Chapter 1

Indefinite integrals

Contents

1.1 Indefinite integral and Antiderivative

1.1.1 An antiderivative of a function

Definition 1.1

Let $f : I \mapsto \mathbb{R}$ be a function defined on an interval I (where I is an open interval).
An antiderivative of f on I is any function F defined and differentiable on I that satisfies the following property:

$$\forall x \in I;\ F'(x) = f(x)$$

Example 1.1

1. Let $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ be the function defined by $f(x) = x^4$, then the function $F : I \to \mathbb{R}$ defined by $F(x) = \frac{1}{5}x^5$ is an antiderivative of f on $I = \mathbb{R}$ since:

$$\forall x \in I;\ F'(x) = x^4 = f(x).$$

2. Let $I =]0,+\infty[$ and $g : I \to \mathbb{R}$ be a function defined by $g(x) = \sqrt{x}$, then the function $G : I \to \mathbb{R}$ defined by $G(x) = \frac{2}{3}x\sqrt{x}$ is an antiderivative of g on $I =]0,+\infty[$ since:

$$\forall x \in I;\ G'(x) = \sqrt{x} = g(x).$$

Remark 1.1 *In the previous examples we see that:*

1. *The function* $F_1(x) = \frac{1}{5}x^5 + 2$ *is also an antiderivative of the function* $f(x) = x^4$ *on* $\mathbb{R}$.
2. *The function* $G_1(x) = \frac{2}{3}x\sqrt{x} + 5$ *is also an antiderivative of* $g(x) = \sqrt{x}$ *on* $]0, +\infty[$.

Proposition 1.1

Let f be a function defined and admit an antiderivative on I then:

1. f has infinitely many antiderivatives on I.
2. If F_1,F_2 are two antiderivatives of f on I, then there exists a constant $c \in \mathbb{R}$ such that:
$$\forall x \in I;\ F_1(x) = F_2(x) + c$$

Proof

1. Let's assume that f admits an antiderivative F on I then, the functions defined by: $G_k(x) = F(x) + k/k \in \mathbb{R}$ are also antiderivatives of f on I since:
$$\forall x \in I;\ G_k'(x) = F'(x) = f(x) \implies f \text{ has infinitely many primitives on } I$$
2. Let F_1,F_2 be two antiderivatives of f on I (I is an open interval) then;
$$\forall x \in I;\ (F_1(x) - F_2(x))' = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$
$$\implies \text{the function } F_1(x) - F_2(x) \text{ is constant on } I$$
$$\text{c-à-d } \exists c \in \mathbb{R};\ \forall x \in I;\ F_1(x) - F_2(x) = c \implies F_1(x) = F_2(x) + c$$

1.1.2 Indefinite integral

Definition 1.2

Let f be a function that is defined and admits an antiderivative on I (I is an open interval). The set of all antiderivatives of f on I is called the indefinite integral of f on I, and is denoted by :
$$\int f(x)\,\mathrm{d}x,\ x \in I$$

Remark 1.2 *If we have: F is an antiderivative of f on I, then we write:*
$$\int f(x)\,\mathrm{d}x = F(x) + c. \quad c \in \mathbb{R}$$

Example 1.2

We have:

- The indefinite integral of the function $\frac{1}{x}$ on $]-\infty,0[$ is defined by:

$$\int \frac{dx}{x} = \ln(-x) + c_1 / c_1 \in \mathbb{R}$$

- The indefinite integral of the function $\frac{1}{x}$ on $]0,+\infty[$ is defined by:

$$\int \frac{dx}{x} = \ln(x) + c_2 / c_2 \in \mathbb{R}$$

$\Longrightarrow$ The indefinite integral of the function $\frac{1}{x}$ on $\mathbb{R}^*$ is defined by:

$$\int \frac{dx}{x} = \ln|x| + c / c \in \mathbb{R}$$

1.1.3 Existence of the indefinite integral

Theorem 1.1

Let $f : I \to \mathbb{R}$ be a function defined on I (I is an open interval), then we have the following implication:

$$f \text{ is continuous on } I \Longrightarrow f \text{ admits an antiderivative on; } I$$

Remark 1.3 *According to Remark (1.2), if f admits a antiderivative on an interval I, then we can define the indefinite integral of f on this interval.*

Example 1.3

We have: $\cos(x)$ is an antiderivative of $\sin(x)$ on $\mathbb{R}$, so we can define the indefinite integral of the function $\sin(x)$ with the following:

$$\int \sin(x)dx = \cos(x) + c / c \in \mathbb{R}$$

Remark 1.4 *The procedure for calculating an indefinite integral is called integration, and we say integrate a function instead of calculating its indefinite integral.*

1.1.4 Properties of the indefinite integral

Proposition 1.2

Let f and g be two real functions, then we have the following properties:

1. $\displaystyle\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
2. $\forall \lambda \in \mathbb{R}$; $\displaystyle\int \lambda f(x)dx = \lambda \int f(x)dx$
3. $\displaystyle\left(\int f(x)dx\right)' = f(x)$
4. $\displaystyle\int f'(x)dx = f(x) + c, \quad c \in \mathbb{R}$

1.1.5 Indefinite integrals of some usual functions

From the derivatives of usual functions, we establish the following table of indefinite integrals of some usual functions

$\int f(x)\,\mathrm{d}x$	Domain of definition
$\int 0\,\mathrm{d}x = c$	$\mathbb{R}$
$\int a\,\mathrm{d}x = ax + c/(a = const)$	$\mathbb{R}$
$\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + c/(n \in \mathbb{N})$	$\mathbb{R}$
$\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + c/(n \in \mathbb{Z} - \{-1\})$	$\mathbb{R}^*$
$\int \frac{1}{x}\,\mathrm{d}x = \ln\lvert x\rvert + c$	$\mathbb{R}^*$
$\int x^\alpha\,\mathrm{d}x = \frac{1}{\alpha+1}x^{\alpha+1} + c/(\alpha \in \mathbb{R} - \{-1\})$	$\mathbb{R}_+^*$
$\int e^x\,\mathrm{d}x = e^x + c$	$\mathbb{R}$
$\int a^x\,\mathrm{d}x = \frac{1}{\ln(a)}a^x + c\ /(a \in \mathbb{R}_+^* - \{1\})$	$\mathbb{R}$
$\int \cos(x)\,\mathrm{d}x = \sin(x) + c$	$\mathbb{R}$

