Chapter 4

Limits and continuous functions

Contents

4.1 Overview concepts:

In this chapter we are going to study real functions of one real variables, or simply functions which are defined on a non-empty part $\mathbb{E}$ of $\mathbb{R}$ to $\mathbb{R}$ with ($\mathbb{E} \subset \mathbb{R}$; or $\mathbb{E} = \mathbb{R}$).

4.1.1 Real function of one real variable

Definition 4.1

Any application from $\mathbb{E}$ to $\mathbb{R}$ is called a numerical function.
If $\mathbb{E} \subset \mathbb{R}$, we say that f is a numerical function of a real variable, or a real function of a real variable.
We write;

$$\begin{array}{rccc} f: & \mathbb{E} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) \end{array}$$

$\mathbb{E}$ is called the domain of definition of f and is denoted by D_f.

Example 4.1

For example, the function defined by:

$$\begin{array}{rccc} f: & \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \dfrac{1}{x} \end{array}$$

is a numerical function of one real variable. In this case the domain of definition of f is $D_f = \mathbb{R}^*$.

4.1.2 The Graph of a function

Definition 4.2

Let $f : D_f \longrightarrow \mathbb{R}$ be a numerical function of a real variable, the Graph of f is a set of ordered pairs of the form $(x, f(x))$. And denote it by Γ_f i.e:

$$\Gamma_f = \{(x, f(x))/x \in D_f\} \subset \mathbb{R}^2$$

Remark 4.1 Γ_f *is a subset of* $\mathbb{R}^2$, *i.e* $\Gamma_f \subset \mathbb{R}^2$

Example 4.2

The graph of $f(x) = \dfrac{1}{x}$ is shown below

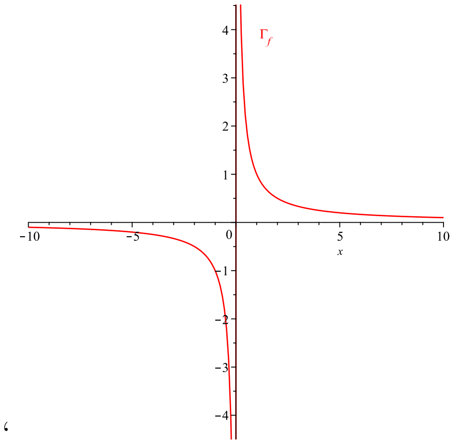


Figure 4.1: Graph of $f(x) = \dfrac{1}{x}$

4.1.3 Operations on Functions

Definition 4.3: (The sum and product of two functions)

Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ two functions defined on D to $\mathbb{R}$

- The sum of f and g is the function defined by $f + g$:

$$\begin{array}{rccl} f+g: & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & (f+g)(x) = f(x) + g(x) \end{array}$$

- The product of f and g is the function defined by $f.g$:

$$\begin{array}{rccl} f.g: & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & (f.g)(x) = f(x).g(x) \end{array}$$

- Let $\lambda \in \mathbb{R}$, the function $\lambda.f$ is defined by:

$$\begin{array}{rccl} \lambda.f: & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & (\lambda.f)(x) = \lambda.f(x) \end{array}$$

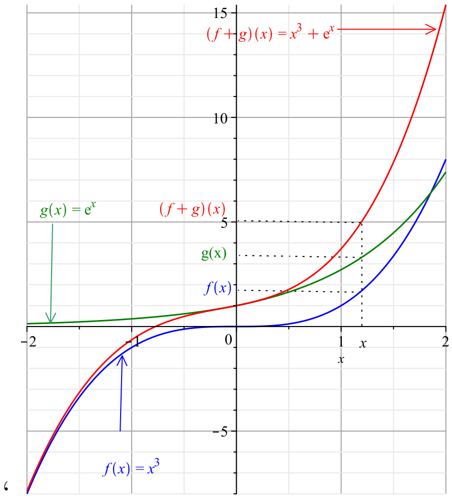


Figure 4.2: Graph of the sum of two functions $f + g$

4.1.4 Monotonicity, parity and periodicity

Definition 4.4

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function.

- The function f is said to be increasing on D_f iff:

$$\forall x, y \in D_f;\ x \leq y \implies f(x) \leq f(y)$$

- The function f is said to be strictly increasing on D_f iff:

$$\forall x, y \in D_f;\ x < y \implies f(x) < f(y)$$

- The function f is said to be decreasing on D_f iff:

$$\forall x, y \in D_f;\ x \leq y \implies f(x) \geq f(y)$$

- The function f is said to be strictly decreasing on D_f iff:

$$\forall x, y \in D_f;\ x < y \implies f(x) > f(y)$$

- The function f is said to be a constant function on D_f iff:

$$\exists a \in \mathbb{R}, \forall x, y \in D_f; f(x) = f(y) = a$$

- The function f is said to be monotonic on D_f if it is either increasing or decreasing on D_f
- The function f is said to be strictly monotonic on D_f if it is either strictly increasing or strictly decreasing on D_f

Example 4.3

1. The $\sqrt{x}$ function is strictly increasing on $[0, +\infty[$.
2. The function $\exp(x)$ is strictly increasing on $\mathbb{R}$ and $\ln(x)$ is strictly increasing on $]0, +\infty[$.
3. The function $\lfloor x \rfloor$ is increasing on $\mathbb{R}$.
4. The function $|x|$ is neither increasing nor decreasing on $\mathbb{R}$.

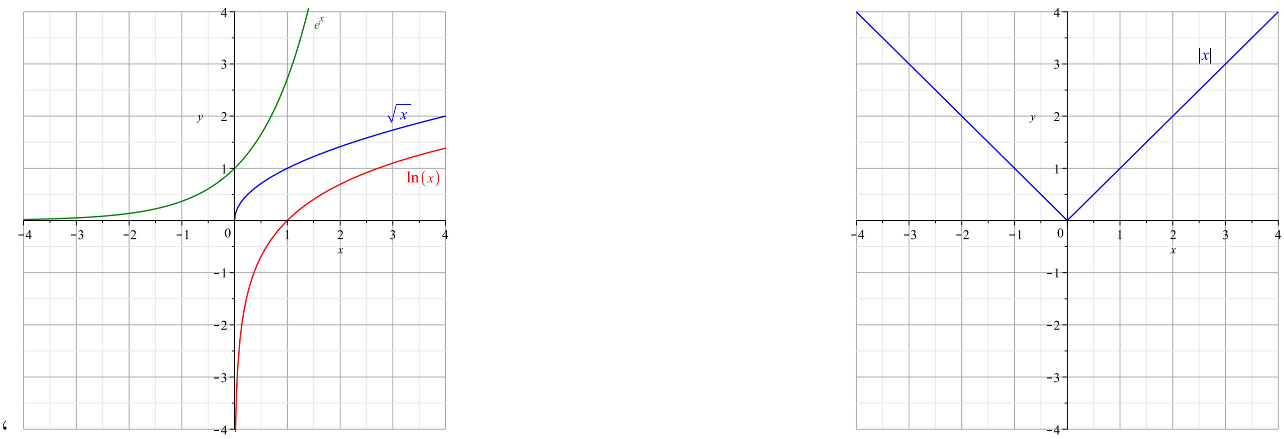


Figure 4.3: The functions $\exp(x)$,$\sqrt{x}$ and $\ln(x)$ (The function $|x|$ on the right)

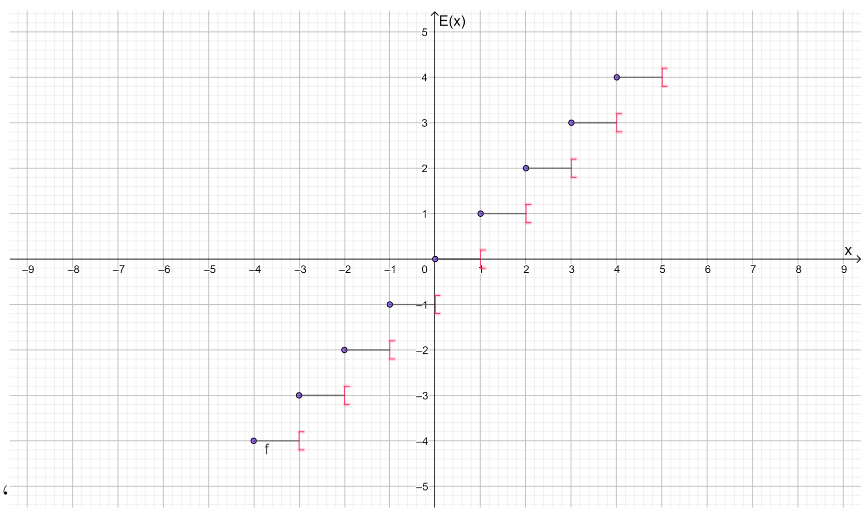


Figure 4.4: The integer part function

Definition 4.5

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function.

- We say that f is even iff: $\begin{cases} \forall x \in \mathbb{R};\ x \in D_f \implies -x \in D_f \\ \forall x \in D_f :\ f(-x) = f(x) \end{cases}$
- We say that f is odd iff: $\begin{cases} \forall x \in \mathbb{R};\ x \in D_f \implies -x \in D_f \\ \forall x \in D_f :\ f(-x) = -f(x) \end{cases}$

Graphical interpretation:

- The graphical representation of an even function has the y-axis as the axis of symmetry.
- The graphical representation of an odd function has the origin of the coordinate system as the centre of symmetry.

Example 4.4

1. Since:
$$\begin{cases} \forall x \in D_f = \mathbb{R} \implies -x \in D_f \\ \forall x \in D_f; f(-x) = (-x)^2 = x^2 = f(x), \end{cases}$$
then the function $f(x) = x^2$ is even.
2. Since:
$$\begin{cases} D_f = \mathbb{R} \\ \forall x \in D_f; f(-x) = -x^3 = -f(x), \end{cases}$$
then the function $f(x) = x^3$ is odd.

