Chapter 3

Sequences of real numbers

3.1 Definitions and examples

Definition 1

A sequence of real numbers is a real-valued function whose domain is the set of natural numbers $\mathbb{N}$ or an infinite subset $\mathcal{N}_1 \subset \mathbb{N}$ to the real numbers i.e:

$$\begin{array}{rcccc} u & : & \mathbb{N} & \longrightarrow & \mathbb{R} \\ & & n & \longmapsto & u(n) \end{array} \quad \text{or} \quad \begin{array}{rcccc} u & : & \mathcal{N}_1 & \longrightarrow & \mathbb{R} \\ & & n & \longmapsto & u(n) \end{array}$$

Notations:

- For $n \in \mathbb{N}$, $u(n)$ is denoted by u_n and is called the general term or n-th term of the sequence.
- The sequence u is denoted by $(u_n)_{n\in\mathbb{N}}$ or $(u_n)_{n\in\mathcal{N}_1}$.

Example 1

1. The sequence $(u_n)_{n\in\mathbb{N}^*}$ defined by: $u_n = \frac{1}{n}$, starts with $u_1 = 1$, and $u_2 = \frac{1}{2}$, $u_3 = \frac{1}{3}$,......

2. The recurrent sequence defined by: $\begin{cases} u_1 = 1 \\ u_n = 1 + \frac{1}{u_{n-1}} \end{cases}$ starts with $u_1 = 1$, and $u_2 = 2$, $u_3 = \frac{3}{2}$,.....

Remark

The ways in which a sequence can be defined.

- By an explicit definition of the general term of the sequence (u_n) i.e.: Express u_n in terms of n. For example, $u_n = \frac{2n+1}{n+7}$.
- By a recurrence formula, i.e. a relationship that links any term in the sequence to the one that precedes it. In this case, to calculate u_n, you need to calculate all the terms that precede it. For example :

$$\begin{cases} u_0 = 2 \\ u_{n+1} = 3u_n - 1 \end{cases}$$

3.2 Bounded sequences

Definition 2

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.

- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded from above iff: $\exists M \in \mathbb{R}, \forall n \in \mathbb{N}; u_n \leq M$
- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded from below iff: $\exists m \in \mathbb{R}, \forall n \in \mathbb{N}; m \leq u_n$
- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded iff: it is bounded from above and bounded from below which means :
$$\exists M \in \mathbb{R}_+, \forall n \in \mathbb{N};\ |u_n| \leq M$$

3.3 Increasing and decreasing sequences

Definition 3

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence

- $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence iff: $\forall n \in \mathbb{N}; u_n \leq u_{n+1}$
- $(u_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence iff: $\forall n \in \mathbb{N}; u_n < u_{n+1}$
- $(u_n)_{n\in\mathbb{N}}$ is a decreasing sequence iff: $\forall n \in \mathbb{N}; u_n \geq u_{n+1}$
- $(u_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence iff: $\forall n \in \mathbb{N}; u_n > u_{n+1}$
- $(u_n)_{n\in\mathbb{N}}$ is monotonic if it is increasing or decreasing.
- $(u_n)_n$ is strictly monotonic if it is strictly increasing or strictly decreasing.
- $(u_n)_{n\in\mathbb{N}}$ is a constant sequence iff $\forall n \in \mathbb{N};\ u_{n+1} = u_n$

3.4 Finite and infinite limit of a numerical sequence

Definition 4: Convergent sequences

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence. We say that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to l iff:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_0 \implies |u_n - l| \leq \varepsilon$$

In this case, we say that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent to the limit l and we note $\lim\limits_{n\to+\infty} u_n = l$

Remark

$$|u_n - l| \leq \varepsilon \Leftrightarrow l - \varepsilon \leq u_n \leq l + \varepsilon \Leftrightarrow u_n \in [l - \varepsilon, l + \varepsilon]$$

The above definition means that for any strictly positive real ε, there exists an integer n_0 (rank) such that: all terms $u_{n_0}, u_{n_0+1}, u_{n_0+2} \dots$ are in the interval $[l - \varepsilon, l + \varepsilon]$.