$\int f(x)\,\mathrm{d}x$	Domain of definition
$\int \sin(x)\,\mathrm{d}x = -\cos(x) + c$	$\mathbb{R}$
$\int \frac{\mathrm{d}x}{\cos^2(x)} = \tan(x) + c$	$\mathbb{R} - \{\frac{\pi}{2} + k\pi/k \in \mathbb{Z}\}$
$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin(x) + c$	$x \in]-1,1[$
$\int \frac{-1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arccos(x) + c$	$x \in]-1,1[$
$\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(x) + c$	$\mathbb{R}$
$\int \mathrm{sh}(x)\,\mathrm{d}x = \mathrm{ch}(x) + c$	$\mathbb{R}$
$\int \mathrm{ch}(x)\,\mathrm{d}x = \mathrm{sh}(x) + c$	$\mathbb{R}$
$\int \frac{\mathrm{d}x}{\mathrm{ch}^2(x)} = \mathrm{th}(x) + c$	$\mathbb{R}$
$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \mathrm{argsh}(x) + c = \ln(x + \sqrt{1+x^2}) + c$	$\mathbb{R}$
$\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \mathrm{argch}(x) + c = \ln(x + \sqrt{x^2-1}) + c$	$]1,+\infty[$
$\int \frac{\mathrm{d}x}{1-x^2} = \mathrm{argth}(x) + c = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + c$	$]-1,1[$
$\int f'(x)f^n(x)\,\mathrm{d}x = \frac{1}{n+1}f^{n+1}(x) + c/n \in \mathbb{N}$	$D_{f'}$
$\int \frac{f'(x)}{f^n(x)}\,\mathrm{d}x = \frac{-1}{(n-1)f^{n-1}(x)} + c/n \in \mathbb{N} - \{1\}$	//
$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln\lvert f(x)\rvert + c$	//

1.2 Direct integration method

This method is based on the properties of indefinite integrals and formulas for transforming functions using primitive tables.

Example 1.4

- Compute $\int (\cos^2(x) + \frac{1}{1+x^2} + \sqrt{x})\,dx$

Solution:
From indefinite integral property we have:

$$\int (\cos^2(x) + \frac{1}{1+x^2} + \sqrt{x})\,dx = \int \cos^2(x)\,dx + \int \frac{1}{1+x^2}\,dx + \int \sqrt{x}\,dx \tag{1.1}$$

1. Using the following trigonometric formula:

$$\cos(2x) = 2\cos^2(x) - 1 \Leftrightarrow \cos^2(x) = \frac{1}{2}\cos(2x) + \frac{1}{2}$$

   On obtain: $\int \cos^2(x)\,dx = \int (\frac{1}{2}\cos(2x) + \frac{1}{2})\,dx = \frac{1}{4}\sin(2x) + \frac{1}{2}x + c_1$

2. Using the table of antiderivatives, we obtain: $\int \frac{1}{1+x^2}\,dx = \arctan(x) + c_2$

3. Using the table of antiderivatives, we also get: $\int \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c_3 = \frac{2}{3}\sqrt{x^3} + c_3$

From steps 1, 2, 3 and equation (1.1) we find:

$$\int (\cos^2(x) + \frac{1}{1+x^2} + \sqrt{x})\,dx = \frac{1}{4}\sin(2x) + \frac{1}{2}x + \arctan(x) + \frac{2}{3}\sqrt{x^3} + c / c \in \mathbb{R}$$

1.3 Integration By Substitution (Change of Variables)

1.3.1 The first formula

Proposition 1.3

Let $f : J \to \mathbb{R}$ and $g : I \to J$ be two functions such that:

- I,J are two open intervals of $\mathbb{R}$.
- f is a continuous function on J.
- g is a differentiable function on I.

If $F(x)$ is an antiderivative of $f(x)$ on J, then $F(g(x))$ is an antiderivative of the function $g'(x)f(g(x))$ on I.

Proof

We have:

$$\forall x \in I;\ [F(g(x))]' = g'(x)F'(g(x)) = g'(x)f(g(x))$$

.

Practical method: To calculate the integral of type $\int g'(x)f(g(x))\,dx$, we follow these steps:

1. We put $t = g(x)$ and $dt = g^{prime}(x)dx$.
2. This gives the integral $\int f(t)\,\mathrm{d}t$.

   If F is an antiderivative of f then $\int f(t)\,\mathrm{d}t = F(t) + c/c \in \mathbb{R}$
3. Replacing t by $g(x)$ and dt by $g^{prime}(x)dx$ gives the following result:

$$\int g'(x)f(g(x))\,\mathrm{d}x = F(g(x)) + c, \quad c \in \mathbb{R}$$

.

Example 1.5

1. Calculate $I = \int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x$.

   We put

$$t = 1 + x^2 \implies dt = 2xdx$$

   Replacing in the integral I gives us:

$$I = \int \frac{1}{2\sqrt{t}}\,\mathrm{d}t = \sqrt{t} + c/c \in \mathbb{R}$$

$$\implies I = \sqrt{1+x^2} + c/c \in \mathbb{R}$$

2. Compute $J = \int \frac{1}{\mathrm{ch}(x)}\,\mathrm{d}x$

   We have:

$$\mathrm{ch}(x) = \frac{e^x + e^{-x}}{2} \Leftrightarrow \frac{1}{\mathrm{ch}(x)} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x}+1}$$

$$\implies J = \int \frac{2e^x}{e^{2x}+1}\,\mathrm{d}x.$$

   let's set:

$$t = e^x \implies dt = e^x dx$$

$$\implies J = 2\int \frac{1}{t^2+1}\,\mathrm{d}t = 2\arctan(t) + c, \quad c \in \mathbb{R}$$

$$\implies J = 2\arctan(e^x) + c, \quad c \in \mathbb{R}$$

1.3.2 The second formula

Proposition 1.4

Let $f : J \to \mathbb{R}$ and $g : I \to J$ be two functions such that:

- f is a continuous function on J.
- g is a bijection from I into J and continuous on I.
- g is differentiable on I and verifies: $\forall x \in I;\ g'(x) \neq 0$

If $H(x)$ is an antiderivative of $g'(x)f(g(x))$ on I, then $H(g^{-1}(x))$ is an antiderivative of $f(x)$ on J.

Proof

We have:

$$\forall x \in J;\ (H(g^{-1}(x)))' = (g^{-1}(x))'H'(g^{-1}(x)) = \frac{1}{g'(g^{-1}(x))}g'(g^{-1}(x))f(g(g^{-1}(x))) = f(x)$$

Practical method: To calculate the integral $\int f(x)\,\mathrm{d}x$ we follow the steps below:

1. We pose $x = g(t)$ and $dx = g'(t)dt$.
2. We obtain the integral $\int f(g(t))g'(t)\,\mathrm{dt}$

   İf $H(t)$ is an antiderivaitive of $f(g(t))g'(t)$ then: $\int f(g(t))g'(t)\,\mathrm{dt} = H(t) + c, \quad c \in \mathbb{R}$
3. Replacing t by $g^{-1}(x)$ and $g'(t)dt$ by dx, this gives: $\int f(x)\,\mathrm{d}x = H(g^{-1}(x)) + c, \quad c \in \mathbb{R}$

Example 1.6

Compute $I = \int \frac{x^2}{\sqrt{1-x^2}},\mathrm{d}x$ on $]-1,1[$.
We make the following change of variable:

$$x = \sin(t) \text{ with } t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\text{ and } dx = \cos(t)dt$$

We obtain: $I = \int \frac{\sin^2(t)}{\sqrt{1-\sin^2(t)}}\cos(t)\,\mathrm{dt} = \int \frac{\sin^2(t)}{|\cos(t)|}\cos(t)\,\mathrm{dt} = \int \sin^2(t)\,\mathrm{dt}$
(since $t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ we have: $cos(t) > 0$)