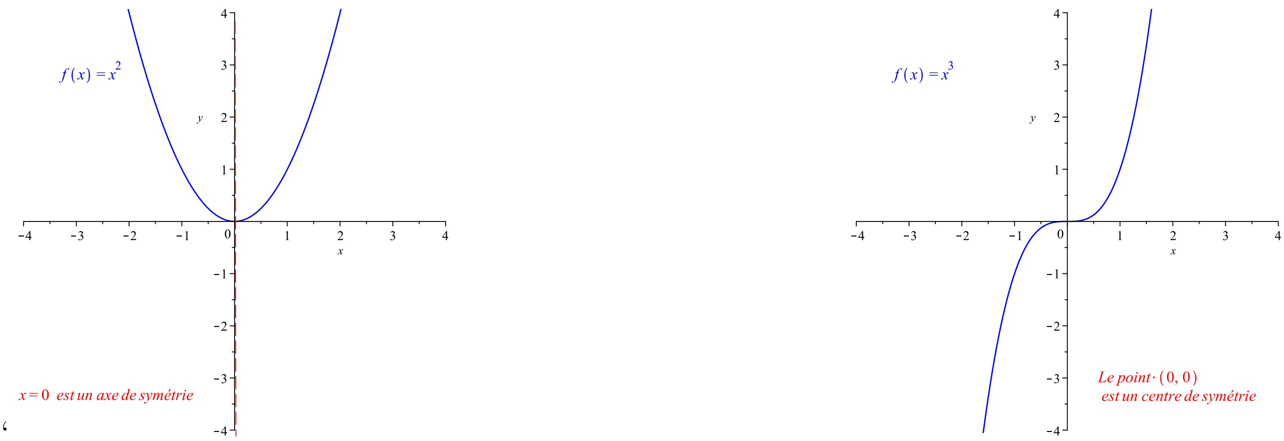


Figure 4.5: The function x^2 and x^3

Definition 4.6

Let $f : D_f \longleftarrow \mathbb{R}$ be a real function.
We say that a function f is periodic, with period $p \in \mathbb{R}_+^*$, if

$$\begin{cases} \forall x \in \mathbb{R};\ x \in D_f \implies x + p \in D_f \\ \forall x \in D_f;\ f(x+p) = f(x) \end{cases}$$

Graphical interpretation:

- If f is a periodic function with period p, then the graph of f is invariant by the translation of vector $p\,\overrightarrow{i}$.

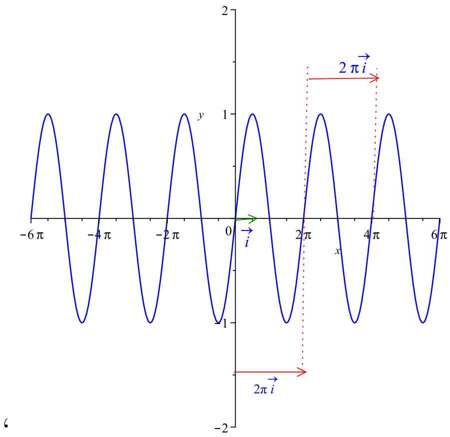


Figure 4.6: $\sin(x)$ is 2π-periodic

4.1.5 Bounded functions

Definition 4.7

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function.

- If there exists $m \in \mathbb{R}$ such that: $m \leq f(x)$ for all $x \in D_f$, then the function f is said to be bounded from below by m. i.e

$$\exists m \in \mathbb{R}, \forall x \in D_f;\ m \leq f(x) \Leftrightarrow \text{the function f is bounded from below}$$

- If there exists $M \in \mathbb{R}$ such that: $f(x) \leq M$ for all $x \in D_f$, then the function f is said to be bounded from above by M. i.e

$$\exists M \in \mathbb{R}, \forall x \in D_f;\ f(x) \leq M \Leftrightarrow \text{the function f is bounded from above}$$

- If there exists $M, m \in \mathbb{R}$ such that: $m \leq f(x) \leq M$ for all $x \in D_f$, then the function f is said to be bounded. i.e

$$\exists M, m \in \mathbb{R}, \forall x \in D_f;\ m \leq f(x) \leq M \Leftrightarrow \text{the function f is bounded}$$

Remark 4.2 *Also, we can say that f is bounded on D_f iff:* $\exists M \in \mathbb{R}_+, \forall x \in D_f;\ |f(x)| \leq M$.

Figure 4.7: The bounded fom below function (in the left) and The bounded from above function (in the right)

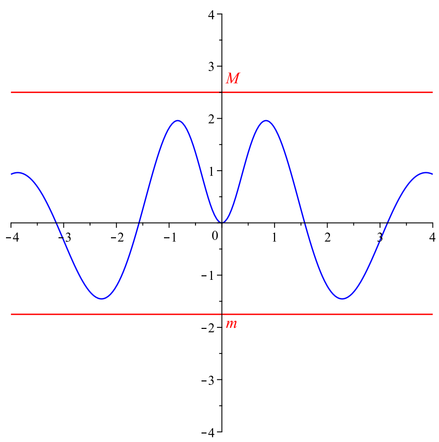


Figure 4.8: bounded function

Definition 4.8

Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ tow functions. We can write:

- $f \leq g$ iff: $\forall x \in D;\ f(x) \leq g(x)$
- $f < g$ iff: $\forall x \in D;\ f(x) < g(x)$
- $f = g$ iff: $\forall x \in D;\ f(x) = g(x)$

Rappel:-

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function. Recall that $f(D_f)$ is the set of all values of f denoted by:

$$f(D_f) = \{f(x)/x \in D_f\}$$

Let's put:

$$\begin{cases} \sup\limits_{x \in D_f} (f(x)) = \sup(f(D_f)) \\ \inf\limits_{x \in D_f} (f(x)) = \inf(f(D_f)) \end{cases}$$

Definition 4.9

- The smallest upper bound of f on D_f is called $\sup\limits_{x\in D_f}(f(x))$ and is denoted by :

$$\sup_{x\in D_f} f = \sup_{x\in D_f}(f(x))$$

- The greatest lower bound of f on D_f is called $\inf\limits_{x\in D_f}(f(x))$ and is denoted by :

$$\inf_{x\in D_f} f = \inf_{x\in D_f}(f(x))$$

Proposition 4.1

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function, then we have the following equivalences:

- f is bounded from above on D_f. $\Leftrightarrow \sup\limits_{x\in D_f} f \in \mathbb{R}$ and we write : $\sup\limits_{x\in D_f} f < +\infty$.
- f is bounded from below on D_f. $\Leftrightarrow \inf\limits_{x\in D_f} f \in \mathbb{R}$ and we write : $\inf\limits_{x\in D_f} f > -\infty$.
- f is bounded on D_f. $\Leftrightarrow \sup\limits_{x\in D_f} f, \inf\limits_{x\in D_f} f \in \mathbb{R}$ and we write : $\sup\limits_{x\in D_f} f < +\infty$ and $\inf\limits_{x\in D_f} f > -\infty$.
- $M = \sup\limits_{x\in D_f}(f(x)) \Leftrightarrow \begin{cases} \forall x \in D_f;\ f(x) \leq M \\ \forall \varepsilon > 0, \exists x_0 \in D_f;\ M - \varepsilon < f(x_0) \end{cases}$
- $m = \inf\limits_{x\in D_f}(f(x)) \Leftrightarrow \begin{cases} \forall x \in D_f;\ m \leq f(x) \\ \forall \varepsilon > 0, \exists x_0 \in D_f;\ f(x_0) < m + \varepsilon \end{cases}$

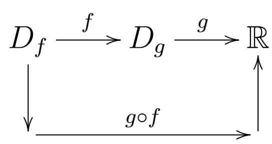
4.1.6 The composition of two functions

Definition 4.10

Consider $f : D_f \longrightarrow \mathbb{R}$ and $g : D_g \longrightarrow \mathbb{R}$ be two functions such that:$f(D_f) \subset D_g$. Then the composition of f and g, denoted by $g \circ f$ is defined as the function:

$$\forall x \in D_f;\ (g \circ f)(x) = g(f(x))$$

The below figure shows the representation of composite functions:



Example 4.5

Let f and g be two functions defined by:

$$f: \begin{array}{rl} \mathbb{R} & \longrightarrow \mathbb{R} \\ x & \longmapsto x^2+1 \end{array} \qquad g: \begin{array}{rl} [-1,+\infty[& \longrightarrow \mathbb{R} \\ x & \longmapsto \sqrt{x+1} \end{array}$$

We have $f(\mathbb{R}) = [1,+\infty[\implies f(D_f) \subset D_g$
So $g \circ f$ defined as follows:

$$\forall x \in \mathbb{R};\ (g \circ f)(x) = g(f(x)) = \sqrt{x^2+2}$$

4.2 Limits of Functions

4.2.1 Limite finie en un point x_0

Definition 4.11

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function, x_0 and l two numbers (with $x_0 \in D_f$ or $x_0 \notin D_f$).
We say that $f(x)$ tends to l when x tends to x_0 iff:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ |x - x_0| < \delta \implies |f(x) - l| < \varepsilon$$

and we write : $\lim\limits_{x \to x_0} f(x) = l$

Remark 4.3

1. *Inequality* $|x - x_0| < \delta \iff x \in]x_0 - \delta, x_0 + \delta[$.
2. *Inequality* $|f(x) - l| < \varepsilon \iff f(x) \in]l - \varepsilon, l + \varepsilon[$.
3. *We can replace inequality "$<$" by "$\leq$" in the definition.*

Graphical interpretation:
For any interval of type $J =]l - \varepsilon, l + \varepsilon[$ with $\varepsilon > 0$, we can find an interval of type $I =]x_0 - \delta, x_0 + \delta[$, such that the graphical representation of f restricted to I is included in J.