Example 2

- The sequence $u_n = \dfrac{n}{n+1}$ converges to 1

Using the definition of convergence, we show that $\lim\limits_{n \to +\infty} u_n = 1$

Let $\varepsilon > 0$ we have:

$$\begin{aligned} & |u_n - 1| \leq \varepsilon \\ \Leftrightarrow\ & |\frac{n}{n+1} - 1| \leq \varepsilon \\ \Leftrightarrow\ & |\frac{n}{n+1} - 1| \leq \varepsilon \\ \Leftrightarrow\ & |1 - \frac{1}{n+1} - 1| \leq \varepsilon \\ \Leftrightarrow\ & \frac{1}{n+1} \leq \varepsilon \\ \Leftrightarrow\ & \frac{1}{\varepsilon} - 1 \leq n \end{aligned}$$

By setting $n_0 = \lfloor \frac{1}{\varepsilon} \rfloor > \frac{1}{\varepsilon} - 1$, we obtain :

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}\, (n_0 = \lfloor \frac{1}{\varepsilon} \rfloor), \forall n \in \mathbb{N}; n \geq n_0 \implies |u_n - 1| \leq \varepsilon$

$\implies (u_n)_{n \in \mathbb{N}}$ converges to $l = 1$

Using Maple, we get the following graph:

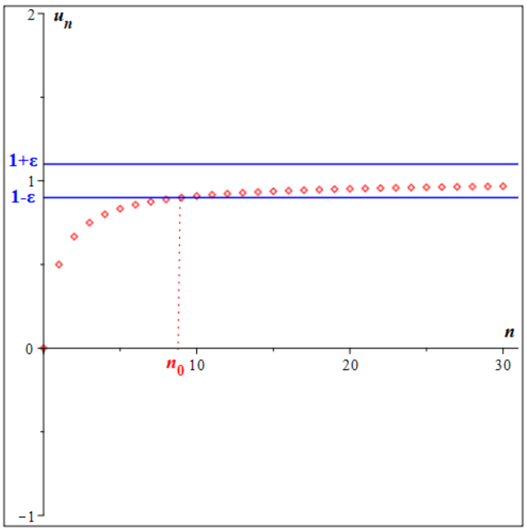

Figure 3.1: $\varepsilon = 0.1$

Definition 5

1. We say that the sequence $(u_n)_{n\in\mathbb{N}}$ tends to $+\infty$ as n tends to infinity and we note $\lim_{n\to+\infty} u_n = +\infty$ iff:

$$\forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 \implies u_n \geq A$$

2. We say that the sequence $(u_n)_{n\in\mathbb{N}}$ tends to $-\infty$ as n tends to infinity and we note $\lim_{n\to+\infty} u_n = -\infty$ iff:

$$\forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 \implies u_n \leq -A$$

Example 3

- Let be the following sequences :

$$\begin{cases} u_n = 2n+1 \\ v_n = -3n+4 \end{cases}$$

We show that $\lim_{n\to+\infty} u_n = +\infty$ and $\lim_{n\to+\infty} v_n = -\infty$

1. Let $A > 0$ on a:

$$\begin{aligned} & u_n \geq A \\ \Leftrightarrow\ & 2n+1 \geq A \\ \Leftrightarrow\ & 2n \geq A-1 \\ \Leftrightarrow\ & 2n \geq \frac{A-1}{2} \end{aligned}$$

Let's put $n_0 = [\frac{A-1}{2}] + 1 > \frac{A-1}{2}$

$\implies (\forall A > 0, \exists n_0 \in \mathbb{N}\,(n_0 = [\frac{A-1}{2}] + 1), \forall n \in \mathbb{N}; n \geq n_0 \implies u_n \geq A)$

2. The same method used for the sequence $(v_n)_{n\in\mathbb{N}}$

Definition 6: divergent sequences

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that the sequence $(u_n)_{n\in\mathbb{N}}$ is divergent if it is not convergent, i.e

$$\forall l \in \mathbb{R}, \exists \varepsilon > 0, \forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}; (n \geq n_0) \wedge (|u_n - l| > \varepsilon)$$

Remark

here are two types of divergence

1. Divergence of infinite type: in this case the sequence converges to $+\infty$ or $-\infty$. For example the sequence with general term $u_n = 2n+4$.

2. Divergence of type limit does not exist: in this case the sequence has no finite or infinite limit.

For example, the sequence with general term $u_n = (-1)^n$

Proof:
We will show that the sequence $(-1)^n$ does not have a finite or infinite limit.

