinh(x) \wedge y_2''(x) = \cosh(x) \end{cases} \Rightarrow y_2'' - y_2 = \cosh(x) - \cosh(x) = 0$$

So y_2 is a solution to the differential equation

2

$$\begin{cases} y(x) = x^2 + 3x \\ y'(x) = 2x + 3 \end{cases} \Rightarrow xy'(x) - y(x) = x(2x+3) - x^2 - 3x = x^2$$

So $y(x)$ is a solution to the differential equation

3

$$\begin{cases} y_1(x) = & e^x \\ y_1'(x) = & e^x \wedge y_1''(x) = e^x \end{cases} \Rightarrow y_1'' + 2y_1' - 3y_1 = e^x + 2e^x - 3e^x = 0$$

So y_1 is a solution to the differential equation

$$\begin{cases} y_2(x) = & e^{-3x} \\ y_2'(x) = & -3e^{-3x} \wedge y_2''(x) = 9e^{-3x} \end{cases} \Rightarrow y_2'' + 2y_2' - 3y_2 = 9e^{-3x} - 6e^{-3x} - 3e^{-3x} = 0$$

So y_2 is a solution to the differential equation

4

$$\begin{cases} y_1(x) = & x^{-2} \\ y_1'(x) = & -2x^{-3} \wedge y_1''(x) = 6x^{-4} \end{cases} \Rightarrow x^2y_1'' + 5xy_1' + 4y_1 = 6x^{-2} - 10x^{-2} + 4x^{-2} = 0$$

So y_1 is a solution to the differential equation

$$\begin{cases} y_2(x) = & x^{-2}\ln(x) \\ y_2'(x) = & \dfrac{1 - 2\ln(x)}{x^3} \wedge \\ y_2''(x) = & \dfrac{-5 + 6\ln(x)}{x^4} \end{cases} \Rightarrow x^2y_2'' + 5xy_2' + 4y_2 = \frac{-5 + 6\ln(x)}{x^2} + \frac{5 - 10\ln(x)}{x^2} + \frac{4\ln(x)}{x^2} = 0$$

So y_2 is a solution to the differential equation

Equations with separable variables

Exercice 2

Solve (explicitly, if possible) the following ordinary differential equations:

1. $x^2y' + y = 2$
2. $(x+2)y^2y' + x^2(y-2) = 0$

Correction

1.

$$\begin{aligned} x^2y' + y + 2 = 0 &\Leftrightarrow x^2dy = (2-y)dx \Leftrightarrow \frac{dx}{x^2} = \frac{dy}{2-y} \\ &\Leftrightarrow \int \frac{dx}{x^2} = \int \frac{dy}{2-y} \Leftrightarrow C - \frac{1}{x} = -\ln|2-y| \\ &\Leftrightarrow C + \frac{1}{x} = \ln|2-y| \Leftrightarrow |2-y| = e^C e^{\frac{1}{x}} \\ &\Leftrightarrow 2 - y = ke^{\frac{1}{x}} \Leftrightarrow y(x) = 2 + \lambda e^{\frac{1}{x}}, \quad \lambda \in \mathbb{R} \end{aligned}$$

2.

$$\begin{aligned} (x+2)y^2y' + x^2(y-2) = 0 &\Leftrightarrow \frac{y^2}{2-y}dy = \frac{x^2}{x+2}dx \\ &\Leftrightarrow \left(-y - 2 - \frac{4}{y-2}\right) dy = \left(x - 2 + \tfrac{4}{x+2}\right) dx \\ &\Leftrightarrow \int \left(-y - 2 - \frac{4}{y-2}\right) dy = \int \left(x - 2 + \frac{4}{x+2}\right) dx \\ &\Leftrightarrow -\tfrac{1}{2}y^2 - 2y - 4\ln|y-2| = \tfrac{1}{2}x^2 - 2x + 4\ln|x+2| + C \\ &\Leftrightarrow \tfrac{1}{2}y^2 + 2y + 4\ln|y-2| + \tfrac{1}{2}x^2 - 2x + 4\ln|x+2| = C \end{aligned}$$

Homogeneous equations.