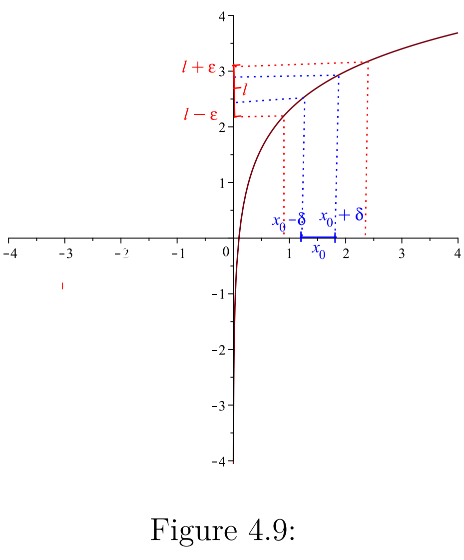


Figure 4.9:

Example 4.6

Show that $\lim\limits_{x\to 2}(4x+1)=9$.
Let $\varepsilon>0$, we have:

$$|(4x+1)-9|\leq\varepsilon\Leftrightarrow|4x-8|\leq\varepsilon\Leftrightarrow 4|x-2|\leq\varepsilon\Leftrightarrow|x-2|\leq\frac{\varepsilon}{4}$$

Let's put $\delta=\frac{\varepsilon}{4}$ we obtain:

$$\forall\varepsilon>0,\exists\delta>0(\delta=\frac{\varepsilon}{4}),\forall x\in\mathbb{R};|x-2|\leq\delta\implies|(4x+1)-9|\leq\varepsilon$$

$$\implies\lim_{x\to 2}(4x+1)=9$$

Proposition 4.2

If a function f has a limit at x_0, then this limit is unique.

Proof 1

By contradiction, suppose that, f has two distinct limits l_1 and l_2. ($l_1\neq l_2$) en x_0.
By setting: $\varepsilon=\frac{1}{3}|l_1-l_2|>0$ because $l_1\neq l_2$.
We have:

$$\begin{cases}\lim\limits_{x\to x_0}f(x)=l_1\\ \quad et\\ \lim\limits_{x\to x_0}f(x)=l_2\end{cases}$$

$$\implies\begin{cases}\exists\delta_1(\varepsilon)>0,\forall x\in D_f;\ |x-x_0|\leq\delta_1\implies|f(x)-l_1|\leq\varepsilon\\ \quad et\\ \exists\delta_2(\varepsilon)>0,\forall x\in D_f;\ |x-x_0|\leq\delta_2\implies|f(x)-l_2|\leq\varepsilon\end{cases}$$

By choosing: $\delta=\min(\delta_1,\delta_2)$ we get:

$$\begin{cases}\forall x\in D_f;\ |x-x_0|\leq\delta\implies|f(x)-l_1|\leq\varepsilon\\ \quad and\\ \forall x\in D_f;\ |x-x_0|\leq\delta\implies|f(x)-l_2|\leq\varepsilon\end{cases}$$

$$\implies\forall x\in D_f;\ |x-x_0|\leq\delta\implies|f(x)-l_1|+|f(x)-l_2|\leq 2\varepsilon\tag{4.1}$$

According to the triangle inequality we have:

$$|l_1-l_2|=|l_1-f(x)+f(x)-l_2|\leq|f(x)-l_1|+|f(x)-l_2|\tag{4.2}$$

$$(3.1)\text{ and }(3.2)\implies|l_1-l_2|\leq 2\varepsilon\implies|l_1-l_2|\leq\frac{2}{3}|l_1-l_2|\implies 1\leq\frac{2}{3}$$

So we end up with a contradiction, this means that f admits a unique limit at point x_0.

4.2.2 Left and Right-Hand Limits

Definition 4.12

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function, x_0 and l be two real numbers. (with $x_0 \in D_f$ or $x_0 \notin D_f$).

- We say that l is the left limit of the function f at a point x_0 iff:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ x_0 - \delta \leq x < x_0 \implies |f(x) - l| \leq \varepsilon$$

  and we write: $\lim\limits_{x \overset{<}{\longrightarrow} x_0} f(x) = l$ or $\lim\limits_{x \to x_0^-} f(x) = l$

- We say that l is the right limit of the function f at a point x_0 iff:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ x_0 < x \leq x_0 + \delta \implies |f(x) - l| \leq \varepsilon$$

  and we write: $\lim\limits_{x \overset{>}{\longrightarrow} x_0} f(x) = l$ or $\lim\limits_{x \to x_0^+} f(x) = l$

Example 4.7

- prove that: $\lim\limits_{x \to 0^+} x\cos(\frac{1}{x}) = 0$

  We have:

$$|x\cos(\frac{1}{x})| \leq |x||\cos(\frac{1}{x})| \leq |x|$$

$$(\text{as } |\cos(\frac{1}{x})| \leq 1)$$

$$\implies |x\cos(\frac{1}{x})| \leq |x| \tag{4.3}$$

  Let $\varepsilon > 0$, with $\delta = \varepsilon$. If we have: $0 < x \leq \delta \Leftrightarrow 0 < x \leq \varepsilon \implies |x| \leq \varepsilon$

  $(3.3) \implies |x\cos(\frac{1}{x})| \leq \varepsilon$.

  So, $\forall \varepsilon > 0, \exists \delta > 0\,(\delta = \varepsilon), \forall x \in \mathbb{R}^*;\ 0 < x \leq \delta \implies |x\cos(\frac{1}{x})| \leq \varepsilon$

  $\implies \lim\limits_{x \to 0^+} x\cos(\frac{1}{x}) = 0$

- Show that $\lim\limits_{x \to 0^-} x\cos(\frac{1}{x}) = 0$ (Using the same technique as above)

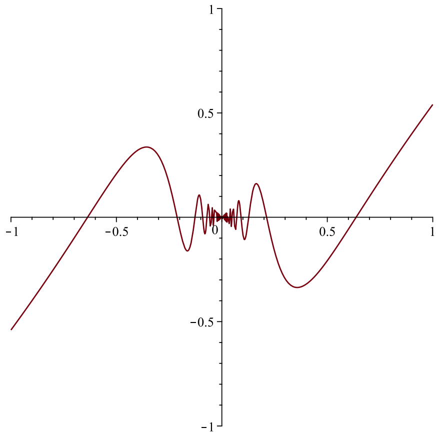


Figure 4.10: Graph of the function $x\cos(\frac{1}{x})$

Theorem 4.1

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function, $x_0, l \in \mathbb{R}$ (with $x_0 \in D_f$ or $x_0 \notin D_f$). The following propositions are equivalent

1. $\lim\limits_{x\to x_0} f(x) = l$
2. $\lim\limits_{x \overset{<}{\longrightarrow} x_0} f(x) = \lim\limits_{x \overset{>}{\longrightarrow} x_0} f(x) = l$

Result:
If we have: $\lim\limits_{x \overset{<}{\longrightarrow} x_0} f(x) \neq \lim\limits_{x \overset{>}{\longrightarrow} x_0} f(x)$ then $\lim\limits_{x\to x_0} f(x)$ doesn't exist.

Example 4.8

Let's consider the function $f(x) = \dfrac{|x|}{x}$. We have:

$$\begin{cases} \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} -\dfrac{x}{x} = -1 \\ \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{x}{x} = 1 \end{cases}$$

$\Longrightarrow \lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$ then $\lim\limits_{x\to 0} f(x)$ doesn't exist.

4.2.3 Infinite limit of a function at x_0.

Definition 4.13

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function and $x_0 \in \mathbb{R}$ (with $x_0 \in D_f$ or $x_0 \notin D_f$)

- It is said that f tends to $+\infty$ when x tends to x_0 iff:

$$\forall A > 0, \exists \delta > 0, \forall x \in D_f;\ |x - x_0| \leq \delta \implies f(x) \geq A$$

and we write $\lim\limits_{x \to x_0} f(x) = +\infty$

- It is said that f tends to $-\infty$ when x tends to x_0 iff:

$$\forall A > 0, \exists \delta > 0, \forall x \in D_f;\ |x - x_0| \leq \delta \implies f(x) \leq -A$$

and we write $\lim\limits_{x \to x_0} f(x) = -\infty$

Example 4.9

Show that $\lim\limits_{x \to 0} \frac{1}{x^2} = +\infty$
Let $A > 0$ we have:

$$\frac{1}{x^2} \geq A \Leftrightarrow x^2 \leq \frac{1}{A} \Leftrightarrow x^2 - \frac{1}{A} \leq 0 \Leftrightarrow x \in \left[-\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{A}}\right] \Leftrightarrow |x| \leq \frac{1}{\sqrt{A}}$$

Putting $\delta = \frac{1}{\sqrt{A}}$ then $\forall x \in D_f;\ |x| \leq \delta \implies \frac{1}{x^2} \geq A$

$\implies \forall A > 0, \exists \delta > 0\,(\delta = \frac{1}{\sqrt{A}}), \forall x \in D_f;\ |x| \leq \delta \implies \frac{1}{x^2} \geq A$ therfore $\lim\limits_{x \to 0} \frac{1}{x^2} = +\infty$