1. By contradiction, suppose that: $\lim_{n\to+\infty}(-1)^n = l / l \in \mathbb{R}$. According to the convergence definition with $\varepsilon = \frac{1}{4}$ we get:

$$\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 \implies u_n \in [l - \frac{1}{4}, l + \frac{1}{4}]$$
$$\implies -1, 1 \in [l - \frac{1}{4}, l + \frac{1}{4}]$$
$$\implies \begin{cases} l - \frac{1}{4} \leq 1 \leq l + \frac{1}{4} \\ l - \frac{1}{4} \leq -1 \leq l + \frac{1}{4} \end{cases}$$
$$\implies \begin{cases} l - \frac{1}{4} \leq 1 \leq l + \frac{1}{4} \\ -l - \frac{1}{4} \leq 1 \leq -l + \frac{1}{4} \end{cases}$$
$$\implies \begin{cases} -\frac{1}{2} \leq 2 \leq \frac{1}{2} \end{cases}$$

It's a contradiction.

2. By contradiction, suppose that: $\lim_{n\to+\infty}(-1)^n = +\infty$. According to the convergence definition with $A = 4$ we get:

$$\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 \implies u_n \geq 4$$
$$\implies u_n \in [4, +\infty[\implies -1, 1 \in [4, +\infty[$$

It's a contradiction.

3. We use the same method for the case: $\lim_{n\to+\infty}(-1)^n = -\infty$

Proposition 1:

If a sequence of real numbers $(u_n)_{n\in\mathbb{N}}$ has a limit, then this limit is unique.

Proof:

By contradiction

Suppose that:; $\begin{cases} \lim_{n\to+\infty} u_n = l_1 \\ \lim_{n\to+\infty} u_n = l_2 \end{cases}$

Taking $\varepsilon = \frac{|l_1 - l_2|}{4}$ with $l_1 \neq l_2$ which implies

$$\begin{cases} \exists n_1 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_1 \implies |u_n - l_1| \leq \varepsilon \\ \exists n_2 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_2 \implies |u_n - l_2| \leq \varepsilon \end{cases}$$

Putting $n_0 = \max(n_1, n_2)$

$$\implies (\forall n \in \mathbb{N}; n \geq n_0 \implies |u_n - l_1| + |u_n - l_2| \leq 2\varepsilon)$$

With $n \geq n_0$ we get

$$\begin{aligned} |l_1 - l_2| &\leq |u_n - l_1| + |u_n - l_2| \leq 2\varepsilon \\ &\implies |1_1 - l_2| \leq 2\varepsilon \\ &\implies \frac{|1_1 - l_2|}{4} \leq \frac{\varepsilon}{2} \\ \implies \varepsilon \leq \frac{\varepsilon}{2} & \quad \text{it's a contradiction} \end{aligned}$$

Proposition 2

If $(u_n)_{n\in\mathbb{N}}$ is a convergent sequence, then $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence.

Proof:

We'll show the following implication:

$$(u_n)_{n\in\mathbb{N}} \text{ is a convergent sequence} \implies (u_n)_{n\in\mathbb{N}} \text{ is bounded}$$

Suppose that $(u_n)_{n\in\mathbb{N}}$ is convergent, then for $\varepsilon = 1$ we have:

$$\begin{aligned} \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_0 &\implies |u_n - l| \leq 1 \\ &\implies m = l - 1 \leq u_n \leq l + 1 = M \end{aligned}$$

So the set $\{u_{n_0}, u_{n_0+1}, ...\}$ is bounded.
On the other hand $A = \{u_0 ....., u_{n_0-2}, u_{n_0-1}\}$ is bounded (because $Card(A) < +\infty$). Then the set of values of (u_n) is: $\{u_0 ....., u_{n_0-2}, u_{n_0-1}, u_{n_0}, u_{n_0+1}, ...\}$ is bounded, this means (u_n) is bounded.