Exercice 3

1. $xy' + x + y = 0$
2. $xy' = y + \sqrt{x^2 + y^2}$

Correction

1. We have :

$$\begin{aligned} xy' + x + y = 0 &\Rightarrow y' = -1 - \frac{y}{x} \\ &\Rightarrow f(x,y) = -1 - \frac{y}{x} \end{aligned}$$

$$\begin{aligned} f(1, \frac{y}{x}) &= -1 - \frac{y}{x} \\ &= f(x,y) \end{aligned}$$

which implies that f is a homogeneous function. then we use the change $y(x) = xz(x)$.

$$\begin{aligned} y(x) = xz(x) &\Rightarrow y' = z + z'x \Rightarrow z + z'x = -1 - z \\ &\Rightarrow \frac{dz}{2z+1} = \frac{-dx}{x} \Rightarrow \int \frac{dz}{2z+1} = \frac{-dx}{x} \\ &\Rightarrow -\ln|x| - \tfrac{1}{2}\ln|2z+1| = C \Rightarrow -\tfrac{1}{2}\ln(x)^2 - \tfrac{1}{2}\ln|2z+1| = C \\ &\Rightarrow \ln(x^2|2z+1|) = C \Rightarrow x^2(2z+1) = C \\ &\Rightarrow x^2(2\frac{y}{x} + 1) = C \Rightarrow y(x) = \frac{C - x^2}{2x} \end{aligned}$$

2 We have :

$$\begin{aligned} xy' = y + \sqrt{x^2+y^2} \;&\Rightarrow\; y' = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2} \\ &\Rightarrow\; f(x,y) = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2} \end{aligned}$$

$$\begin{aligned} f(1,\frac{y}{x}) &= \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2} \\ &= f(x,y) \end{aligned}$$

which implies that f is a homogeneous function. then we use the change $y(x) = xz(x)$.

$$\begin{aligned} y(x) = xz(x) &\Rightarrow\; y' = z + z'x \Rightarrow z + z'x = z + \sqrt{1+z^2} \\ &\Rightarrow\; \frac{dz}{\sqrt{1+z^2}} = \frac{dx}{x} \Rightarrow \int \frac{dz}{\sqrt{1+z^2}} = \int \frac{dx}{x} \\ &\Rightarrow\; \operatorname{argsinh}(z) - \ln|x| = C \Rightarrow \ln(z+\sqrt{1+z^2}) - \ln|x| = C \\ &\Rightarrow\; \ln\left(\frac{z+\sqrt{1+z^2}}{|x|}\right) = C \Rightarrow \frac{z+\sqrt{1+z^2}}{x} = C \\ &\Rightarrow\; Cx - z = \sqrt{1+z^2} \Rightarrow C^2x^2 - 2Cxz = 1 \\ &\Rightarrow\; C^2x^2 - 2Cy(x) = 1 \Rightarrow y(x) = \frac{C^2x^2-1}{2C} \end{aligned}$$

Exercice 4

For problems 1 to 4, determine the order of the differential equation, then specify if it's linear or non-linear.

1 $x^2y'' + xy' + 3y = \cos(x)$

2 $\dfrac{d^2y}{dx^2} + \cos(y-x) = e^x$

3 $\dfrac{d^3y}{dx^3} + x\dfrac{dy}{dx} + (\cos^2 x)y = \sin(x)$

4 $3y^{(4)} - y^{(3)} + \sqrt{2}y'' + y' + y = 2$

Correction

1 The order is $n = 2$. Since F is linear, the differential equation is linear.

$$F(y) = x^2y'' + xy' + 3y$$

Let y_1, y_2 be two functions which are twice differentiable and $\alpha \in \mathbb{R}$:

$$\begin{aligned} F(\alpha y_1 + y_2) &= x^2(\alpha y_1 + y_2)'' + x(\alpha y_1 + y_2)' + 3(\alpha y_1 + y_2) \\ &= \alpha x^2 y_1'' + x^2 y_2'' + \alpha x y_1' + x y_2' + 3\alpha y_1 + 3y_2 \\ &= \alpha(x^2y_1'' + xy_1' + 3y_1) + x^2y_2'' + xy_2' + 3y_2 \\ &= \alpha F(y_1) + F(y_2) \end{aligned}$$

2 The order is $n = 2$, Since the map F is non-linear, the differential equation is non-linear. ($F(0) = \cos(x) \neq 0$).

$$F(y) = y'' + \cos(y-x)$$

3 The order is $n = 3$, $F(y) = y^{(3)} + xy' + (\cos^2 x)y$, F is a linear map.

4 The order is $n = 4$, $F(y) = 3y^{(4)} - y^{(3)} + \sqrt{2}y'' + y' + y$, F is a linear map.

Linear equations by integrating factor method.