On the other hand we have: $\sin^2(t) = \frac{1}{2} - \frac{1}{2}\cos(2t) \implies I = \int (\frac{1}{2} - \frac{1}{2}\cos(2t))\,\mathrm{dt}$
$\implies I = \frac{1}{2}t - \frac{1}{4}\sin(2t) + c, \quad c \in \mathbb{R}$. We have: $x = \sin(t) \implies t = \arcsin(x)$
Replacing t by $\arcsin(x)$, this gives: $I = \frac{1}{2}\arcsin(x) - \frac{1}{4}\sin(2\arcsin(x)) + c, \quad c \in \mathbb{R}$.
Using the following trigonometric formula, we obtain:

$$\begin{aligned} &\sin(2\arcsin(x)) = 2\sin(\arcsin(x))\cos(\arcsin(x)) \\ \Leftrightarrow\ &\sin(2\arcsin(x)) = 2\sin(\arcsin(x))\sqrt{1-\sin^2(\arcsin(x))} \\ \Leftrightarrow\ &\sin(2\arcsin(x)) = 2x\sqrt{1-x^2} \end{aligned}$$

$$\implies I = \frac{1}{2}\arcsin(x) - \frac{1}{2}x\sqrt{1-x^2} + c/c \in \mathbb{R}$$

1.4 Integration by parts method

Proposition 1.5

Let f and g be two functions in class C^1 on an open interval I, we have:

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

Proof

Let f and g be two functions in class C^1 on I, we have:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$\Longrightarrow \int (f(x)g(x))'\,\mathrm{d}x = \int f'(x)g(x)\,\mathrm{d}x + \int f(x)g'(x)\,\mathrm{d}x$$

$$\Longrightarrow \int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

Example 1.7

1. Compute $\int xe^{-2x}\,\mathrm{d}x$ on $\mathbb{R}$.
   Let's put:

$$\begin{array}{rcl} f(x) = x & \longrightarrow & f'(x) = 1 \\ g'(x) = e^{-2x} & \longrightarrow & g(x) = -\frac{1}{2}e^{-2x} \end{array}$$

   According to the integration by parts formula, we obtain:

$$\int xe^{-2x}\,\mathrm{d}x = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,\mathrm{d}x$$

$$\Longrightarrow \int xe^{-2x}\,\mathrm{d}x = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c/c \in \mathbb{R}$$

2. Compute $\int \sqrt{1-x^2}\,\mathrm{d}x$ on $]-1,1[$.
   Let's put:

$$\begin{array}{rcl} f(x) = \sqrt{1-x^2} & \longrightarrow & f'(x) = -\dfrac{x}{\sqrt{1-x^2}} \\ g'(x) = 1 & \longrightarrow & g(x) = x \end{array}$$

$$\Longrightarrow \int \sqrt{1-x^2}\,\mathrm{d}x = x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}$$

   Example (1.6) gives us:

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{3}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x) + c/c \in \mathbb{R}$$

1.5 Integration of a rational function

Definition 1.3

Let $P(x)$ and $Q(x)$ be two polynomials of degree n_1 and n_2 respectively. A rational function is any function of kind $f(x) = \dfrac{P(x)}{Q(x)}$

- If $n_1 \geq n_2$ we say that f is an improper rational fraction.
- If $n_1 < n_2$ we say that f is a proper rational fraction.

Proposition 1.6

Let $f(x) = \dfrac{P(x)}{Q(x)}$ be an improper rational fraction (i.e. $P(x)$ and $Q(x)$ are two polynomials of degree n_1 and n_2 respectively with $n_1 \geq n_2$).
The Euclidean division of $P(x)$ by $Q(x)$ yields:

$$f(x) = N(x) + \frac{R(x)}{Q(x)}$$

tq: $\begin{cases} N(x) \text{ is a polynomial of degree } n_1 - n_2. \\ R(x) \text{ is a polynomial of degree strictly less than } n_2. \\ \dfrac{R(x)}{Q(x)} \text{ is a proper rational fraction.} \end{cases}$

We deduce that:

$$\int f(x)\,\mathrm{d}x = \int \frac{P(x)}{Q(x)}\,\mathrm{d}x = \int N(x)\,\mathrm{d}x + \int \frac{R(x)}{Q(x)}\,\mathrm{d}x$$

Example 1.8

If we want to integrate the following improper fraction: $f(x) = \dfrac{3x^3 + 2x - 5}{3x^2 - 5x - 2}$, we perform Euclidean division as follows:

$$\begin{array}{r|l} 3x^3 + 2x - 5 & 3x^2 - 5x - 2 \\ \hline -3x^3 + 5x^2 + 2x & x + \frac{5}{3} \\ \hline 5x^2 + 4x - 5 & \\ -5x^2 + \frac{25}{3} + \frac{10}{3} & \\ \hline \frac{37}{3}x - \frac{5}{3} & \end{array}$$

$$\Longrightarrow \quad f(x) = x + \frac{5}{3} + \frac{\frac{37}{3}x - \frac{5}{3}}{3x^2 - 5x - 2}$$

$$\Longrightarrow \int f(x)\,\mathrm{d}x = \int \frac{3x^3 + 2x - 5}{2x^2 - 5x - 2}\,\mathrm{d}x = \int (x + \frac{5}{3})\,\mathrm{d}x + \int \frac{\frac{37}{3}x - \frac{5}{3}}{3x^2 - 5x - 2}\,\mathrm{d}x$$

Remark 1.5 *The difficulty of integrating a rational fraction is restricted to integrating a rational fraction proper.*

1.5.1 Decomposition of a proper fraction into simple elements

Theorem 1.2: (A fundamental theorem on the decomposition of a proper fraction)

Let $f(x) = \dfrac{P(x)}{Q(x)}$ be an proper rational fraction such that:
$Q(x) = c(x-x_1)^{m_1}(x-x_2)^{m_2}....(x-x_k)^{m_k}(x^2+p_1x+q_1)^{n_1}(x^2+p_2x+q_2)^{n_2}....(x^2+p_lx+q_l)^{n_l}$
where:

- $c,x_1,x_2,...,x_k \in \mathbb{R}$, $p_1,p_2,...,p_l \in \mathbb{R}$ et $q_1,q_2,....,q_l \in \mathbb{R}$
- $m_1,m_2,....,m_k \in \mathbb{N}$ et $n_1,n_2,....,n_l \in \mathbb{N}$
- $\forall i \in \{1,2,...,l\};\ \Delta_i = p_i^2 - 4q_i < 0$

