Chapter 5

Differential Calculus of Functions of One Variable

Contents

5.1 The Derivative of a Function at a Point

Definition 5.1

Let f be a function defined in the neighborhood of x_0. We say that f is differentiable at a point x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists in $\mathbb{R}$. When this limit exists, it is denoted by $f'(x_0)$ and called the derivative of f at x_0.

Remark 5.1 *If we put $x - x_0 = h$, the quantity $\frac{f(x) - f(x_0)}{x - x_0}$ becomes $\frac{f(x_0 + h) - f(x_0)}{h}$. So we can define the notion of differentiability of f at x_0 in the following way:*

$$f \text{ is differentiable at the point } x_0 \Leftrightarrow \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ exists in } \mathbb{R}$$

Notations:

We can use the notations $f'(x_0)$, $Df(x_0)$, $\frac{df}{dx}(x_0)$ to designate the derivative of f at x_0.

Example 5.1

1. The function $f(x) = x^2$ is differentiable at any point $x_0 \in \mathbb{R}$ and the derivative $f'(x_0) = 2x_0$. As an explanation, given $x_0 \in \mathbb{R}$ we have:

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h} = \lim_{h\to 0}(h+2x_0) = 2x_0.$$

2. The function $f(x) = \sin(x)$ is differentiable at any point $x_0 \in \mathbb{R}$ and the derivative $f'(x_0) = \cos(x_0)$. As an explanation, given $x_0 \in \mathbb{R}$ we have:

$$\begin{aligned}\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} &= \lim_{h\to 0} \frac{\sin(x_0+h)-\sin(x_0)}{h} \\ &= \lim_{h\to 0} \cos\left(\frac{2x_0+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = \cos(x_0)\end{aligned}$$

Definition 5.2: (Left and right derivative)

1. Let f be a function defined on an interval of type $[x_0, x_0+\alpha[$ with $\alpha > 0$. We say that f is right-differentiable at x_0 iff:

$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$$

exists in $\mathbb{R}$. This limit is denoted by $f'_r(x_0)$ and is called the right derivative of f at x_0.

2. Let f be a function defined on an interval of type $]x_0-\alpha, x_0]$ with $\alpha > 0$. We say that f is left-differentiable at x_0 iff:

$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}$$

exists in $\mathbb{R}$. This limit is denoted by $f'_l(x_0)$ and is called the left derivative of f at x_0.

Proposition 5.1

Let f be a function defined in the neighborhood of x_0, we have:

$$f \text{ is differentiable at } x_0 \iff \begin{cases} f \text{ is differentiable on the right and left at } x_0 \\ \quad\text{and} \\ f'_r(x_0) = f'_l(x_0) \end{cases}$$

Example 5.2

Let $f(x) = |x|$, we have:

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} -\frac{h}{h} = -1 = f'_l(0)$$
$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1 = f'_r(0)$$

$\Longrightarrow$ The function f is differentiable on the right and on the left at $x_0 = 0$ and moreover $f'_r(0) = 1$ and

$$f'_l(0) = -1, \text{ so } f'_l(0) \neq f'_r(0) \implies f \text{ is not differentiable at } x_0 = 0$$

5.1.1 Geometrical interpretation

The figure below shows the graph of a function $y = f(x)$:

The ratio $\dfrac{f(x_0+h)-f(x_0)}{h} = \tan(\theta)$ is the slope of the straight line joining point $A(x_0, f(x_0))$ to point $B(x_0+h, f(x_0+h))$ on the graph. When $h \to 0$, this line tends towards the tangent (AC) to the curve at a point $A(x_0, f(x_0))$. So we get:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \tan(\alpha) = \frac{CD}{AD}$$

is the slope of the tangent to the curve at point $A(x_0, f(x_0))$.

