Chapter 2

Definite integrals

Contents

2.1 Introduction

This chapter contains the method for constructing the definite (or Riemann) integral of a function f defined and bounded on an interval of type $[a,b]$ and its fundamental properties. Geometrically, the notion of the definite integral of a continuous and positive function f on $[a,b]$ is interpreted as a measure of the portion of the plane lying between (Γ_f) the graph of the function f, the x axis and the straight lines $x = a$, $x = b$.

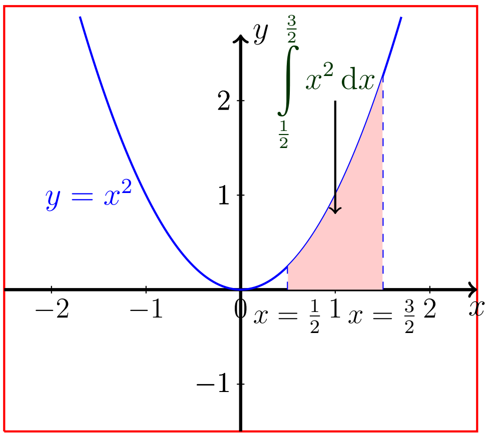


Figure 2.1 – *Geometrical interpretation of the definite integral of* $f(x) = x^2$ *on* $\left[\frac{1}{2},\frac{3}{2}\right]$

2.2 Partitions and Darboux sums

2.2.1 Subdivision of an interval

Definition 2.1

A partition of the interval $[a,b]$ is any finite sequence $P = \{x_0,x_1,...,x_n\}$ of real numbers satisfying the following conditions:

1. $\forall i \in \{1,2,...,n\}$; $x_i \in [a,b]$
2. $x_0 = a$ and $x_n = b$
3. $a = x_0 < x_1 < .... < x_{n-1} < x_n = b$

Remark:

1. Let $P = \{x_0,x_1,....,x_n\}$ be a partition of the interval $[a,b]$ then, P contains $n+1$ points (called P partition nodes) and determines n interval $[x_{i-1},x_i]/i \in \{1....,n\}$
2. The real $h = \max\limits_{1\leq i\leq n}(x_i - x_{i-1})$ is called the norm of partition P.
3. Let $P = \{x_0,x_1,....,x_n\}$ and $P' = \{x'_0,x'_1,....,x'_m\}$ two partitions of the interval $[a,b]$. We say that P' is a refinement of P if:

$$\{x_0,x_1,....,x_n\} \subset \{x'_0,x'_1,....,x'_m\}$$

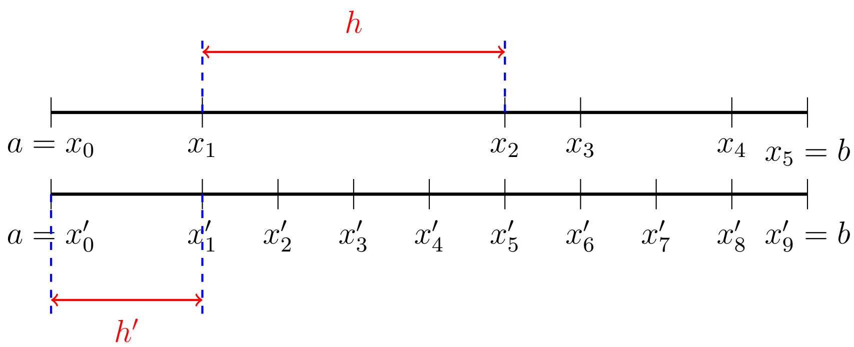


Figure 2.2 – $P = \{x_0,x_1,...,x_5\}$ *and* $P' = \{x'_0,x'_1,....,x'_9\}$ *are two partitions of* $[a,b]$

Remark 2.1 *In the figure above we have:*

- P *is a partition of* $[a,b]$ *with a largest step* h
- P' *is also a partition of* $[a,b]$ *with a largest step* h'

We note that: $P \subset P'$ *and* $h' < h$, *so in this case* P' *is a refinement of* P

Example 2.1

Let $P = \{x_0,x_1,...,x_n\}$ a partition of $[a,b]$ defined by:

$$x_k = a + k\left(\frac{b-a}{n}\right) / k = 0,...,n$$

In this case, P is called: the uniform partition of $[a,b]$, with $h = \frac{b-a}{n}$