Then $f(x) = \dfrac{P(x)}{Q(x)}$ decomposes into simple fractions in the following form:

$$\begin{aligned}
f(x) = \frac{P(x)}{Q(x)} &= \frac{A_{1,1}}{x-x_1} + \frac{A_{1,2}}{(x-x_1)^2} + ... + \frac{A_{1,m_1}}{(x-x_1)^{m_1}} \\
&+ \frac{A_{2,1}}{x-x_2} + \frac{A_{2,2}}{(x-x_2)^2} + ... + \frac{A_{2,m_2}}{(x-x_1)^{m_2}} \\
&+ .... \\
&+ \frac{A_{k,1}}{x-x_k} + \frac{A_{k,2}}{(x-x_k)^2} + ... + \frac{A_{k,m_2}}{(x-x_1)^{m_k}} \\
&+ \frac{B_{1,1}x + C_{1,1}}{x^2+p_1x+q_1} + \frac{B_{1,2}x + C_{1,2}}{(x^2+p_1x+q_1)^2} + ... + \frac{B_{1,n_1}x + C_{1,n_1}}{(x^2+p_1x+q_1)^{n_1}} \\
&+ \frac{B_{2,1}x + C_{2,1}}{x^2+p_2x+q_2} + \frac{B_{2,2}x + C_{2,2}}{(x^2+p_2x+q_2)^2} + ... + \frac{B_{2,n_2}x + C_{2,n_2}}{(x^2+p_2x+q_2)^{n_2}} \\
&+ .... \\
&+ \frac{B_{l,1}x + C_{l,1}}{x^2+p_lx+q_l} + \frac{B_{l,2}x + C_{l,2}}{(x^2+p_lx+q_l)^2} + ... + \frac{B_{l,n_l}x + C_{l,n_l}}{(x^2+p_lx+q_l)^{n_l}}
\end{aligned}$$

where $A_{i,j}$,$B_{i,j}$ and $C_{i,j}$ are real constants.

Remark 1.6 *According to the previous theorem, we can decompose any proper fraction into a finite sum of the following simple elements:*

1. *A simple element of the first kind is of the form* $\dfrac{A}{(x-a)^m}$ *such that:* $A,a \in \mathbb{R}$ *and* $m \in \mathbb{N}^*$

2. *A simple element of the second kind is of the form* $\dfrac{Bx+C}{(x^2+px+q)^n}$ *such that:* $n \in \mathbb{N}^*,B,C,p,q \in \mathbb{R}$ *and* $\Delta = p^2 - 4q < 0$

Example 1.9

Decompose the following fractions into simple elements.

1. $f(x) = \dfrac{P(x)}{Q(x)} = \dfrac{2x^3 + 4x^2 + x + 2}{(x-1)^2(x^2 + x + 1)}$

2. $g(x) = \dfrac{P(x)}{Q(x)} = \dfrac{1 + 2x^2}{x^2(1 + x^2)^2}$

Solution:

01) $f(x)$ is a proper fraction, according to the fundamental theorem of decomposition we have:

$$\frac{2x^3 + 4x^2 + x + 2}{(x-1)^2(x^2 + x + 1)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B_1x + C_1}{x^2 + x + 1}$$

$$\Leftrightarrow \frac{2x^3 + 4x^2 + x + 2}{(x-1)^2(x^2 + x + 1)} = \frac{A_1(x-1)(x^2+x+1) + A_2(x^2+x+1) + (x-1)^2(B_1x + C_1)}{(x-1)^2(x^2 + x + 1)}$$

$$\Leftrightarrow \frac{2x^3 + 4x^2 + x + 2}{(x-1)^2(x^2 + x + 1)} = \frac{(A_1 + B_1)x^3 + (A_2 + C_1 - 2B_1)x^2 + (A_2 + B_1 - 2C_1)x - A_1 + A_2 + C_1}{(x-1)^2(x^2 + x + 1)}$$

By identifying the numerator coefficients, we obtain:

$$\begin{cases} A_1 + B_1 = 2 \\ A_2 + C_1 - 2B_1 = 4 \\ A_2 + B_1 - 2C_1 = 1 \\ -A_1 + A_2 + C_1 = 2 \end{cases}$$

$$\Longrightarrow A_1 = 2,\ A_2 = 3,\ B_1 = 0 \text{ et } C_1 = 1$$

$$\Longrightarrow \frac{2x^3 + 4x^2 + x + 2}{(x-1)^2(x^2 + x + 1)} = \frac{2}{(x-1)} + \frac{3}{(x-1)^2} + \frac{1}{x^2 + x + 1}$$

02) $g(x)$ is a proper fraction, according to the fundamental theorem of decomposition we have:

$$\begin{aligned} \frac{1 + 2x^2}{x^2(1 + x^2)^2} &= \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1x + C_1}{x^2 + 1} + \frac{B_2x + C_2}{(x^2+1)^2} \\ &= \frac{A_1x(x^2+1)^2 + A_2(x^2+1)^2 + x(B_1x + C_1)(x^2+1) + x^2(B_2x + C_2)}{x^2(1+x^2)^2} \\ &= \frac{A_1x^5 + (A_2 + B_1)x^4 + (2A_1 + B_2 + C_1)x^3 + (2A_2 + B_1 + C_2)x^2 + (A_1 + C_1)x + A_2}{x^2(1+x^2)^2} \end{aligned}$$

By identifying the numerator coefficients, we obtain:

$$\begin{cases} A_1 = 0 \\ A_2 + B_1 = 0 \\ 2A_1 + B_2 + C_1 = 0 \\ 2A_2 + B_1 + C_2 = 2 \\ A_1 + C_1 = 0 \\ A_2 = 1 \end{cases}$$

$$\Longrightarrow A_1 = 0, A_2 = 1, B_1 = 0, B_2 = 0, \ C_1 = -1, \text{ et } C_2 = 1$$

.

$$\Longrightarrow \frac{1+2x^2}{x^2(1+x^2)^2} = \frac{1}{x^2} - \frac{1}{x^2+1} + \frac{1}{(x^2+1)^2}$$

1.5.2 Integration of simple elements

1. A simple element of first kind: $U(x) = \dfrac{A}{(x-a)^m}$ such that: $A, a \in \mathbb{R}$ and $m \in \mathbb{N}^*$, then we have:

$$\begin{cases} \displaystyle\int \frac{A}{x-a}\,\mathrm{d}x = A\ln|x-a| + c/c \in \mathbb{R} & \text{Si } m = 1 \\ \displaystyle\int \frac{A}{(x-a)^m}\,\mathrm{d}x = -\frac{A}{(m-1)(x-a)^{m-1}} + c/c \in \mathbb{R} & \text{Si } m \neq 1 \end{cases}$$

2. A simple element of second kind: $V(x) = \dfrac{Bx+C}{(x^2+px+q)^n}$ such that:
$n \in \mathbb{N}^*$, $B, C, p, q \in \mathbb{R}$ and $\Delta = p^2 - 4q < 0$

How to calculate ?
To calculate the integral of type: $I = \displaystyle\int \frac{Bx+C}{(x^2+px+q)^n}$,we follow the steps below:

- We have:

$$\begin{aligned} x^2+px+q &= (x+\frac{1}{2}p)^2 + q - \frac{1}{4}p^2 \\ &= (x+\frac{1}{2}p)^2 - \frac{1}{4}(p^2-4q) \\ &= (x+\frac{1}{2}p)^2 - \frac{1}{4}\Delta \\ &= -\frac{\Delta}{4}\left[-\frac{4}{\Delta}(x+\frac{1}{2}p)^2 + 1\right] \\ &= -\frac{\Delta}{4}\left[\left(\frac{2x+p}{\sqrt{-\Delta}}\right)^2 + 1\right] \end{aligned}$$