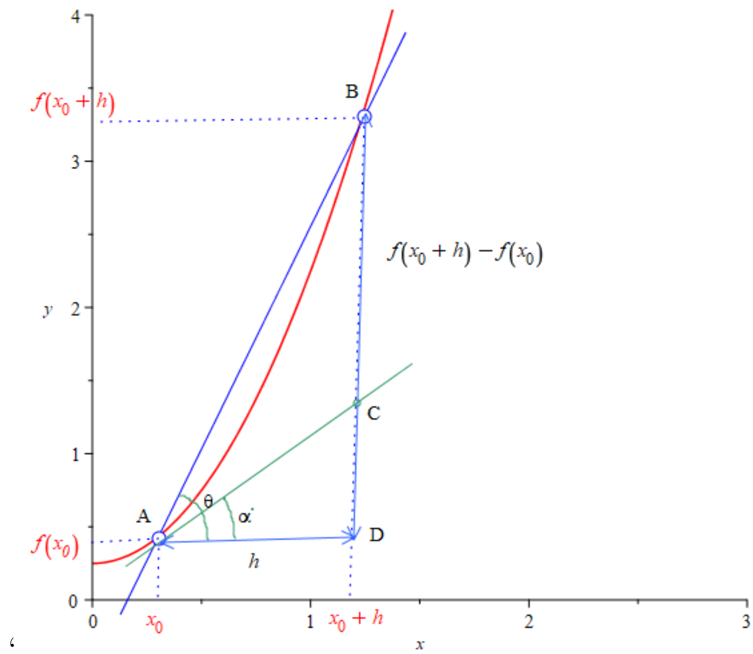

Figure 5.1: Geometrical Interpretation of Differentiability at a point x_0

Remark 5.2 *According to the figure above, the equation of the tangent to the curve $y = f(x)$ at the point $A(x_0, f(x_0))$ is $y - f(x_0) = f'(x_0)(x - x_0)$*

Proposition 5.2

Let f be a function differentiable at a point x_0, then f is continuous at x_0.

Proof:
We have: $\lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) (x - x_0)$
Since f is differentiable at x_0 we get:
$\lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} f'(x_0)(x - x_0) = 0 \implies f$ is continuous at x_0

Remark 5.3 *The opposite of this theorem is incorrect. A function can be continuous at a point x_0 without being differentiable at the same point. For example, the function $x \mapsto |x|$ is continuous at $x_0 = 0$ but not differentiable at the same point.*

5.2 Differential on an interval. Derivative function.

Definition 5.3

Let f be a function defined on an open interval I. We say that f is differentiable on I if: it is differentiable at any point on I. The function defined on I by: $x \mapsto f'(x)$ is called the derivative function or simply the derivative of the function f and is denoted by f' ou $\frac{df}{dx}$.

Remark 5.4 *let f be a function defined on an interval I and $a, b \in \mathbb{R} \cup \{+\infty, -\infty\}$ then:*

- *We say that f is differentiable on $I = [a, b]$ iff: it is differentiable on the open interval $]a, b[$ and differentiable on the right at a and on the left at b.*
- *We say that f is differentiable on $I = [a, b[$ if: it is differentiable on the open interval $]a, b[$ and differentiable on the right at a.*
- *We say that f is differentiable on $I =]a, b]$ if: it is differentiable on the open interval $]a, b[$ and differentiable on the left at b.*

5.3 Operations on differentiable functions

Proposition 5.3: (At a point)

Let f, g be two functions differentiable at x_0, then we have:

- $f + g$ is differentiable at x_0 et $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
- $f.g$ is differentiable at x_0 et $(f.g)'(x_0) = f'(x_0).g(x_0) + f(x_0).g'(x_0)$
- If we have: $f(x_0) \neq 0$, alors $\frac{1}{f}$ is differentiable at x_0 et $\left(\frac{1}{f}\right)'(x_0) = -\frac{f'(x_0)}{f(x_0)^2}$
- If we have: $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0).g(x_0) - f(x_0).g'(x_0)}{g(x_0)^2}$$

Proposition 5.4: (On an interval)

Let f and g be two functions differentiable on an open interval I then:

- $f+g$ is differentiable on I and $(f+g)' = f' + g'$
- $f.g$ is differentiable on I and $(f.g)' = f'.g + f.g'$
- If $f \neq 0$ on I, $\frac{1}{f}$ is differentiable on I and $\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$
- If $g \neq 0$ on I, $\frac{f}{g}$ is differentiable on I and

$$\left(\frac{f}{g}\right)' = \frac{f'.g - f.g'}{g^2}$$

Proposition 5.5: Differentiability and composition

Let $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ be two functions where I and J are two open intervals such that: $f(I) \subset J$

- **Differentiability at a point:** If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0 and $(g \circ f)'(x_0) = f'(x_0).g'(f(x_0))$
- **differentiability on an interval:** If f is differentiable on I and g is differentiable on J, then $g \circ f$ is differentiable on I and $(g \circ f)' = f'.(g' \circ f)$

Proposition 5.6: Differentiability and inverse function

Let $f : I \longrightarrow J$ be a bijective and differentiable function at $x_0 \in I$. Then f^{-1} is differentiable at $y_0 = f(x_0)$ if and only if $f'(x_0) \neq 0$ and in this case: $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.

Proposition 5.7

Let $f : I \longrightarrow J$ be a bijective and differentiable function on I. If $f' \neq 0$ on I, then f^{-1} is differentiable on J and we have : $(f^{-1})' = \frac{1}{f' \circ f^{-1}}$

5.4 Mean value Theorem

Theorem 5.1: (Rolle's theorem)

Let f be a function defined on $[a, b]$. If we have:

1. f is continuous on $[a, b]$.
2. f is differentiable on $]a, b[$
3. $f(a) = f(b)$

then there exists a real number $c \in]a, b[$ such that $f'(c) = 0$

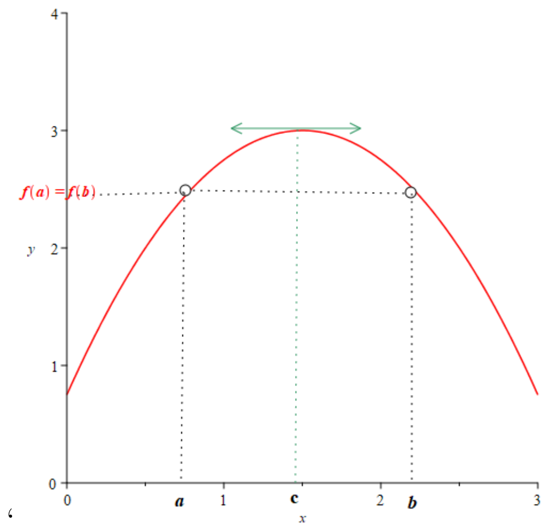

Figure 5.2: Geometrical interpretation of Rolle's theorem

Theorem 5.2: (Mean value Theorem)

Let f be a function defined on $[a, b]$, if we have:

1. f is continuous on $[a, b]$.
2. f is differentiable on $]a, b[$

then there exists a real number $c \in]a, b[$ such that:

$$f(b) - f(a) = f'(c)(b - a)$$

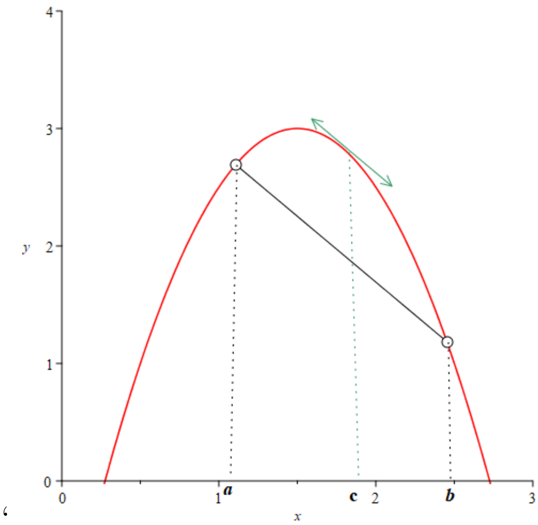

Figure 5.3: Geometrical interpretation of the mean value theorem

Consequence:(second form of the mean value theorem)

Let f be a function defined on I, $h > 0$ and $x_0 \in I$ such that $x_0 + h \in I$, then if we have:

1. f is continuous on $[x_0, x_0 + h]$.