Exercice 5

1. $y' + 4y = e^{-3x}$
2. $y' + 2xy = x$

Correction

1.

$$y' + 4y = e^{-3x} \Rightarrow a(x) = 4 \Rightarrow \mu(x) = e^{4x}$$

which implies that

$$\begin{aligned} e^{4x}y' + 4e^{4x}y = e^{x} &\Rightarrow (ye^{4x})' = e^{x} \\ &\Rightarrow ye^{4x} = e^{x} + C \\ &\Rightarrow y(x) = e^{-3x} + Ce^{-4x} \end{aligned}$$

2.

$$y' + 2xy = x \Rightarrow a(x) = 2x \Rightarrow \mu(x) = e^{(x^2)}$$

which implies that

$$\begin{aligned} y'e^{(x^2)} + 2xe^{(x^2)}y = xe^{(x^2)} &\Rightarrow \left(y(x)e^{(x^2)}\right)' = xe^{(x^2)} \\ &\Rightarrow y(x)e^{(x^2)} = \int xe^{(x^2)}dx + C \\ &\Rightarrow y(x)e^{(x^2)} = \frac{1}{2}\int (2x)e^{(x^2)}dx + C \\ &\Rightarrow y(x)e^{(x^2)} = \frac{1}{2}e^{(x^2)} + C \\ &\Rightarrow y(x) = \frac{1}{2} + Ce^{-(x^2)} \end{aligned}$$

Linear equations via the constant variation method.

Exercice 6

1. $y' + 4y = e^{-3x}$
2. $y' + 2xy = x$

Correction

1. The homogeneous equation (the equation associated without a second member) is $y' + 4y = 0$, which implies that $y_h = Ce^{-4x}$. We're looking for a particular solution in the form $y_p = C(x)e^{-4x}$. So we write

$$y_p(x) = C(x)e^{-4x} \Rightarrow y_p'(x) = C'(x)e^{-4x} - 4C(x)e^{-4x}$$

Hence :

$$\begin{aligned} y_p'(x) + 4y_p(x) = e^{-3x} &\Rightarrow C'(x)e^{-4x} - 4C(x)e^{-4x} + 4C(x)e^{-4x} = e^{-3x} \\ &\Rightarrow C'(x) = e^{x} \\ &\Rightarrow C(x) = e^{x} \Rightarrow y_p(x) = e^{-3x} \end{aligned}$$

Finally

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= Ce^{-4x} + e^{-3x} \end{aligned}$$

2 The homogeneous equation (the equation associated without a second member) is $y' + 2xy = 0$, which means that $y_h = Ce^{-(x^2)}$. We're looking for a particular solution in the form $y_p = C(x)e^{-(x^2)}$. So we write

$$y_p = C(x)e^{-(x^2)} \Rightarrow y_p'(x) = C'(x)e^{-(x^2)} - 2xC(x)e^{-(x^2)}$$

Hence :

$$\begin{aligned} y_p'(x) + 2xy_p(x) = x &\Rightarrow C'(x)e^{-(x^2)} - 2xC(x)e^{-(x^2)} + 2xC(x)e^{-(x^2)} = x \\ &\Rightarrow C'(x)e^{-(x^2)} = x \Rightarrow C'(x) = xe^{(x^2)} \\ &\Rightarrow C(x) = \int xe^{(x^2)}dx = \frac{1}{2}\int 2xe^{(x^2)}dx = \frac{1}{2}e^{(x^2)} \\ &\Rightarrow y_p(x) = \frac{1}{2}e^{(x^2)}e^{-(x^2)} = \frac{1}{2} \end{aligned}$$

Finally

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= Ce^{-(x^2)} + \frac{1}{2} \end{aligned}$$