- By changing the variable as follows:

$$t = \frac{2x+p}{\sqrt{-\Delta}} \Longrightarrow x = \frac{\sqrt{-\Delta}}{2}t - \frac{p}{2} \quad \text{and} \quad dx = \frac{\sqrt{-\Delta}}{2}dt$$

We get:

$$I = \frac{B}{2}\left(\frac{2}{\sqrt{-\Delta}}\right)^{2n-2}\int \frac{2t}{(t^2+1)^n}\,\mathrm{dt} + \left(\frac{2}{\sqrt{-\Delta}}\right)^{2n-1}\left(\frac{2C-Bp}{2}\right)\int \frac{1}{(t^2+1)^n}\,\mathrm{dt}$$

let's put: $I_n = \displaystyle\int \frac{2t}{(t^2+1)^n}\,\mathrm{dt}$ and $J_n = \displaystyle\int \frac{1}{(t^2+1)^n}\,\mathrm{dt} \Longrightarrow I = \alpha I_n + \beta J_n$

$$\text{with:} \quad \alpha = \frac{B}{2}\left(\frac{2}{\sqrt{-\Delta}}\right)^{2n-2} \quad \text{et} \quad \beta = \left(\frac{2}{\sqrt{-\Delta}}\right)^{2n-1}\left(\frac{2C-Bp}{2}\right)$$

- Calculation of I_n and J_n.
  (a) We have:

$$\begin{cases} I_1 = \int \frac{2t}{t^2+1}\,\mathrm{dt} = \ln(t^2+1) + c/c \in \mathbb{R} \\ \\ I_n = \int \frac{2t}{(t^2+1)^n}\,\mathrm{dt} = -\frac{1}{(n-1)(t^2+1)^{n-1}} + c/c \in \mathbb{R} \quad \text{et } n > 1 \end{cases}$$

  (b) J_n is calculated by the following recurrence formula:

$$\begin{cases} J_1 = \arctan(t) + c, \quad c \in \mathbb{R} \\ \\ J_{n+1} = \frac{2n-1}{2n} J_n + \frac{t}{2n(t^2+1)^n} \end{cases}$$

Proof

- For $n = 1$, we have: $J_1 = \int \frac{1}{t^2+1}\,\mathrm{dt} \implies J_1 = \arctan(t) + c/c \in \mathbb{R}$
- For $n > 1$, using integration by parts in the following way: Let's put:

$$\begin{aligned} f(t) = \tfrac{1}{(t^2+1)^n} &\longrightarrow f'(t) = -\tfrac{2nt}{(t^2+1)^{n+1}} \\ g'(t) = 1 &\longrightarrow g(t) = t \end{aligned}$$

$$\begin{aligned} &\implies J_n = \int \frac{1}{(t^2+1)^n}\,\mathrm{dt} = \frac{t}{(t^2+1)^n} + 2n \int \frac{t^2}{(t^2+1)^{n+1}}\,\mathrm{dt} \\ &\implies J_n = \frac{t}{(t^2+1)^n} + 2n \int \left(\frac{1}{(t^2+1)^n} - \frac{1}{(t^2+1)^{n+1}} \right) \mathrm{dt} \\ &\implies J_n = \frac{t}{(t^2+1)^n} + 2nJ_n - 2nJ_{n+1} \\ &\implies J_{n+1} = \frac{2n-1}{2n} J_n + \frac{t}{2n(t^2+1)^n} \end{aligned}$$

- Finally, we replace t by $\frac{2}{\sqrt{-\Delta}}t + \frac{p}{\sqrt{-\Delta}}$ we obtain the result.

Example 1.10

- Calculate $\int \frac{2x^3+4x^2+x+2}{(x-1)^2(x^2+x+1)}\,\mathrm{d}x$

Solution:

- **Step 01:**(The decomposition into simple elements) From example (1.9) we have:

$$\frac{2x^3+4x^2+x+2}{(x-1)^2(x^2+x+1)} = \frac{2}{x-1} + \frac{3}{(x-1)^2} + \frac{1}{x^2+x+1}$$

- **Step 02:**(Integration of simple elements obtained after decomposition)
  From the property of indefinite integrals we have:

$$\int \frac{2x^3+4x^2+x+2}{(x-1)^2(x^2+x+1)}\,\mathrm{d}x = \int \frac{2}{x-1}\,\mathrm{d}x + \int \frac{3}{(x-1)^2}\,\mathrm{d}x + \int \frac{1}{x^2+x+1}\,\mathrm{d}x$$

1. We immediately obtain:

$$\begin{cases} \displaystyle\int \frac{2}{x-1}\,\mathrm{d}x = 2\ln|x-1| + c_1 \\ \text{et} \\ \displaystyle\int \frac{3}{(x-1)^2}\,\mathrm{d}x = -\frac{3}{x-1} + c_2 \end{cases}$$

2. For the integral $J = \displaystyle\int \frac{1}{x^2+x+1}\,\mathrm{d}x$ we follow the following method:
   We have:

$$\begin{aligned} x^2+x+1 &= \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\frac{4}{3}\left(x+\frac{1}{2}\right)^2 + 1\right] \\ &= \frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1\right] \end{aligned}$$

   A change of variable is applied as follows:

$$t = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}} \Longrightarrow \begin{cases} x = \frac{\sqrt{3}}{2}t - \frac{1}{2} \\ \text{et} \\ dx = \frac{\sqrt{3}}{2}dt \end{cases}$$

   We get:

$$J = \int \frac{1}{x^2+x+1}\,\mathrm{d}x = \frac{2}{\sqrt{3}}\int \frac{1}{t^2+1}\,\mathrm{d}t = \frac{2}{\sqrt{3}}\arctan(t) + c_3$$

   So we replace t by $\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}$, we find:

$$J = \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + c_3$$

$$\Longrightarrow \int \frac{2x^3+4x^2+x+2}{(x-1)^2(x^2+x+1)}\,\mathrm{d}x = 2\ln|x-1| - \frac{3}{x-1} + \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + c/c \in \mathbb{R}$$

1.6 Integration of irrational functions

The integration of several irrational functions can be transformed by a suitable change of variable to the integration of a rational fraction. For more details on this integration method, we need the following definitions.

1.6.1 Polynomials and fractions of two variables

Definition 1.4

A polynomial of two variables x,y of degree n is defined by the following expression:

$$\begin{aligned} P(x,y) = a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + ... + a_{n,0}x^n \\ + a_{n-1,1}x^{n-1}y + ... + a_{1,n-1}xy^{n-1} + a_{0,n}y^n \end{aligned}$$

Example 1.11

1. A polynomial of two variables x,y of degree 1 is defined by:

$$P(x,y) = a_{0,0} + a_{1,0}x + a_{0,1}y$$

2. A polynomial of two variables x,y of degree 3 is defined by:

$$P(x,y) = a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3$$

Definition 1.5

The quotient of two polynomials of variables x,y is called a rational fraction of two variables x,y. i.e. a rational fraction $\mathfrak{R}(x,y)$ of two variables is defined by:

$$\mathfrak{R}(x,y) = \frac{P(x,y)}{Q(x,y)}$$

Where $P(x,y)$ and $Q(x,y)$ are two polynomials of two variables x,y.