2. f is derivable on $]x_0, x_0 + h[$

then there exists a $\theta \in]0, 1[$ such that:

$$f(x_0 + h) - f(x_0) = f'(x_0 + \theta.h)h$$

Example 5.3

By using the mean value theorem, show that:

$$\forall x > 0;\ \sin(x) \leq x$$

By putting $f(t) = t - \sin(t)$ we get:

$$\forall x > 0 \text{ we have:} \begin{cases} f \text{ is continuous on } [0, x] \\ \quad \text{and} \\ f \text{ is differentiable on }]0, x[\end{cases}$$

According to the mean value theorem, there exists $c \in]0, x[$ such that:

$$f(x) - f(0) = f'(c)(x - 0)$$

$$\iff x - \sin(x) = (1 - \cos(c))x \iff \sin(x) = \cos(c)x$$

$$\implies \sin(x) \leq x \text{ (as } \cos(c) \leq 1)$$

Theorem 5.3: Generalized mean value theorem

Let f and g be two real functions defined on $[a, b]$ such that:

1. f and g are continuous on $[a, b]$.
2. f and g are differentiable on $]a, b[$.

Then there exists a real number $c \in]a, b[$ such that:

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

Proposition 5.8: (Variations of a function)

Let f be a continuous function on $[a, b]$ and differentiable on $]a, b[$, we have:

1. If $f'(x) > 0$ on $]a, b[$, then f is strictly increasing on $[a, b]$.
2. If $f'(x) \geq 0$ on $]a, b[$, then f is increasing on $[a, b]$.
3. If $f'(x) < 0$ on $]a, b[$, then f is strictly decreasing on $[a, b]$.
4. If $f'(x) \leq 0$ on $]a, b[$, then f is decreasing on $[a, b]$.
5. If $f'(x) = 0$ on $]a, b[$, then f is constant on $[a, b]$.

5.4.1 L'Hôpital's rules

Theorem 5.4: (First rule of L'Hôpital)

Let f and g be two continuous functions on I (where I is a neighborhood of x_0), differentiable on $I - \{x_0\}$ and satisfying the following conditions:

1. $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0$
2. $\forall x \in I - \{x_0\};\ g'(x) \neq 0$

Then:

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l \implies \lim_{x \to x_0} \frac{f(x)}{g(x)} = l$$

Example 5.4

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$$

Remark 5.5 *The converse is generally false. For example:* $f(x) = x^2 \cos(\frac{1}{x})$, $g(x) = x$. *We have:* $\lim\limits_{x \to 0} \frac{f(x)}{g(x)} = \lim\limits_{x \to 0} x \cos(\frac{1}{x}) = 0$. *While* $\lim\limits_{x \to 0} \frac{f'(x)}{g'(x)} = \lim\limits_{x \to 0} (2x \cos(\frac{1}{x}) + \sin(\frac{1}{x}))$ *does not exist (since:* $\lim\limits_{x \to 0} \sin(\frac{1}{x})$ *does not exist)*

Remark 5.6 *Also, the Hopital's rules is true when* $x \to \pm\infty$

Theorem 5.5: (Second rule of L'Hôpital)

Let f and g be two functions defined on I (where I is a neighborhood of x_0), differentiable on $I - \{x_0\}$ and satisfying the following conditions:

1. $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \pm\infty$
2. $\forall x \in I - \{x_0\}$; $g'(x) \neq 0$

Then:

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l \implies \lim_{x \to x_0} \frac{f(x)}{g(x)} = l$$

Example 5.5

$$\lim_{x \to +\infty} \frac{x^n}{e^x} = \lim_{x \to +\infty} \frac{nx^{n-1}}{e^x} = \lim_{x \to +\infty} \frac{n(n-1)x^{n-2}}{e^x} = \text{.........} = \lim_{x \to +\infty} \frac{n!x^0}{e^x} = 0$$