3.5 Finding Limits: Properties of Limits

Theorem 1

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ two convergent sequences with: $\lim\limits_{n\to+\infty} u_n = l$ and $\lim\limits_{n\to+\infty} v_n = l'$. The properties of limits are summarized as follows:

1. $\lim\limits_{n\to+\infty} \lambda u_n = \lambda l$ with $\lambda \in \mathbb{R}$
2. $\lim\limits_{n\to+\infty} (u_n + v_n) = l + l'$
3. $\lim\limits_{n\to+\infty} u_n v_n = ll'$
4. If $u_n \neq 0$ for $n \geq n_0$ and $l \neq 0$ then $\lim\limits_{n\to+\infty} \frac{1}{u_n} = \frac{1}{l}$
5. If $v_n \neq 0$ for $n \geq n_0$ and $l' \neq 0$ then $\lim\limits_{n\to+\infty} \frac{u_n}{v_n} = \frac{l}{l'}$

Remark

$\lim_{n\to+\infty} u_n = l \implies \lim_{n\to+\infty} |u_n| = |l|$. **Be careful** the reverse is not true. For example, if we take the sequence $u_n = (-1)^n$ we have $\lim_{n\to+\infty} |u_n| = 1$ but $\lim_{n\to+\infty} u_n$ doesn't exist.

Proposition 3: Infinite limit's operations

Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ two sequences with: $\lim_{n\to+\infty} u_n = +\infty$ and $\lim_{n\to+\infty} v_n = +\infty$ then:

1. $\lim_{n\to+\infty} (u_n + v_n) = +\infty$

2. If $\forall n \geq n_0$, $u_n \neq 0$ then $\lim_{n\to+\infty} \frac{1}{u_n} = 0$

3.6 Limits and inequalities

Theorem 2

1. Let $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ be two convergent sequences, then:

$$\text{If } \exists n_0 \in \mathbb{N}, \forall n \geq n_0;\ u_n \leq v_n \quad \text{this implies } \lim_{n\to+\infty} u_n \leq \lim_{n\to+\infty} v_n$$

2. If, we have $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ two sequences which verify:- $\begin{cases} \exists n_0 \in \mathbb{N}, \forall n \geq n_0;\ u_n \leq v_n \\ \quad \text{and} \\ \lim_{n\to+\infty} u_n = +\infty \end{cases}$

this implies $\lim_{n\to+\infty} v_n = +\infty$

3. Squeeze Theorem : If $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ three sequences with:

$$\begin{cases} \exists n_0 \in \mathbb{N}, \forall n \geq n_0;\ u_n \leq v_n \leq w_n \\ \quad \text{and} \\ \lim_{n\to+\infty} u_n = \lim_{n\to+\infty} w_n = l \end{cases}$$

then the sequence $(v_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to+\infty} v_n = l$

3.7 Convergence theorems

Theorem 3: Convergence of monotonic sequences

- If a sequence of real numbers is increasing and bounded from above, then it converges.
- If a sequence of real numbers is decreasing and bounded from below, then it converges.

Example 4

Let $(u_n)_{n\in\mathbb{N}}$ be a numerical sequence defined by: $\begin{cases} u_0 = 1 \\ u_{n+1} = \dfrac{1+u_n^2}{2} \end{cases}$.

1. Prove that $\forall n \in \mathbb{N};\ u_n \leq 1$
2. Deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent.

- by using proof by induction, we have for $n=0$, $u_0 = \frac{1}{2} \leq 1$ so the proposition is true. Let's assume that the proposition is true for $k \in \{1,\dots,n\}$ and we'll show that $u_{n+1} \leq 1$. According to the assumption we have:

$$u_n \leq 1 \implies u_n^2 \leq 1 \implies 1+u_n^2 \leq 2 \implies \frac{1+u_n^2}{2} \leq 1 \implies u_{n+1} \leq 1$$

So, assertion $\forall n \in \mathbb{N};\ u_n \leq 1$ is true.

- On a $\forall n \in \mathbb{N}; u_{n+1} - u_n = \dfrac{1+u_n^2}{2} - u_n = \dfrac{(u_n-1)^2}{2} \geq 0$
- Since $(u_n)_{n\in\mathbb{N}}$ is increasing and bounded from above so $(u_n)_{n\in\mathbb{N}}$ is convergent.

Definition 7: Adjacent sequences

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two real sequences. We say that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent iff:

$$\begin{cases} (u_n)_{n\in\mathbb{N}} \quad \text{is increasing} \\ \qquad \text{and} \\ (v_n)_{n\in\mathbb{N}} \quad \text{is decreasing} \\ \qquad \text{and} \\ \lim\limits_{n\to+\infty} (u_n - v_n) = 0 \end{cases}$$

Theorem 4:

If the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent then they converge to the same limit.