Example 1.12

1. $\mathfrak{R}(x,y) = \dfrac{x + xy^2 + y + 5}{x^3y + y^2 + 1}$

2. If we have: $\mathfrak{R}(x,y) = \dfrac{x + xy + 1}{x + 4xy + 1}$, then $\mathfrak{R}(x,\sqrt{x^2+1}) = \dfrac{x + x\sqrt{x^2+1} + 1}{x + 4x\sqrt{x^2+1} + 1}$

3. If we have: $\mathfrak{R}(x,y) = \dfrac{x^2 + xy + 1}{x + 4xy + 1}$, then:

$$\mathfrak{R}(\cos(x), \sin(x)) = \frac{\cos^2(x) + \cos(x)\sin(x) + 1}{\cos(x) + 4\cos(x)\sin(x) + 1}$$

Remark 1.7 *The fraction $\mathfrak{R}(x,\sqrt{x^2+1})$ is an irrational function of the variable x.*

1.6.2 The process of calculating this type: $\displaystyle\int \mathfrak{R}\left(x, \sqrt[n]{\frac{a_1x + b_1}{a_2x + b_2}}\right) \mathrm{d}x /\ a_1b_2 - a_2b_1 \neq 0$

In this case, we follow the steps below:

1. let's set $t = \sqrt[n]{\dfrac{a_1x + b_1}{a_2x + b_2}} \implies t^n = \dfrac{a_1x + b_1}{a_2x + b_2}$

   After computation, we obtain : $x = \dfrac{b_2t^n - b_1}{a_1 - a_2t^n}$

2. Therfore

$$dx = \left(\frac{b_2t^n - b_1}{a_1 - a_2t^n}\right)' dt = \frac{nb_2(a_1 - a_2t^n)t^{n-1} + na_2(b_2t^n - b_1)t^{n-1}}{(a_1 - a_2t^n)^2} dt$$

$$\Longrightarrow dx = \frac{n(a_1b_2 - a_2b_1)t^{n-1}}{(a_1 - a_2t^n)^2} dt$$

3. Replacing x by $\frac{b_2t^n - b_1}{a_1 - a_2t^n}$ and dx by $\frac{n(a_1b_2 - a_2b_1)t^{n-1}}{(a_1 - a_2t^n)^2}dt$ in the integral, we obtain:

$$\int \mathfrak{R}\left(x, \sqrt[n]{\frac{a_1x + b_1}{a_2x + b_2}}\right) dx = \int R(t)\, dt$$

With $R(t)$ is a rational fraction with variable t.

4. Finally, we calculate the integral $\int R(t)\, dt$ then we replace in the result t by $\sqrt[n]{\frac{a_1x + b_1}{a_2x + b_2}}$.

Example 1.13

Compute $I = \int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x$.

Solution:

We note that this integral is of type: $\int \mathfrak{R}\left(x, \sqrt[n]{\frac{a_1x+b_1}{a_2x+b_2}}\right)\mathrm{d}x$ with:

$n=2$, $a_1=1$, $b_1=4$, $a_2=0$, $b_2=1$ et $a_1b_2-a_2b_1 \neq 0$

So we'll take the following steps:

1. We put $t=\sqrt{x+4} \implies t^2=x+4 \implies x=t^2-4$
2. From the previous step dx=2$t$dt
3. By replacing in the integral, we obtain:
$$I=\int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x = 2\int \frac{t^2}{t^2-4}\,\mathrm{d}t \tag{1.2}$$
4. By using the rational fraction decomposition method we calculate the integral $\int \frac{t^2}{t^2-4}\,\mathrm{d}t$.
$$\frac{t^2}{t^2-4}=\frac{t^2-4+4}{t^2-4}=1+\frac{4}{t^2-4}=1+\frac{1}{t-2}-\frac{1}{t+2}$$
$$\implies \int \frac{t^2}{t^2-4}\,\mathrm{d}t=\int \mathrm{d}t+\int \frac{1}{t-2}\,\mathrm{d}t-\int \frac{1}{t+2}\,\mathrm{d}t$$
which implies that $I=2\int \frac{t^2}{t^2-4}\,\mathrm{d}t = 2t+2\ln\left|\frac{t-2}{t+2}\right|+c/c\in\mathbb{R}$.
5. Finally, we replace t in the result by $\sqrt{x+4}$ and we obtain:
$$I=2\sqrt{x+4}+2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right|+c/c\in\mathbb{R}$$

Remark 1.8 *We can apply the previous method to calculate integrals of type:*

$$\int \mathfrak{R}(x, \sqrt[n_1]{\left(\frac{a_1x+b_1}{a_2x+b_2}\right)^{m_1}}, \sqrt[n_2]{\left(\frac{a_1x+b_1}{a_2x+b_2}\right)^{m_2}}, \ldots, \sqrt[n_k]{\left(\frac{a_1x+b_1}{a_2x+b_2}\right)^{m_k}})\,\mathrm{d}x \;/a_1b_2-a_2b_1\neq 0$$

by posing: $t^\alpha=\frac{a_1x+b_1}{a_2x+b_2}$ *with:* $\alpha = LCM(n_1,n_2,\ldots,n_k)$
($LCM(n_1,n_2,\ldots,n_k)$ is the smallest common multiple of $n_1,n_2,\ldots,n_k$)

Example 1.14

Compute $I = \int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}\,\mathrm{d}x$

Solution:

We note that the type of this integral is:

$$\int \mathfrak{R}\left(x, \sqrt[n_1]{\left(\frac{a_1x+b_1}{a_2x+b_2}\right)^{m_1}}, \sqrt[n_2]{\left(\frac{a_1x+b_1}{a_2x+b_2}\right)^{m_2}}\right)\mathrm{d}x$$

With: $n_1 = 2, n_2 = 4, m_1 = 1, m_2 = 3$ et $a_1 = 1, b_1 = 0, a_2 = 0, b_2 = 0$ such that: $a_1b_2 - a_2b_1 \neq 0$

We have: $\alpha = \text{LCM}(2{,}4) = 4$, so we effect a change of variable as follows:

$$t^4 = x \implies \mathrm{d}x = 4t^3\mathrm{d}t$$

By replacing in the integral I we obtain:

$$I = 4\int \frac{t^5}{t^3+1}\,\mathrm{d}t$$

Calculating the integral $\int \frac{t^5}{t^3+1}\,\mathrm{d}t$ by the decomposition method

We have:

$$\frac{t^5}{t^3+1} = t^2 - \frac{t^2}{t^3+1}$$

$$\implies \int \frac{t^5}{t^3+1}\,\mathrm{d}t = \int t^2\,\mathrm{d}t - \frac{1}{3}\int \frac{3t^2}{t^3+1}\,\mathrm{d}t$$

$$\implies I = \frac{4}{3}t^3 - \frac{4}{3}\ln|t^3+1| + c, \quad c \in \mathbb{R}$$

Finally, we replace t by $\sqrt[4]{x}$ to obtain the result:

$$I = \frac{4}{3}x^{\frac{3}{4}} - \frac{4}{3}\ln|x^{\frac{3}{4}}+1| + c, \quad c \in \mathbb{R}$$