5.5 Higher Order Derivatives

Definition 5.4

Let f be a function differentiable on I, then f' is called the 1st-order derivative of f; if f' is differentiable on I, then its derivative is called the 2nd-order derivative of f and is denoted by f'' or $f^{(2)}$. Recursively, we define the derivative of order n of f as follows:

$$\begin{cases} f^{(0)} = f \\ \\ (f^{(n-1)})' = f^{(n)} \end{cases}$$

Another notations used are: $D_n f, \dfrac{d^n f}{dx^n}$ for $f^{(n)}$

Example 5.6

$$\sin^{(n)}(x) = \sin(x + n\frac{\pi}{2}) \quad \text{and} \quad \cos^{(n)}(x) = \cos(x + n\frac{\pi}{2})$$

Definition 5.5: (Class Functions: C^n)

Let n be a non-zero natural number. A function f defined on I is said to be of class C^n or n times continuously differentiable if it is n times differentiable and $f^{(n)}$ is continuous on I, and we note $f \in C^n(I)$.

Remark 5.7 *A function f is said to be "of class C^0" if it is continuous on I.*

Definition 5.6: (Class Functions: C^∞)

A function f is said to be of class C^∞ on I if it is in the class C^n. $\forall n \in \mathbb{N}$

5.5.1 n-th derivative of a product (Leibniz rule)

Theorem 5.6

Let f and g be two functions n times differentiable on I, then fg is n times differentiable on I and we have:

$$\forall x \in I;\ (f.g)^n(x) = \sum_{k=0}^{n} C_n^k f^{(n-k)}(x) g^{(k)}(x)$$

$$\text{with: } C_n^k = \frac{n!}{k!(n-k)!}$$

Example 5.7

Compute $(x^2 \sin(2x))^{(3)}$ According to Leibniz' formula, we have:

$$\begin{aligned}(x^2 \sin(2x))^{(3)} =& \sum_{k=0}^{3} C_3^k (x^2)^{(3-k)} (\sin(2x))^{(k)} \\ =& C_3^0 (x^2)^{(3)} (\sin(2x))^{(0)} + C_3^1 (x^2)^{(2)} (\sin(2x))^{(1)} \\ & + C_3^2 (x^2)^{(1)} (\sin(2x))^{(2)} + C_3^3 (x^2)^{(0)} (\sin(2x))^{(3)} \\ =& 12\cos(2x) - 24x\sin(2x) - 8x^2\cos(2x)\end{aligned}$$

5.6 Taylor's formulas

Theorem 5.7: (Taylor's formula with Lagrange remainder)

Let $x_0 \in [a, b]$ et $f : [a, b] \longrightarrow \mathbb{R}$ be a function that checks:

1. $f \in C^n$ on $[a, b]$.
2. $f^{(n)}$ is differentiable on $]a, b[$.

then, $\forall x \in [a, b]$ (with $x \neq x_0$), $\exists c \in [a, b]$ such that:

$$\begin{aligned}f(x) =& f(x_0) + \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ & + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}\end{aligned}$$

This expression is the Taylor formula of order n with the Lagrange remainder

$$R_n(x, x_0) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Theorem 5.8: (Taylor Mac-Laurin formula)

If we set $x_0 = 0$ in the Taylor-Lagrange formula, we obtain:
$\exists\theta \in]0,1[$ such that:

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$$

This is Taylor Mac-Laurin's formula.

Remark 5.8 *In practice, the Taylor Mac-Laurin formula is used to calculate the approximate values.*

Example 5.8

Show that for every $x \in \mathbb{R}_+$:

$$x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$

Let $x \geq 0$, Applying the Taylor Mac-Laurin formula of order 2 to the function $f(x) = \ln(1+x)$, we find:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} \; / \theta \in]0,1[$$

Since $x \geq 0$ then,

$$x - \frac{x^2}{2} \leq x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$$

$$\Longrightarrow x - \frac{x^2}{2} \leq \ln(1+x) \tag{5.1}$$

On the other hand $\frac{x^3}{3(1+\theta x)^3} \leq \frac{x^3}{3}$

$$\Longrightarrow x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$\Longrightarrow \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} \tag{5.2}$$

from (5.1) and (5.2) we get:

$$x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$