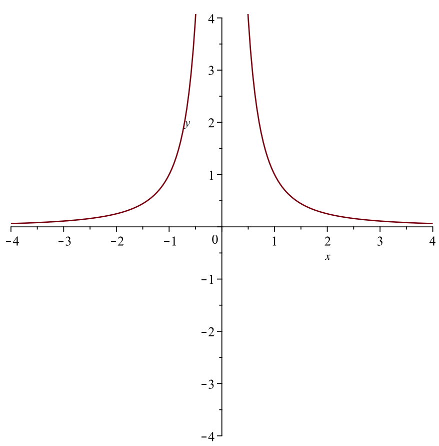


Figure 4.11: The graph of a function $\frac{1}{x^2}$

4.2.4 Finite limit of a function at $-\infty$ and $+\infty$

Definition 4.14

- Let f be a function defined on an interval of type $]-\infty, a]$ (i.e $]-\infty, a] \subset D_f$) We say that f tends to l ($l \in \mathbb{R}$) when x tends to $-\infty$ iff:

$$\forall \varepsilon > 0, \exists B > 0, \forall x \in D_f;\ x \leq -B \implies |f(x) - l| \leq \varepsilon$$

and we write $\lim\limits_{x \to -\infty} f(x) = l$

- Let f be a function defined on an interval of type $[a, +\infty[$ (i.e. $[a, +\infty[\subset D_f$). We say that f tends to l ($l \in \mathbb{R}$) when x tends to $+\infty$ iff:

$$\forall \varepsilon > 0, \exists B > 0, \forall x \in D_f;\ x \geq B \implies |f(x) - l| \leq \varepsilon$$

and we write $\lim\limits_{x \to +\infty} f(x) = l$

Example 4.10

prove that $\lim\limits_{x \to +\infty} \dfrac{x}{x+1} = 1$ Let $\varepsilon > 0$, we have:

$$\left|\frac{x}{x+1} - 1\right| \leq \varepsilon \Leftrightarrow \left|\frac{1}{x+1}\right| \leq \varepsilon \Leftrightarrow |x+1| \geq \frac{1}{\varepsilon}$$

$$\Leftrightarrow \begin{cases} x+1 \geq \dfrac{1}{\varepsilon} \\ or \\ x+1 \leq -\dfrac{1}{\varepsilon} \end{cases} \Leftrightarrow \begin{cases} x \geq \dfrac{1}{\varepsilon} - 1 \\ or \\ x \leq -1 - \dfrac{1}{\varepsilon} \end{cases}$$

We set $B = \dfrac{1}{\varepsilon} - 1$, if $x \geq B \implies \left|\dfrac{x}{x+1} - 1\right| \leq \varepsilon$

So, $\forall \varepsilon > 0, \exists B > 0\,(B = \dfrac{1}{\varepsilon} - 1), \forall x \in D_f;\ x \geq B \implies \left|\dfrac{x}{x+1} - 1\right| \leq \varepsilon$

$\implies \lim\limits_{x \to +\infty} \dfrac{x}{x+1} = 1.$

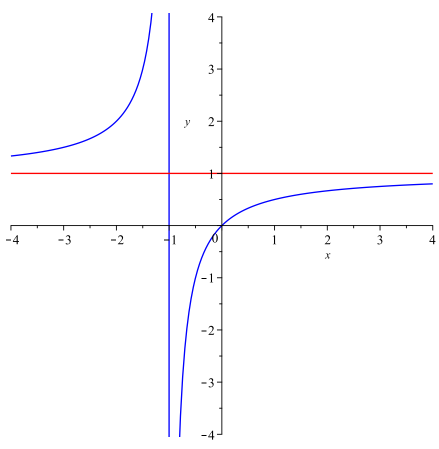


Figure 4.12: The graph of a function $\dfrac{x}{x+1}$

4.2.5 Infinite limit of a function at $+\infty$ and $-\infty$

Definition 4.15

- Let f be a function defined on an interval of type $[a, +\infty[$ (i.e. $[a, +\infty[\subset D_f$). We say that f tends to $+\infty$ when x tends to $+\infty$ if:

$$\forall A > 0, \exists B > 0, \forall x \in D_f;\ x \geq B \implies f(x) \geq A$$

  and we write: $\lim_{x \to +\infty} f(x) = +\infty$

- Let f be a function defined on an interval of type $]-\infty, a]$ (i.e. $]-\infty, a] \subset D_f$) . We say that f tends to $+\infty$ when x tends to $-\infty$ if:

$$\forall A > 0, \exists B > 0, \forall x \in D_f;\ x \leq -B \implies f(x) \geq A$$

  and we write: $\lim_{x \to -\infty} f(x) = +\infty$

- Let f be a function defined on an interval of type $[a, +\infty[$ (i.e. $[a, +\infty[\subset D_f$) . We say that f tends to $-\infty$ when x tends to $+\infty$ if:

$$\forall A > 0, \exists B > 0, \forall x \in D_f;\ x \geq B \implies f(x) \leq -A$$

  and we write: $\lim_{x \to +\infty} f(x) = -\infty$

- Let f be a function defined on an interval of type $]-\infty, a]$ (i.e. $]-\infty, a] \subset D_f$) . We say that f tends to $-\infty$ when x tends to $-\infty$ if:

$$\forall A > 0, \exists B > 0, \forall x \in D_f;\ x \leq -B \implies f(x) \leq -A$$

  and we write: $\lim_{x \to -\infty} f(x) = -\infty$

Notation: Let $\overline{\mathbb{R}}$ denote the set defined by:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$$

$\overline{\mathbb{R}}$ is called the extended real line.

4.2.6 Relationship between limits and sequences

Theorem 4.2

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function, $x_0 \in \mathbb{R}$ (with $x_0 \in D_f$ or $x_0 \notin D_f$) and $l \in \overline{\mathbb{R}}$. The following properties are equivalent:

1. $\lim_{x \to x_0} f(x) = l$
2. For any sequence $(x_n)_{n \in \mathbb{N}}$ in D_f such that: $\forall n \in \mathbb{N}; x_n \neq x_0$ and $\lim_{n \to +\infty} x_n = x_0$, then we have $\lim_{n \to +\infty} f(x_n) = l$

Proof 2

- First, we prove implication $(1 \implies 2)$.
  Let $\varepsilon > 0$,

$$\exists \delta_\varepsilon > 0, \forall x \in D_f;\ |x - x_0| \leq \delta_\varepsilon \implies |f(x) - l| \leq \varepsilon \tag{4.4}$$

$$(\text{As} \lim_{x \to x_0} f(x) = l)$$

$$\delta_\varepsilon > 0 \implies \exists n_0(\delta_\varepsilon) \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_0 \implies |x_n - x_0| \leq \delta_\varepsilon \tag{4.5}$$

$$(\text{Because} \lim_{n \to +\infty} x_n = x_0)$$

$$(3.4) \text{ et } (3.5) \implies |f(x_n) - l| \leq \varepsilon$$

$$\implies \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_0 \implies |f(x_n) - l| \leq \varepsilon$$

$$\implies \lim_{n \to +\infty} f(x_n) = l$$

- Next, we prove the implication $(2 \implies 1)$ by contradiction proof,
  we assume that for any sequence $(x_n)_{n\in\mathbb{N}} \subset D_f$ that converges to x_0 we have $f(x_n)$ converges to l and $\lim_{x \to x_0} f(x) \neq l$.

$$\lim_{x \to x_0} f(x) \neq l \Leftrightarrow \exists \varepsilon > 0, \forall \delta > 0, \exists x^* \in D_f;\ |x^* - x| \leq \delta \wedge |f(x^*) - l| \geq \varepsilon \tag{4.6}$$

We set: $\delta = \frac{1}{n} / n \in \mathbb{N}^*$

$$(3.6) \implies \forall n \in \mathbb{N}^*, \exists x_n \in D_f; (|x_n - x_0| \leq \frac{1}{n}) \wedge (|f(x_n) - l| > \varepsilon)$$

So we have found a sequence $(x_n)_{n\in\mathbb{N}^*} \subset D_f$ that converges to x_0.

(since $\forall n \in \mathbb{N}^*; |x_n - x_0| \leq \frac{1}{n}$) et $f(x_n)$ doesn't converge to l (as $\forall n \in \mathbb{N}^*; |f(x_n) - l| > \varepsilon$), which contradicts our hypothesis.

Remark 4.4 *If there are two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ of D_f such that:*

$$\begin{cases} \lim_{n \to +\infty} x_n = x_0 \\ \quad \text{and} \\ \lim_{n \to +\infty} y_n = x_0 \end{cases} \wedge \lim_{n \to +\infty} f(x_n) \neq \lim_{n \to +\infty} f(y_n)$$

Then $\lim_{x \to x_0} f(x)$ doesn't exist.

Example 4.11

Let $\begin{array}{cccc} f: & \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \sin(\frac{1}{x}) \end{array}$

We have $\lim\limits_{x \to 0} \sin(\frac{1}{x})$ doesn't exist because: If we set $x_n = \dfrac{1}{n\pi}$ et $y_n = \dfrac{1}{\frac{\pi}{2} + 2n\pi}$

The sequences $(x_n)_{n\in\mathbb{N}^*}$, $(y_n)_{n\in\mathbb{N}^*}$ in $\mathbb{R}^*$ and $\lim\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} y_n = 0$

On the other hand, we have: $\begin{cases} \lim\limits_{n\to+\infty} f(x_n) = \lim\limits_{n\to+\infty} \sin(n\pi) = 0 \\ \quad et \\ \lim\limits_{n\to+\infty} f(y_n) = \lim\limits_{n\to+\infty} \sin(\frac{\pi}{2} + 2n\pi) = 1 \end{cases}$

$$\Longrightarrow \lim_{n\to+\infty} f(x_n) \neq \lim_{n\to+\infty} f(y_n)$$

$$\Longrightarrow \lim_{x\to x_0} f(x) \text{ doesn't exist.}$$

4.2.7 Limits operations

Proposition 4.3: (The limit of sum of two or more functions)

Let f, g be two functions and $x_0 \in \overline{\mathbb{R}}$. Then we have:

$\lim\limits_{x\to x_0} f(x)$	$\lim\limits_{x\to x_0} g(x)$	$\lim\limits_{x\to x_0} (f(x)+g(x))$
$l_1 \in \mathbb{R}$	$l_2 \in \mathbb{R}$	$l_1 + l_2$
$l_1 \in \mathbb{R}$	$\pm\infty$	$\pm\infty$
$+\infty$	$+\infty$	$+\infty$
$-\infty$	$-\infty$	$-\infty$
$+\infty$	$-\infty$	Indeterminate form
$-\infty$	$+\infty$	Indeterminate form

Proposition 4.4: (The limit of product of two or more functions)

Let f, g be two functions and $x_0 \in \overline{\mathbb{R}}$. Then we have:

$\lim\limits_{x\to x_0} g(x)$ / $\lim\limits_{x\to x_0} f(x)$	$l_2 > 0$	$l_2 < 0$	0	$+\infty$	$-\infty$
$l_1 > 0$	$\lim\limits_{x\to x_0} f x)g(x) = l_1 l_2$	$l_1 l_2$	0	$+\infty$	$-\infty$
$l_1 < 0$	$l_1 l_2$	$l_1 l_2$	0	$-\infty$	$+\infty$
0	0	0	0	Indeterminate form	Indeterminate form
$+\infty$	$+\infty$	$-\infty$	Indeterminate form	$+\infty$	$-\infty$
$-\infty$	$-\infty$	$+\infty$	Indeterminate form	$-\infty$	$+\infty$

Proposition 4.5: (The limit of quotient of two functions)

Let f, g be two functions defined on D with $g(x) \neq 0$ on D and $x_0 \in \overline{\mathbb{R}}$. Then we have:

$\lim_{x\to x_0} f(x)$ \ $\lim_{x\to x_0} g(x)$	$l_2 > 0$	$l_2 < 0$	0^+	0^-	$+\infty$	$-\infty$
$l_1 > 0$	$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{l_1}{l_2}$	$\frac{l_1}{l_2}$	$+\infty$	$-\infty$	0	0
$l_1 < 0$	$\frac{l_1}{l_2}$	$\frac{l_1}{l_2}$	$-\infty$	$+\infty$	0	0
0^+	0	0	Indeterminate form	Indeterminate form	0	0
0^-	0	0	Indeterminate form	Indeterminate form	0	0
$+\infty$	$+\infty$	$-\infty$	$+\infty$	$-\infty$	IF	IF
$-\infty$	$-\infty$	$+\infty$	$-\infty$	$+\infty$	IF	IF

Remark 4.5 *According to the previous propositions, the indeterminate forms are:* $+\infty - \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$. *Also we can deduce the other forms which are:* 0^0, ∞^0, 1^∞

4.2.8 Limit of Composite Functions

Proposition 4.6

Let $f : D_f \longrightarrow \mathbb{R}$, $g : D_g \longrightarrow \mathbb{R}$ and $x_0, y_0, l \in \overline{\mathbb{R}}$.

If we have: $\begin{cases} \lim_{x\to x_0} f(x) = y_0 \\ \quad et \\ \lim_{x\to y_0} g(x) = l \end{cases}$ then $\lim_{x\to x_0} (f \circ g)(x) = l$

Example 4.12

Let $\begin{array}{rccc} f: & \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \dfrac{e^x - 1}{x} \end{array}$ and $\begin{array}{rccc} g: & \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \ln(x) \end{array}$

We have: $\begin{cases} \lim_{x\to 0} f(x) = \lim_{x\to 0} \left(\frac{e^x - 1}{x}\right) = 1 \\ \quad et \\ \lim_{x\to 1} g(x) = \lim_{x\to 1} \ln(x) = 0 \end{cases}$

Then $\lim_{x\to 0} (g \circ f)(x) = \lim_{x\to 0} \ln\left(\frac{e^x - 1}{x}\right) = 0$

4.2.9 Finding Limits: Properties of Limits

Proposition 4.7

1. If we have: $\lim_{x \to x_0} f(x) = l$ then there exists $\alpha > 0$ such that the function f is bounded on $]x_0 - \alpha, x_0 + \alpha[$.

2. If we have: $f(x) \leq g(x)$ in the neighbourhood of x_0 and $\lim_{x \to x_0} f(x) = l_1$, $\lim_{x \to x_0} g(x) = l_2$ then $l_1 \leq l_2$.

3. The Squeeze Theorem: Let f, g, h be three functions with the following property $f(x) \leq g(x) \leq h(x)$ in the neighbourhood of x_0.

$$\text{If we have: } \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l \text{ then } \lim_{x \to x_0} g(x) = l$$

4. Let f, g two functions which verify $f(x) \leq g(x)$ in the neighbourhood of x_0

$$\text{If we have: } \begin{cases} \lim_{x \to x_0} f(x) = +\infty \text{ then } \lim_{x \to x_0} g(x) = +\infty \\ \lim_{x \to x_0} g(x) = -\infty \text{ then } \lim_{x \to x_0} f(x) = -\infty \end{cases}$$

5. Let f be a bounded function in the neighborhood of x_0 and g a function verifying $\lim_{x \to x_0} g(x) = 0$ then $\lim_{x \to x_0} f(x)g(x) = 0$.

Definition 4.16: (important definition)

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function. We say that f is defined in the neighborhood of x_0 iff: there exists an interval of the following type $I =]x_0 - \varepsilon, x_0 + \varepsilon[$ such that: $I \subset D_f$. (I is an interval with center x_0 and radius ε).

4.3 Continuous Functions

4.3.1 Continuity at a point x_0

Definition 4.17

Let f be a function defined in the neighborhood of x_0. We say that f is continuous at the point x_0 iff:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

$$\text{i.e} \qquad \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ |x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \varepsilon$$

Example 4.13

Let f be a function defined by:

$$f(x) = \begin{cases} x \sin(\frac{1}{x^2}), & \text{si } x \neq 0 \\ 0 & \text{si } x = 0 \end{cases}$$

Show that f is continuous at $x_0 = 0$.

1. $D_f = \mathbb{R} \implies f$ is defined in the neighbourhood of $x_0 = 0$.
2. We'll show that $\lim_{x \to 0} f(x) = 0$.
   We have:
   $$\forall x \in \mathbb{R}^*;\ |x \sin(\frac{1}{x^2}| \leq |x|$$
   If we choose $\delta = \varepsilon$ (with $\varepsilon > 0$), we find:
   $$\forall \varepsilon > 0, \exists \delta > 0\ (\delta = \varepsilon), \forall x \in \mathbb{R};\ |x| \leq \delta \implies |x \sin(\frac{1}{x^2}| \leq \varepsilon$$
   $$\implies \lim_{x \to 0} f(x) = 0 \implies f \text{ is continuous at } x_0$$

4.3.2 Left and right continuity at a point x_0

Definition 4.18

- Let f be a function defined on an interval of kind $[x_0, x_0 + h[$ with $h > 0$ (i.e.;$[x_0, x_0 + h[\subset D_f$). A function f is right continuous at a point x_0 iff:
  $$\lim_{x \overset{>}{\longrightarrow} x_0} f(x) = f(x_0)$$
  $$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ x_0 \leq x < x_0 + \delta \implies |f(x) - f(x_0)| \leq \varepsilon$$
- Let f be a function defined on an interval of kind $[x_0 - h, x_0[$ with $h > 0$ (i.e.;$[x_0 - h, x_0[\subset D_f$). A function f is left continuous at a point x_0 iff:
  $$\lim_{x \overset{<}{\longrightarrow} x_0} f(x) = f(x_0)$$
  $$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ x_0 - \delta < x \leq x_0 \implies |f(x) - f(x_0)| \leq \varepsilon$$

Example 4.14

Let f be a function defined by:

$$f(x) = \begin{cases} \dfrac{\sin(x)}{|x|}, & \text{si } x \neq 0 \\ 1 & \text{si } x = 0 \end{cases}$$

1. We'll study the right continuity of f at $x_0 = 0$
   - We have $D_f = \mathbb{R} \implies f$ is defined in the right of $x_0 = 0$
   - $\lim\limits_{x \overset{>}{\longrightarrow} 0} \dfrac{\sin(x)}{|x|} = \lim\limits_{x \overset{>}{\longrightarrow} 0} \dfrac{\sin(x)}{x} = 1 = f(1)$

   So f is right continuous at x_0.
2. The continuity of f at the left of $x_0 = 0$.
   - We have $D_f = \mathbb{R} \implies f$ is defined in the left of $x_0 = 0$.
   - $\lim\limits_{x \overset{<}{\longrightarrow} 0} \dfrac{\sin(x)}{|x|} = \lim\limits_{x \overset{<}{\longrightarrow} 0} -\dfrac{\sin(x)}{x} = -1 \neq f(1)$

   So f is not left continuous at x_0.

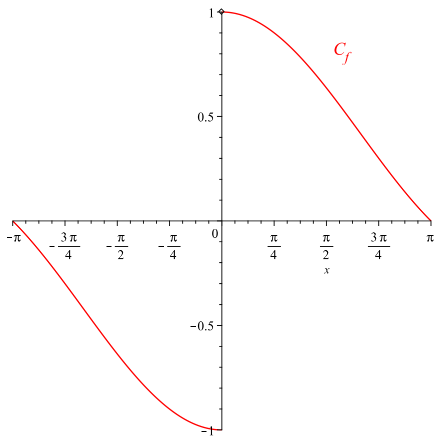


Figure 4.13: Graph of the function f

Theorem 4.3

Let f be a function defined in the neighborhood of x_0. The following two propositions are equivalent:

1. f is left and right continuous at x_0.
2. f is continuous at x_0.