1.6.3 The process of calculating this type: $\int \mathfrak{R}(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x / a \neq 0$

To calculate integrals of this kind, there's a general method called the Euler substitution method. The principle of this method is based on transforming the integral $\int \mathfrak{R}(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x$ into a rational fraction integral using the following three types of variable change:

1. **1st case: if** $a > 0$
   We make the following change of variable:

$$\sqrt{ax^2+bx+c} = t + \sqrt{a}x \quad \text{or} \quad \sqrt{ax^2+bx+c} = t - \sqrt{a}x$$

   Let's study the case $\sqrt{ax^2+bx+c} = t - \sqrt{a}x$

$$\implies ax^2 + bx + c = t^2 - 2\sqrt{a}tx + ax^2$$

$$\Longrightarrow \boxed{x = \frac{t^2 - c}{2t\sqrt{a} + b}}$$

$$\text{so,} \quad \mathrm{d}x = \left(\frac{t^2 - c}{2t\sqrt{a} + b}\right)' \mathrm{d}t = \frac{2t^2\sqrt{a} + 2bt + 2\sqrt{ac}}{(b + 2t\sqrt{a})^2}\mathrm{d}t$$

$$\Longrightarrow \boxed{\mathrm{d}x = 2\left(\frac{t^2\sqrt{a} + bt + \sqrt{ac}}{(b + 2t\sqrt{a})^2}\right)\mathrm{d}t}$$

$$\text{Finally; } \sqrt{ax^2 + bx + c} = t - \sqrt{a}x = t - \sqrt{a}\left(\frac{t^2 - c}{2t\sqrt{a} + b}\right)$$

$$\Longrightarrow \boxed{\sqrt{ax^2 + bx + c} = \frac{\sqrt{a}t^2 + bt + \sqrt{ac}}{2t\sqrt{a} + b}}$$

By replacing the expressions x,$\mathrm{d}x$ and $\sqrt{ax^2 + bx + c}$ in the integral, we obtain an integral of the type: $\int R(t)\,\mathrm{d}t$ where $R(t)$ is a rational fraction. For return to x in the result, we replace t by $\sqrt{ax^2 + bx + c} + \sqrt{a}x$.

Example 1.15

Compute $I = \int \frac{x^2}{\sqrt{x^2 + 9}}\,\mathrm{d}x$

Solution:

We put:

$$\sqrt{x^2 + 9} = t - x \Longrightarrow x^2 + 9 = t^2 - 2tx + x^2$$

$$\Longrightarrow \boxed{x = \frac{t^2 - 9}{2t}} \quad \text{and} \quad \boxed{\mathrm{d}x = \left(\frac{t^2 + 9}{2t^2}\right)\mathrm{d}t}$$

We have: $\sqrt{x^2 + 9} = t - x$ then

$$\sqrt{x^2 + 9} = t - \frac{t^2 - 9}{2t} \Longrightarrow \boxed{\sqrt{x^2 + 9} = \frac{t^2 + 9}{2t}}$$

By replacing in the integral, we obtain:

$$\begin{aligned} I &= \int \frac{x^2}{\sqrt{x^2 + 9}}\,\mathrm{d}x = \frac{1}{4}\int \frac{(t^2 - 9)^2}{t^3}\,\mathrm{d}t \\ &= \frac{1}{4}\int t\,\mathrm{d}t - \frac{18}{4}\int \frac{1}{t}\,\mathrm{d}t + \frac{81}{4}\int \frac{1}{t^3}\,\mathrm{d}t \\ &= \frac{1}{8}t^2 - \frac{18}{4}\ln|t| - \frac{81}{8t^2} + c/c \in \mathbb{R} \end{aligned}$$

We replace t by $\sqrt{x^2 + 9} + x$, we obtain:

$$I = \frac{1}{8}(\sqrt{x^2 + 9} + x)^2 - \frac{18}{4}\ln|\sqrt{x^2 + 9} + x| - \frac{81}{8(\sqrt{x^2 + 9} + x)^2} + c, \quad c \in \mathbb{R}$$

2. **2nd case: if** $c > 0$
   We apply the following change of variable:

Let $\sqrt{ax^2+bx+c} = xt + \sqrt{c}$ or $\sqrt{ax^2+bx+c} = xt - \sqrt{c}$

We'll study the case: $\sqrt{ax^2+bx+c} = xt + \sqrt{c}$
Following the same process as above, we obtain:

$$\boxed{x = \frac{2\sqrt{c}-b}{a-t^2}}, \quad \boxed{\mathrm{d}x = 2\frac{\sqrt{c}t^2 - bt + a\sqrt{c}}{(a-t^2)^2}\mathrm{dt}}, \quad \text{et} \quad \boxed{\sqrt{ax^2+bx+c} = \frac{\sqrt{c}t^2 - bt + \sqrt{c}}{a-t^2}}$$

By replacing the expressions for x,dx and $\sqrt{ax^2+bx+c}$ in the integral, we obtain an integral of the type: $\int R(t)\,\mathrm{dt}$ where $R(t)$ is a rational fraction.

Replace t by $\dfrac{\sqrt{ax^2+bx+c}-\sqrt{c}}{x}$ in order to return to x in the result. (with the hypothesis $x \neq 0$).

Example 1.16

Compute $I = \displaystyle\int \frac{1}{x\sqrt{x^2-x+1}}\,\mathrm{d}x$

Solution:

We have: I of type $\int \mathfrak{R}(x,\sqrt{ax^2+bx+c})\,\mathrm{d}x$ with $c > 0$.

Let's put:

$$\sqrt{x^2-x+1} = xt + 1$$

$$\Longrightarrow x = \frac{2t+1}{1-t^2}, \mathrm{d}x = \frac{2(t^2+t+1)}{(1-t^2)^2}\mathrm{dt}, \text{ and } \sqrt{x^2-x+1} = \frac{t^2+t+1}{1-t^2}$$

Replacing in I, we obtain:

$$I = \int \frac{1}{2t+1}\,\mathrm{dt} = \frac{1}{2}\ln|2t+1| + c/c \in \mathbb{R}$$

We replace t in the result by $\dfrac{\sqrt{x^2-x+1}-1}{x}/x \neq 0$, we obtain:

$$I = \frac{1}{2}\ln\left|\frac{2\sqrt{x^2-x+1}-2+x}{x}\right| + c/c \in \mathbb{R}$$

3. **3rd case if** $\Delta = b^2 - 4ac > 0$
In this case, the polynomial ax^2+bx+c has two roots $x_1,x_2 \in \mathbb{R}$ and $x_1 \neq x_2$, so we can write: $ax^2+bx+c = a(x-x_1)(x-x_2)$. Then we put:

$$\sqrt{ax^2+bx+c} = \pm t(x-x_1) \quad \text{or} \quad \sqrt{ax^2+bx+c} = \pm t(x-x_2)$$

We'll study the case: $\sqrt{ax^2+bx+c} = t(x-x_1)$

$$\Longrightarrow ax^2+bx+c = a(x-x_1)(x-x_2) = t^2(x-x_1)^2$$

$$\Longrightarrow a(x-x_2) = t^2(x-x_1)$$

$$\Longrightarrow x = \frac{ax_2 - x_1t^2}{a-t^2}, \quad \mathrm{d}x = \frac{2a(x_1-x_2)t}{(a-t^2)^2}\mathrm{dt}, \quad \text{and} \quad \sqrt{ax^2+bx+c} = \frac{a(x_2-x_1)t}{a-t^2}$$

By replacing the expressions for x,dx and $\sqrt{ax^2+bx+c}$ in the integral, we obtain an integral of type: $\int R(t)\,\mathrm{d}t$, where $R(t)$ is a rational fraction. To return to x in the result, replace t by $\dfrac{\sqrt{ax^2+bx+c}}{x-x_1}$.