Example 5

The sequences $u_n = \sum\limits_{k=1}^{n} \dfrac{1}{k^2}$ and $v_n = u_n + \dfrac{2}{n}$ are adjacent :

- $u_{n+1} - u_n = \sum\limits_{k=1}^{n+1} \dfrac{1}{k^2} - \sum\limits_{k=1}^{n} \dfrac{1}{k^2} = \dfrac{1}{(n+1)^2} \geq 0 \implies (u_n)_{n\in\mathbb{N}}$ is increasing
- $v_{n+1} - v_n = \dfrac{1}{(n+1)^2} + \dfrac{2}{n+1} - \dfrac{2}{n} = -\dfrac{(n+2)}{n(n+1)^2} \leq 0 \implies (v_n)_{n\in\mathbb{N}}$ is decreasing

- $\lim_{n\to+\infty}(u_n - v_n) = \lim_{n\to+\infty}(-\frac{2}{n}) = 0$

Therefore the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are convergent to the same limits.

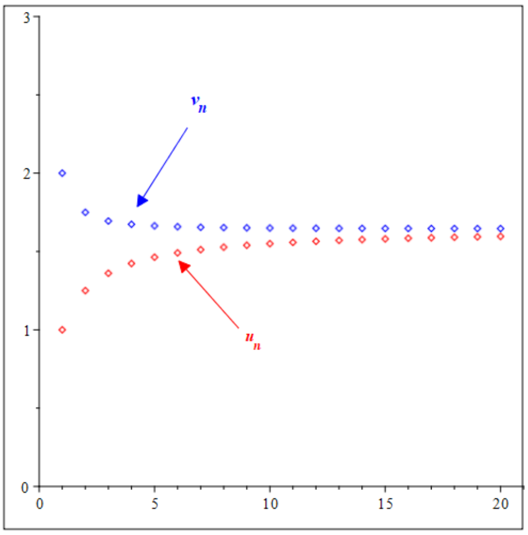

Figure 3.2: (u_n) and (v_n) are adjacent

Definition 8: Cauchy sequence

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.
$(u_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence in $\mathbb{R}$ iff:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N};\ p, q \geq n_0 \implies |u_p - u_q| \leq \varepsilon$$

Remark

$|u_p - u_q| \leq \varepsilon \Leftrightarrow$ the distance between u_p, and u_q is less than ε.
So the definition above means that:- for any strictly positive real ε, there exists n_0 (rank), such that the distance between each two terms u_p,u_q (with $p, q \geq n_0$) is less than ε.

Using Maple, we obtain the following graph of a Cauchy sequence:

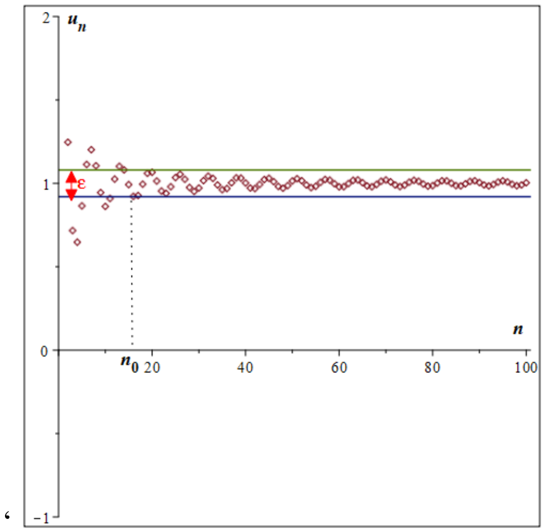

Figure 3.3: $u_n = \dfrac{\cos(n)+\sin(n)+n}{n}$, $\varepsilon = 0.08$

Example 6

- $u_n = \dfrac{1}{n}$ is a Cauchy sequence

Let $p, q \in \mathbb{N}^*$ with $p \leq q$ then we have:

$$|u_p - u_q| = \left|\frac{1}{p} - \frac{1}{q}\right| \leq \left|\frac{1}{p}\right| + \left|\frac{1}{q}\right| \quad \text{according to the triangular inequality}$$

$$\Longrightarrow |u_p - u_q| \leq \frac{2}{p} \quad (\text{ because: } \quad \frac{1}{q} \leq \frac{1}{p})$$

Let $\varepsilon > 0$, we put $n_0 = \left[\dfrac{2}{\varepsilon}\right] + 1 > \dfrac{2}{\varepsilon}$

So, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}^*$, $\forall p, q \in \mathbb{N}^*$; $p, q \geq n_0 \implies |u_p - u_q| \leq \varepsilon$

Theorem 5:

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence then:
$(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence $\Longleftrightarrow$ $(u_n)_{n\in\mathbb{N}}$ is convergent

3.8 Subsequence

Definition 9

The sequence $(u_{\phi(n)})_{n\in\mathbb{N}}$ is a subsequence of the sequence $(u_n)_{n\in\mathbb{N}}$ if $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ is a strictly increasing sequence of of natural numbers.