Remark 4.6 *Our example (3.13) shows that f is right continuous at $x_0 = 0$ and is not left continuous at $x_0 = 0$. which implies that f is not continuous at $x_0 = 0$.*

4.3.3 Continuous extension to a point

Definition 4.19

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function and $x_0 \in \mathbb{R}$ Such that: $x_0 \notin D_f$. If $\lim_{x \to x_0} f(x) = l \in \mathbb{R}$.exists, but $f(x_0)$ is not defined, we define a new function:

$$\begin{array}{rccl} \tilde{f}: & D_f \cup \{x_0\} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ l & \text{if } x = x_0 \end{cases} \end{array}$$

which is continuous at x_0. It is called the continuous extension of f to x_0.

Example 4.15

Let

$$\begin{array}{rccl} f: & \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \dfrac{\sin(x)}{x} \end{array}$$

Can we extend the function f to be continuous at $x_0 = 0$.

We have: $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{\sin(x)}{x} = 1 \implies f$ has a finite limit at $x_0 = 0$
So f is extendable by continuity at $x_0 = 0$ and the extension by continuity of f at $x_0 = 0$ is defined by:

$$\begin{array}{rccl} \tilde{f}: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \tilde{f}(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \end{array}$$

4.3.4 Operations on continuous functions at x_0

Theorem 4.4

Let f, g be two continuous functions at a point $x_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Then:

1. The function $|f|$ is continuous at x_0.
2. The functions λf, $f + g$ and fg are continuous at x_0.
3. If $g(x_0) \neq 0$ then $\dfrac{f}{g}$ is continuous at x_0.

Proposition 4.8

Let $f : D_f \longrightarrow \mathbb{R}$ and $g : D_g \longrightarrow \mathbb{R}$ be two functions such that: $f(D_f) \subset D_g$.

If we have:

$$\begin{cases} f \text{ is defined in a neighborhood of } x_0 \text{ and continuous at } x_0 \\ \quad\quad \text{et} \\ g \text{ is defined in a neighborhood of } y_0 = f(x_0) \text{ and continuous at } y_0 \end{cases}$$

Then $(g \circ f)(x)$ is continuous at x_0.

4.3.5 The sequential continuity theorem

Theorem 4.5

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function. The following two statements are equivalent:

1. f is continuous at x_0.
2. for each sequence $(x_n)_{n\in\mathbb{N}} \subset D_f$ such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Proof 3

The proof of this theorem follows from theorem (3.2)

Proposition 4.9

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function and $x_0 \in D_f$.
If f is continuous at x_0 and $f(x_0) \neq 0$ then there exists a neighborhood $(\mathcal{V})$ of x_0 such that:

$$\forall x \in \mathcal{V};\ f(x) \neq 0$$

Proof 4

We have f is continuous at x_0 so,

1. f is defined in a neighborhood of x_0

$$\Leftrightarrow \exists \eta > 0 \text{ such that: } I =]x_0 - \eta, x_0 + \eta[\subset D_f \tag{4.7}$$

2. $\lim_{x \to x_0} f(x) = f(x_0)$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f;\ |x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \varepsilon \tag{4.8}$$

If we put: $\varepsilon = \frac{1}{2}|f(x_0)|$ then:

$$\exists \delta > 0, \forall x \in D_f;\ |x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \frac{1}{2}|f(x_0)|$$

$$(3.7) \implies \exists \delta > 0, \forall x \in I;\ |x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \frac{1}{2}|f(x_0)|$$

$$\implies \forall x \in I \cap]x_0 - \eta, x_0 + \eta[;\ |f(x) - f(x_0)| \leq \frac{1}{2}|f(x_0)| \tag{4.9}$$

Let's put : $\mathcal{V} = I \cap]x_0 - \delta, x_0 + \delta[$, then $\mathcal{V}$ is a neighborhood of x_0.
On the other hand, according to the triangular inequality, we have:

$$|f(x_0)| - |f(x)| \leq ||f(x_0)| - |f(x)|| \leq |f(x_0) - f(x)| \leq \frac{1}{2}|f(x_0)|$$

$$(3.9) \implies \forall x \in \mathcal{V}; |f(x_0)| - |f(x)| \leq \frac{1}{2}|f(x_0)|$$

$$\implies \forall x \in \mathcal{V};\ |f(x)| \geq \frac{1}{2}|f(x_0)| \neq 0$$

So there is a neighbourhood $\mathcal{V}$ of x_0 such that: $\forall x \in \mathcal{V}; f(x) \neq 0$.

4.3.6 Continuity over an interval

Definition 4.20

1. f is said to be continuous on an open interval of type $]a, b[$ iff: it is continuous at any point on the interval $]a, b[$.

2. f is said to be continuous on an interval of type $[a, b]$ iff: it is continuous on $]a, b[$ and continuous to the right of a and to the left of b.

3. f is said to be continuous on an interval of type $]a, b]$ iff: it is continuous on $]a, b[$ and continuous to the left of b.

4. f is said to be continuous on an interval of type $[a, b[$ iff: it is continuous on $]a, b[$ and continuous to the right of a.

4.3.7 Uniform continuity

Definition 4.21

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function. We say that f is uniformly continuous on D_f iff:

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0, \forall x, y \in D_f;\ |x - y| \leq \delta(\varepsilon) \implies |f(x) - f(y)| \leq \varepsilon$$

Remark 4.7 *Note that uniform continuity is a property of the function over the set D_f, while continuity can be defined at a point $x_0 \in D_f$. The number δ depends only on ε in the case of uniform continuity, but in the case of continuity at a point x_0, δ depends on ε and x_0.*

Example 4.16

Show that the function $f(x) = \sqrt{x}$ is uniformly continuous on $\mathbb{R}_+$.
Solution: Step 01: In this step, we'll show that:
$\forall x, y \in \mathbb{R}_+ : \sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ and $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$

1. Let $x, y \in \mathbb{R}_+$ we have: $0 \leq 2\sqrt{x}\sqrt{y} \Leftrightarrow x + y \leq x + 2\sqrt{x}\sqrt{y} + y \Leftrightarrow x + y \leq \left(\sqrt{x} + \sqrt{y}\right)^2 \Leftrightarrow \sqrt{x+y} \leq |\sqrt{x} + \sqrt{y}| = \sqrt{x} + \sqrt{y}$
   So $\forall x, y \in \mathbb{R}_+;\ \sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.

2. Let $x, y \in \mathbb{R}_+$ we have:

$$\begin{aligned} (x-y) \leq |x-y| &\implies y + (x-y) \leq y + |x-y| \\ &\implies \sqrt{x} \leq \sqrt{y + |x-y|} \leq \sqrt{y} + \sqrt{|x-y|} \\ &\implies \sqrt{x} - \sqrt{y} \leq \sqrt{|x-y|} \end{aligned} \tag{4.10}$$

   on the other hand, we have:

$$\begin{aligned} (y-x) \leq |y-x| &\implies x + (y-x) \leq x + |x-y| \\ &\implies \sqrt{y} \leq \sqrt{x + |x-y|} \leq \sqrt{x} + \sqrt{|x-y|} \\ &\implies -\sqrt{|x-y|} \leq \sqrt{x} - \sqrt{y} \end{aligned} \tag{4.11}$$

   (3.10) et (3.11) $\implies \forall x, y \in \mathbb{R}_+;\ |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$

Step 02:
In this step we will show the uniform continuity of the function $f(x) = \sqrt{x}$
Let $\varepsilon > 0$ and $x, y \in \mathbb{R}_+$ we have:

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$$

Let's put $\delta = \varepsilon^2$ then:

$$|x-y| \leq \delta \implies \sqrt{|x-y|} \leq \varepsilon \implies |\sqrt{x} - \sqrt{y}| \leq \varepsilon$$

$$\implies \forall \varepsilon > 0, \exists \delta > 0\ (\delta = \varepsilon^2), \forall x, y \in \mathbb{R}_+;\ |x-y| \leq \delta \implies |\sqrt{x} - \sqrt{y}| \leq \varepsilon$$

therefore, $f(x) = \sqrt{x}$ is uniformly continuous on $\mathbb{R}_+$.

Example 4.17

Show that the function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.
Solution:
$f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$

$$\Leftrightarrow \exists \varepsilon > 0, \forall \delta > 0, \exists x, y \in \mathbb{R};\ |x-y| \leq \delta \wedge |x^2 - y^2| > \varepsilon$$

Let's put: $\varepsilon = 1$
Let $\delta > 0$, we will confirm the existence of $x, y \in \mathbb{R}$ such that:$|x-y| \leq \delta \wedge |x^2 - y^2| > \varepsilon$
Let's take : $y = x + \frac{1}{2}\delta \implies x - y = \frac{1}{2}\delta \implies |x-y| = \frac{1}{2}\delta \leq \delta$
$|x^2 - y^2| > 1 \Leftrightarrow |x^2 - x^2 - x\delta - \frac{1}{4}\delta^2| > 1 \Leftrightarrow |\frac{1}{4}\delta^2 + x\delta| > 1$
If we choose $x = \frac{1}{\delta} + \frac{3}{4}\delta$, then $y = \frac{1}{\delta} + \frac{3}{4}\delta + \frac{1}{2}\delta = \frac{1}{\delta} + \frac{5}{4}\delta$

$$\implies \begin{cases} |x-y| = \frac{1}{2}\delta \leq \delta \\ \wedge \\ |x^2 - y^2| = |1 + \delta^2| > 1 \end{cases}$$

$$\implies \exists \varepsilon > 0\ (\varepsilon = 1), \forall \delta > 0, \exists x, y \in \mathbb{R}\ (x = \frac{1}{\delta} + \frac{3}{4}\delta, y = \frac{1}{\delta} + \frac{5}{4}\delta);\ \begin{cases} |x-y| \leq \delta \\ \wedge \\ |x^2 - y^2| > 1 \end{cases}$$

$\implies f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proposition 4.10

Let $f : D_f \longrightarrow \mathbb{R}$ be a real function function, then we have the following implication:

$$f \text{ is uniformly continuous on } D_f \implies f \text{ is continuous on } D_f$$

Remark 4.8 *The converse is false: a function can be continuous on D_f without being uniformly continuous on D_f. From example (3.16) we have: $f(x) = x^2$ is continuous on $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$.*

4.3.8 Theorems about continuous functions

Theorem 4.6: (Heine's theoem)

Every continuous function on an interval of type $[a, b]$ is uniformly continuous on this interval.