Example 1.17

Compute $I=\int\sqrt{2x-x^2}\,\mathrm{d}x$

Solution: We have
$2x-x^2=x(2-x)$ i.e. the polynomial $2x-x^2$ has two roots $x_1=0$ and $x_2=2$
let's set:

$$\sqrt{2x-x^2}=xt$$

$$\Longrightarrow\ 2x-x^2=x^2t^2\ \Longrightarrow\ x=\frac{2}{t^2+1},\ \mathrm{d}x=-\frac{4t}{(t^2+1)^2}\mathrm{d}t;\ \text{and}\ \sqrt{2x-x^2}=\frac{2t}{t^2+1}$$

Replacing in I gives us:

$$I=-8\int\frac{t^2}{(t^2+1)^3}\,\mathrm{d}t$$

On the other hand, we have :n a :

$$\frac{t^2}{(t^2+1)^3}=\frac{t^2+1-1}{(t^2+1)^3}=\frac{1}{(t^2+1)^2}-\frac{1}{(t^2+1)^3}$$
$$\Longrightarrow\int\frac{t^2}{(t^2+1)^3}\mathrm{d}t=\int\frac{1}{(t^2+1)^2}\mathrm{d}t-\int\frac{1}{(t^2+1)^3}\mathrm{d}t$$
$$\Longrightarrow\int\frac{t^2}{(t^2+1)^3}\mathrm{d}t=J_2-J_3$$

According to the previous recurrence formula, we have:

$$\begin{cases}J_1=\arctan(t)+c, & c\in\mathbb{R}\\ J_{n+1}=\frac{2n-1}{2n}J_n+\frac{t}{2n(t^2+1)^n}\end{cases}\quad\Longrightarrow\quad\begin{cases}J_2=\frac{1}{2}J_1+\frac{t}{2(t^2+1)}\\ J_3=\frac{3}{4}J_2+\frac{t}{4(t^2+1)^2}\end{cases}$$

$$\Longrightarrow\ J_2-J_3=\frac{1}{4}J_2-\frac{t}{4(t^2+1)^2}$$

$$\Longrightarrow\ J_2-J_3=\frac{1}{8}\arctan(t)+\frac{t}{8(t^2+1)}-\frac{t}{4(t^2+1)^2}+c,\quad c\in\mathbb{R}$$

Finally, we find:

$$I=-8(J_2-J_3)=-\arctan(t)-\frac{t}{t^2+1}+\frac{2t}{(t^2+1)^2}+k,\quad k\in\mathbb{R}$$

1.7 Integration of trigonometric functions

In the calculation of trigonometric function integrals, we distinguish the following cases:

1.7.1 Integrals of type $\int R(\cos(x))\sin(x)\,dx$ or $\int R(\sin(x))\cos(x)\,dx$ with $R(x)$ is a rational fraction

- If we have: $I = \int R(\cos(x))\sin(x)\,dx$, we make a change of variable of the form:

$$t = \cos(x) \quad \text{and} \quad dt = -\sin(x)dx$$

- If we have: $I = \int R(\sin(x))\cos(x)\,dx$, we make a change of variable of the form:

$$t = \sin(x) \quad \text{and} \quad dt = \cos(x)dx$$

Example 1.18

calculate $I = \int \frac{\cos^3(x)}{\sin^2(x)}\,dx$

Solution:

We have: $\frac{\cos^3(x)}{\sin^2(x)} = \frac{1-\sin^2(x)}{\sin^2(x)}\cos(x) \implies I$ of type $\int R(\sin(x))\cos(x)\,dx$

So we perform the following change of variable:

$$t = \sin(x) \implies dt = \cos(x)dx$$

$$\implies I = \int \frac{1-t^2}{t^2}\,dt = \int \frac{1}{t^2}dt - \int dt$$

$$\implies I = -\frac{1}{t} - t + c, \quad c \in \mathbb{R}$$

Finally, if we replace t by $\sin(x)$ we obtain:

$$I = -\frac{1}{\sin(x)} - \sin(x) + c, \quad c \in \mathbb{R}$$

1.7.2 Integrals of type $\int \mathfrak{R}(\cos(x), \sin(x))\,dx$

The following change of variable can be used for integrals of this kind:

We put

$$t = \tan\left(\frac{x}{2}\right), \ \cos(x) = \frac{1-t^2}{1+t^2}, \ \sin(x) = \frac{1-t^2}{1+t^2}, \text{ and } \quad dx = \frac{2dt}{1+t^2}$$

By replacing the expressions dx, $\cos(x)$ and $\sin(x)$ in the integral, we obtain an integral of type: $\int R(t)\,dt$ where $R(t)$ is a rational fraction. To return to the variable x in the result, replace t by $\tan\left(\frac{x}{2}\right)$

Proof

1. We put $t = \tan\left(\frac{x}{2}\right)$, then $\mathrm{d}t = \frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right)\mathrm{d}x = \frac{1}{2}\left(1 + t^2\right)\mathrm{d}x$

$$\implies \mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t$$

2. We have: $\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \implies \sin(x) = 2\tan\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)$
   On the other hand $\tan^2\left(\frac{x}{2}\right) + 1 = \dfrac{1}{\cos^2\left(\frac{x}{2}\right)} \implies \cos^2\left(\frac{x}{2}\right) = \dfrac{1}{t^2+1}$
   $\implies \sin(x) = \dfrac{2t}{t^2+1}$
3. We have: $\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \cos^2\left(\frac{x}{2}\right)\left(1 - \tan^2\left(\frac{x}{2}\right)\right)$
   $\implies \cos(x) = \dfrac{1-t^2}{1+t^2}$

Example 1.19

Compute $I = \displaystyle\int \frac{1}{1-\cos(x)}\mathrm{d}x$
We put: $t = \tan\left(\frac{x}{2}\right)$, $\cos(x) = \dfrac{1-t^2}{1+t}$ and $\mathrm{d}x = \dfrac{2}{1+t^2}\mathrm{d}t$
Replacing in I gives us:

$$I = \int \frac{1}{t^2}\mathrm{d}t = -\frac{1}{t} + c, \quad c \in \mathbb{R}$$

. Finally, we replace t by $\tan\left(\frac{x}{2}\right)$, we find:

$$I = -\frac{1}{\tan\left(\frac{x}{2}\right)} + c, \quad c \in \mathbb{R}$$