Example 7

1. $u_n = (-1)^n \longrightarrow \begin{cases} u_{2n} = (-1)^{2n} = 1 \\ \\ u_{2n+1} = (-1)^{2n+1} = -1 \end{cases}$

   $(u_{2n})_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ are subsequences taken from $(u_n)_{n\in\mathbb{N}}$

2. $v_n = \cos(\frac{n\pi}{3}) \longrightarrow v_{3n} = \cos(n\pi) = (-1)^n$

   $(v_{3n})_{n\in\mathbb{N}}$ is a sub-sequence of $(v_n)_{n\in\mathbb{N}}$

Proposition 4:

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers:

1. If $\lim\limits_{n\to+\infty} u_n = l$, then for any subsequence $(u_{\phi(n)})_{n\in\mathbb{N}}$; $\lim\limits_{n\to+\infty} u_{\phi(n)} = l$
2. If $(u_n)_{n\in\mathbb{N}}$ admits a divergent subsequence then $(u_n)_{n\in\mathbb{N}}$ is divergent
3. If $(u_n)_{n\in\mathbb{N}}$ has two subsequences converging to distinct limits then $(u_n)_{n\in\mathbb{N}}$ is divergent.

Example 8

the sequence with general term $u_n = (-1)^n$ is divergent:
We have:

$$\begin{cases} u_{2n} = 1 \\ \text{and} \\ u_{2n+1} = -1 \end{cases} \Longrightarrow \begin{cases} \lim\limits_{n\to+\infty} u_{2n} = 1 \\ \text{and} \\ \lim\limits_{n\to+\infty} u_{2n+1} = -1 \end{cases}$$

So,$(u_{2n})_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ are two subsequences of $(u_n)_{n\in\mathbb{N}}$ which converge to distinct limits, therefore $(u_n)_{n\in\mathbb{N}}$ is divergent.

Theorem 6: Bolzano-Weierstrass Property

Every bounded sequence has a convergent sub-sequence.

Definition 9: Cluster Points of the sequence

A cluster Point of a numerical sequence $(u_n)_{n\in\mathbb{N}}$ is any scalar which is the limit of a subsequence of $(u_n)_{n\in\mathbb{N}}$.

Example 9

- Let's consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by: $u_n = \cos(n\frac{\pi}{2})$

$$\begin{cases} u_{4n} = \cos(2n\pi) = 1 \implies \lim_{n\to+\infty} u_{4n} = 1 \\ u_{4n+1} = \cos(\frac{\pi}{2}) = 0 \implies \lim_{n\to+\infty} u_{4n+1} = 0 \\ u_{4n+2} = \cos(\pi) = -1 \implies \lim_{n\to+\infty} u_{4n+2} = -1 \\ u_{4n+3} = \cos(3\frac{\pi}{2}) = 0 \implies \lim_{n\to+\infty} u_{4n+3} = 0 \end{cases}$$

So the sequence $(u_n)_{n\in\mathbb{N}}$ is divergent. The numbers $1,-1,0$ are the cluster points of the sequence $(u_n)_{n\in\mathbb{N}}$.

3.9 Limit inferior and limit superior

Definition 10

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.
Denoting by S = The set of cluster points of the sequence $(u_n)_{n\in\mathbb{N}}$.
We define the limit superior (resp. inferior) of $(u_n)_{n\in\mathbb{N}}$ as

$$\begin{cases} \limsup u_n = \sup S \\ \\ \liminf u_n = \inf S \end{cases}$$

Example 10

Let $(u_n)_{n\in\mathbb{N}}$ defined by: $u_n = (-1)^n$
The set of all cluster points of the sequence $(u_n)_{n\in\mathbb{N}}$ is $S = \{1,-1\}$
so, $\limsup u_n = 1$, and $\liminf u_n = -1$