Proof 5

In this theorem, we'll show the following implication:
f is continuous on $[a,b] \implies f$ is uniformly continuous on $[a,b]$. By contradiction, we assume that f is continuous on $[a,b]$ and not uniformly continuous on $[a,b]$.
f is not uniformly continuous on $[a,b] \Leftrightarrow$

$$\exists \varepsilon_0, \forall \delta > 0, \exists x, y \in [a,b]; (|x-y| \leq \delta) \wedge (|f(x) - f(y)| > \varepsilon_0)$$

Let's put: $\delta = \dfrac{1}{n}$ tq: $n \in \mathbb{N}^*$

$$\implies \forall n \in \mathbb{N}^*, \exists x_n, y_n \in [a,b];\ |x_n - y_n| \leq \frac{1}{n} \wedge |f(x_n) - f(y_n)| > \varepsilon_0 \tag{4.12}$$

So we have constructed two sequences $(x_n)_{n\in\mathbb{N}^*}$ and $(y_n)_{n\in\mathbb{N}^*}$ which are included in $[a,b]$.
$(x_n)_{n\in\mathbb{N}^*} \subset [a,b] \implies (x_n)_{n\in\mathbb{N}^*}$ is a bounded sequence.
According to bolzano weierstrass's theorem, there exists a sub-sequence $(x_{\phi(n)})_{n\in\mathbb{N}^*}$ such that: $\lim\limits_{n\to+\infty} x_{\phi(n)} = l$ with $l \in [a,b]$.
On the one hand we have: $|x_{\phi(n)} - y_{\phi(n)}| \leq \dfrac{1}{\phi(n)} \implies \lim\limits_{n\to+\infty}(x_{\phi(n)} - y_{\phi(n)}) = 0$

$$\text{So } \begin{cases} \lim\limits_{n\to+\infty}(x_{\phi(n)} - y_{\phi(n)}) = 0 \\ \quad\text{and} \\ \lim\limits_{n\to+\infty} x_\phi(n) = l \end{cases} \implies \lim_{n\to+\infty} y_\phi(n) = l$$

f is continuous at $l \implies \exists \eta > 0, \forall x, y \in [a,b];\ |x-y| \leq \eta \implies |f(x) - f(l)| \leq \dfrac{\varepsilon_0}{3}$.
The sequences $(x_\phi(n))_{n\in\mathbb{N}^*}$ and $(y_\phi(n))_{n\in\mathbb{N}^*}$ converges to l

$$\implies \begin{cases} \exists n_0 \in \mathbb{N}^*, \forall n \in \mathbb{N}^*;\ n \geq n_0 \implies |x_{\phi(n)} - l| \leq \eta \implies |f(x_{\phi(n)}) - f(l)| \leq \dfrac{\varepsilon_0}{3} \\ \quad\text{and} \\ \exists n_1 \in \mathbb{N}^*, \forall n \in \mathbb{N}^*;\ n \geq n_1 \implies |y_{\phi(n)} - l| \leq \eta \implies |f(y_{\phi(n)}) - f(l)| \leq \dfrac{\varepsilon_0}{3} \end{cases}$$

If we put: $n^* = \max(n_0, n_1)$ we get:

$$\exists n^* \in \mathbb{N}^*, \forall n \in \mathbb{N}^*;\ n \geq n^* \implies |f(x_{\phi(n)}) - f(l)| + |f(y_{\phi(n)}) - f(l)| \leq \frac{2\varepsilon_0}{3}$$

According to the triangular inequality we have:

$$|f(x_{\phi(n)}) - f(y_{\phi(n)})| \leq |f(x_{\phi(n)}) - f(l)| + |f(y_{\phi(n)}) - f(l)| \leq \frac{2\varepsilon_0}{2}$$

.

$$\implies \forall n \geq n^* \text{ we have: } |f(x_{\phi(n)}) - f(y_{\phi(n)})| \leq \frac{2\varepsilon_0}{2} \tag{4.13}$$

$$(3.12) \text{ and } (3.13) \implies \forall n \geq n^*; \varepsilon_0 < |f(x_{\phi(n)}) - f(y_{\phi(n)})| \leq \frac{2\varepsilon_0}{2}$$

$$\implies \varepsilon_0 < \frac{2\varepsilon_0}{2} \quad \text{is a contradiction}$$

so the multiplication (f is continuous on $[a,b] \implies f$ is uniformly continuous on $[a,b]$) is true.

Theorem 4.7: (Weirstrass's theorem)

Let f be a continuous function on $[a, b]$, then:

$$\begin{cases} f \text{ is bounded on } [a,b] \\ \quad\text{and} \\ \exists x_1, x_2 \in [a,b] \text{ tq:} f(x_1) = \min\limits_{x\in[a,b]}(f(x)) \text{ and } f(x_2) = \max\limits_{x\in[a,b]}(f(x)) \end{cases}$$

(i.e. f is bounded and reaches its bounds on $[a, b]$.)

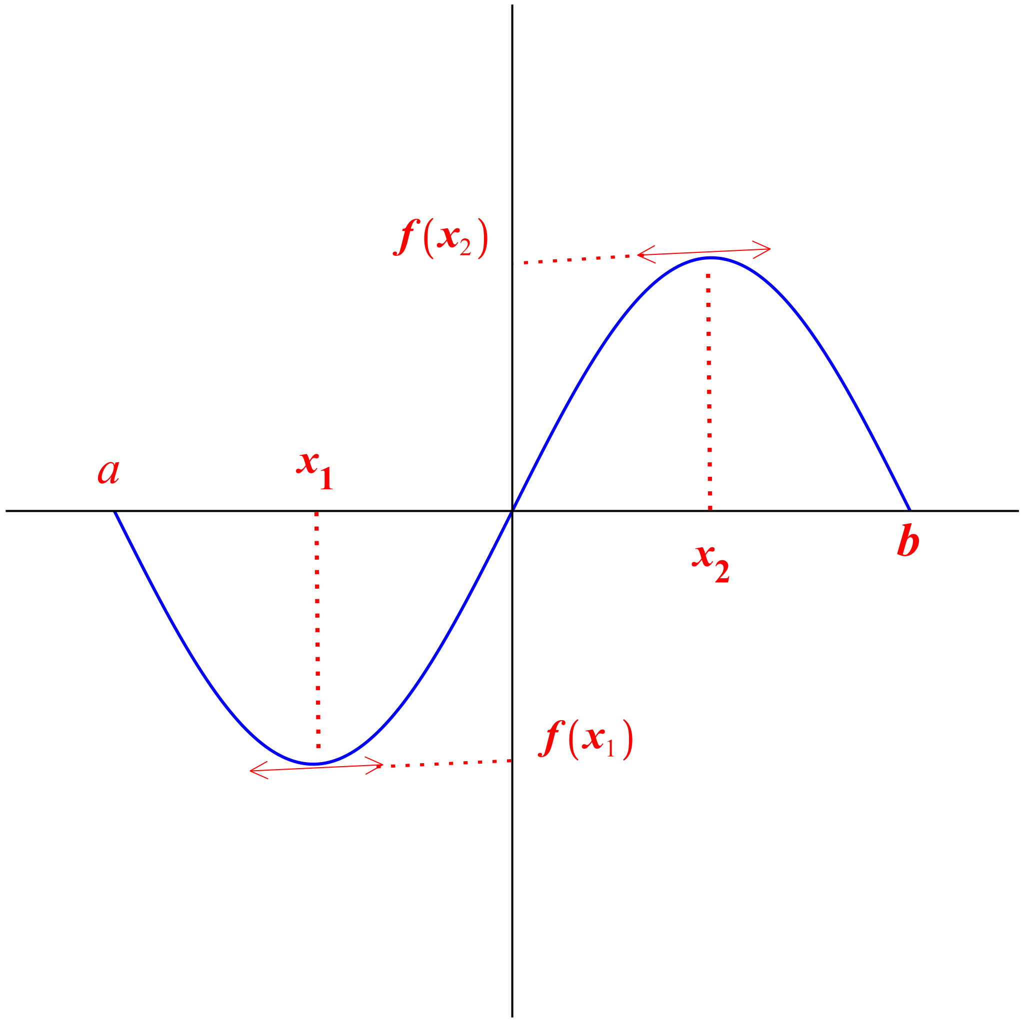


Figure 4.14: A continuous function on $[a, b]$

Proof 6

1. Let's assume that f is not bounded on $[a, b] \Leftrightarrow$

$$\forall n \in \mathbb{N}^*, \exists x_n \in [a, b] \text{ tq: } |f(x_n)| > n \tag{4.14}$$

So we constructed a sequence $(x_n)_{n\in\mathbb{N}^*} \subset [a, b] \implies (x_n)_{n\in\mathbb{N}^*}$ is bounded.
According to B.W's theorem there exists a sub-sequence $(x_{\phi(n)})_{n\in\mathbb{N}^*}$ of $(x_n)_{n\in\mathbb{N}^*}$ such that:

$$\lim_{n\to+\infty} x_{\phi(n)} = l \text{ avec } l \in [a, b]$$

$l \in [a, b] \implies f$ is continuous at $l \implies \lim\limits_{n\to+\infty} f(x_{\phi(n)}) = f(l) \in \mathbb{R}$
(3.14) $\implies \forall n \in \mathbb{N}^*;\ |f(x_{\phi(n)}| > \phi(n) \implies \lim\limits_{n\to+\infty} f(x_{\phi(n)}) = +\infty$ is a contradiction
$\implies f$ is bounded.

2. We put $\begin{cases} m = \inf\limits_{x\in[a,b]}(f(x)) = \inf(f([a, b])) \\ \quad\text{and} \\ M = \sup\limits_{x\in[a,b]}(f(x)) = \sup(f([a, b])) \end{cases}$

From the definition of sup and inf we have:

$$\forall \varepsilon > 0, \begin{cases} \exists x^* \in [a, b];\ f(x^*) < m + \varepsilon \\ \quad\text{and} \\ \exists y^* \in [a, b];\ M - \varepsilon < f(y^*) \end{cases}$$

Let's put: $\varepsilon = \dfrac{1}{n} /\ n \in \mathbb{N}^*$, we get:

$$\forall n \in \mathbb{N}^*; \begin{cases} \exists x_n \in [a, b];\ f(x_n) < m + \dfrac{1}{n} \\ \quad\text{and} \\ \exists y_n \in [a, b];\ M < f(y_n) + \dfrac{1}{n} \end{cases}$$

So we constructed two sequences $(x_n)_{n\in\mathbb{N}^*}$ and $(y_n)_{n\in\mathbb{N}^*}$ which are included in $[a, b] \implies (x_n)_{n\in\mathbb{N}^*}$ and $(y_n)_{n\in\mathbb{N}^*}$ are bounded. According to B.W's theorem we have:

$$\begin{cases} \exists(x_{\phi(n)})_{n\in\mathbb{N}^*} \text{ such that: } \lim\limits_{n\to+\infty} x_{\phi(n)} = \alpha /\ \alpha \in [a, b] \\ \quad\text{and} \\ \exists(y_{\sigma(n)})_{n\in\mathbb{N}^*} \text{ such tnat: } \lim\limits_{n\to+\infty} y_{\sigma(n)} = \beta /\ \beta \in [a, b] \end{cases}$$

$$\alpha, \beta \in [a, b] \implies \begin{cases} f \text{ is continuous at } \alpha \implies \lim\limits_{n\to+\infty} f(x_{\phi(n)}) = f(\alpha) \\ \quad\text{and} \\ f \text{ is continuous at } \beta \implies \lim\limits_{n\to+\infty} f(y_{\sigma(n)}) = f(\beta) \end{cases}$$

$\implies \forall n \in \mathbb{N}^*; \begin{cases} f(x_{\phi(n)}) - \dfrac{1}{n} < m \leq f(x_{\phi(n)}) \\ \quad\text{and} \\ f(y_{\sigma(n)}) \leq M < f(y_{\sigma(n)}) + \dfrac{1}{n} \end{cases}$ Passing to the limits we obtain: $m = f(\alpha) = \inf\limits_{x\in[a,b]}(f(x)) = \min\limits_{x\in[a,b]}(f(x))$ with $\alpha \in [a, b]$.
and $M = f(\beta) = \sup\limits_{x\in[a,b]}(f(x)) = \max\limits_{x\in[a,b]}(f(x))$ with $\beta \in [a, b]$.

Theorem 4.8: (Bolzano-Cauchy)

Let f be a continuous function on the interval $[a, b]$ tq: $f(a).f(b) \leq 0$, then there exists at least $c \in [a, b]$ verifying $f(c) = 0$.

Proof 7

Assume that $f(a) < 0$ et $f(b) > 0$. Let's put: $F = \{x \in [a, b]/f(x) \leq 0\}$.
Since $(F \subset [a, b])$, the set F is bounded above.
According to the completeness axiom for the real numbers, we have: $\exists c \in \mathbb{R};\ \sup(F) = c$ with $a \leq c \leq b$ (since $b \in \text{Upper}(F)$ and $a \in F$).

1. $c = \sup(F) \implies \forall \varepsilon > 0, \exists x^* \in F;\ c - \varepsilon < x^* \leq c$
   Let's take $\varepsilon = \dfrac{1}{n}$
   $$\implies \forall n \in \mathbb{N}^*, \exists x_n \in F;\ c - \frac{1}{n} < x_n \leq c \tag{4.15}$$
   So we constructed a sequence $(x_n)_{n \in \mathbb{N}^*} \subset F$
   According to (3.15) $\lim\limits_{n \to +\infty} x_n = c$ (Squeeze theorem).
   f is continuous at $c \implies \lim\limits_{n \to +\infty} f(x_n) = f(c)$.
   On the other hand, we have: $(x_n)_{n \in \mathbb{N}^*} \subset F \implies \forall n \in \mathbb{N}^*;\ f(x_n) \leq 0 \implies f(c) \leq 0$
2. Let's consider the sequence $y_n = c + \dfrac{b - c}{n}/n \in \mathbb{N}^*$.
   We have: $y_{n+1} - y_n = -\dfrac{b - c}{n(n+1)} \leq 0 \implies (y_n)_{n \in \mathbb{N}^*}$ is decreasing, then:
   $$\forall n \in \mathbb{N}^*;\ c < y_n \leq y_1 = b$$
   $\implies (y_n)_{n \in \mathbb{N}^*}$ is a sequence in $[a, b]$ which converges to c.
   f is continuous at $c \implies \lim\limits_{n \to +\infty} f(y_n) = f(c)$.
   On the other hand, we have: $\forall n \in \mathbb{N}^*;\ c < y_n \implies f(y_n) > 0 \implies f(c) > 0$.

Finally, from (1) and (2) we get: $\exists c \in [a, b];\ f(c) = 0$

Example 4.18

Let
$$\begin{aligned} f:\ [0, 2\pi] &\longrightarrow \mathbb{R} \\ x &\longmapsto \sin(x) + (x - 1)\cos(x) \end{aligned}$$

1. The function $f(x)$ is continuous on $[0, 2\pi]$ (since f is a sum of two continuous functions on $[0, 2\pi]$)
2. $f(0) = -1$ and $f(2\pi = 2\pi - 1 > 0 \implies f(0)f(2\pi) < 0$

According to B.C's theorem, there exists at least one real $c \in [0, 2\pi]$ such that: $f(c) = 0$

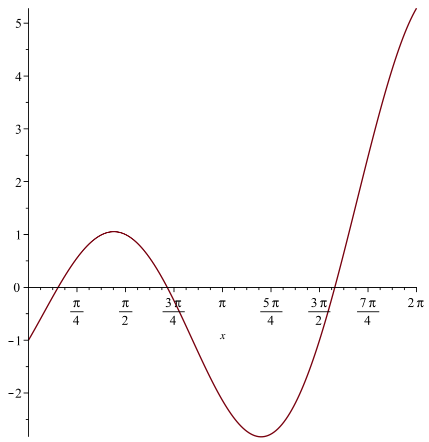


Figure 4.15: The graph of $f(x) = \sin(x) + (x-1)\cos(x)$ on the interval $[0, 2\pi]$.

Theorem 4.9: (The Intermediate Value Theorem)

Let f be a continuous function on $[a, b]$ we have:

1. If $f(a) < f(b)$ then $\forall \gamma \in [f(a), f(b)], \exists c \in [a, b]$ such that: $f(c) = \gamma$
2. If $f(b) < f(a)$ then $\forall \gamma \in [f(b), f(a)], \exists c \in [a, b]$ such that: $f(c) = \gamma$

Proposition 4.11

Let $f : I \longrightarrow \mathbb{R}$ be a continuous function on interval I (where I is an arbitrary interval). Then $f(I)$ is an interval.

Proof 8

Let $y_1, y_2 \in f(I)$ such that: $y_1 < y_2 \implies \exists x_1, x_2 \in I$ such that: $y_1 = f(x_1) \wedge y_2 = f(x_2)$.
Let's put: $a = \min(x_1, x_2)$ and $b = \max(x_1, x_2)$. We have: $a, b \in I$.
Let $y \in [y_1, y_2] \implies \exists c \in [a, b];\ f(c) = y$ (I.V.Th).
We have: $[a, b] \subset I$ (as I is an interval) $\implies y = f(c) \in f(I)$.
$\forall y_1, y_2 \in f(I), \forall y \in \mathbb{R};\ y \in [y_1, y_2] \implies y \in f(I) \implies f(I)$ is an interval.

Remark 4.9 *If f is a continuous function on $[a, b]$ then, $f([a, b]) = [m, M]$ with $m = \min_{x\in[a,b]}(f(x))$ and $M = \max_{x\in[a,b]}(f(x))$*

4.3.9 Monotonic functions and continuity

Theorem 4.10

Let $f : I \longrightarrow \mathbb{R}$ be a function (I is an interval). If f is strictly monotone on the interval I, then f is injective on I.

Proof 9

Let's show that f is injective. consider $x_1, x_2 \in I;\ x_1 \neq x_2$

1. Si $x_1 < x_2$ et f is strictly increasing $\Longrightarrow f(x_1) < f(x_2) \Longrightarrow f(x_1) \neq f(x_2)$.
2. Si $x_1 < x_2$ et f is strictly decreasing

$$\Longrightarrow f(x_1) > f(x_2) \Longrightarrow f(x_1) \neq f(x_2)$$

.

The same technique is used for $x_1 > x_2$.
So $\forall x_1, x_2 \in I;\ x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2) \Longrightarrow f$ is injective.

Theorem 4.11

Let $f : I \longrightarrow \mathbb{R}$ be a function defined and monotone on interval I. Then the following two statements are equivalent

1. f is continuous on I.
2. $f(I)$ is an interval.

Theorem 4.12: (bijection theorem)

Let $f : I \longrightarrow \mathbb{R}$ be a function.
If f is strictly monotone and continuous on I, then

1. f is a bijection from I into $J = f(I)$.
2. The inverse function $f^{-1} : J = f(I) \longrightarrow I$ is strictly monotonic and continuous on J (and varies in the same direction as f).