2.2.2 Darboux sums

Let $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$ (i.e. $\sup\limits_{x\in[a,b]} f(x)$ and $\inf\limits_{x\in[a,b]} f(x)$ exist in $\mathbb{R}$) and $P = \{x_0,x_1,....,x_n\}$ a partition of $[a,b]$.
Define:

$$\begin{cases} m_k = \inf\limits_{x\in[x_{k-1},x_k]} f(x) \\ M_k = \sup\limits_{x\in[x_{k-1},x_k]} f(x) \end{cases} \quad \text{with} \quad k = 1,...,n$$

Definition 2.2

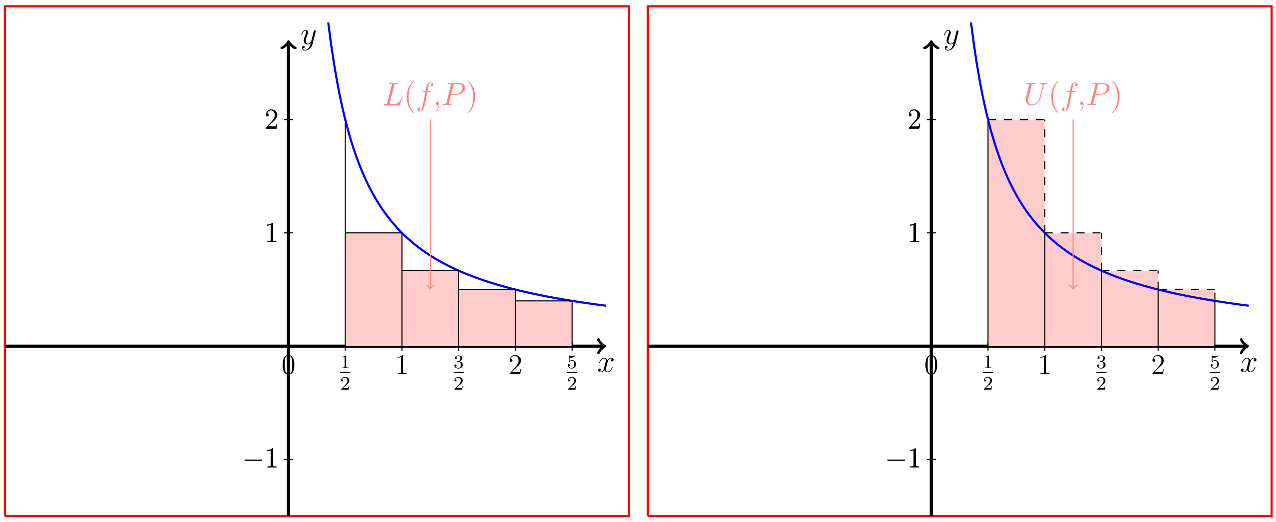
We call

1. Lower Darboux sum associated to f and P the number

$$L(f,P) = \sum_{k=1}^{k=n} m_k(x_k - x_{k-1})$$

2. Upper Darboux sum associated to f and P the number

$$U(f,P) = \sum_{k=1}^{k=n} M_k(x_k - x_{k-1})$$



(a) $L(f,P)$ *for* $f(x) = \frac{1}{x}$ *and* $P = \{\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2}\}$ (b) $U(f,P)$ *for* $f(x) = \frac{1}{x}$ *and* $P = \{\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2}\}$

Figure 2.3 – *Darboux sums*

2.2.3 Properties of Darboux sums

Proposition 2.1

Let $f : [a,b] \to \mathbb{R}$ be a bounded function, Darboux sums satisfy the following properties:

1. for every partition P of $[a,b]$: $L(f,P) \leq U(f,P)$.
2. let P and P' be two partitions of $[a,b]$ with $P \subset P'$, then:

$$\begin{cases} U(f,P) \geq U(f,P') \\ L(f,P) \leq L(f,P') \end{cases}$$

3. If P and P' are any two partitions of $[a,b]$, then

$$L(f,P) \leq U(f,P')$$

4. Let P be a partition of $[a,b]$, $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$, then:

$$m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a)$$

Proof

1. Let's prove (3).
   We have P and P' are two partitions of $[a,b]$ then, $P \cup P'$ is a partition of $[a,b]$. So it is a refinement of P and P'. From (1) and (2) we get:

$$L(f,P) \leq L(f,P\cup P') \leq U(f,P\cup P') \leq U(f,P')$$

$$\implies L(f,P) \leq U(f,P')$$

2. Proof of (4).

   (a) Let $P = \{x_0,x_1,\ldots,x_n\}$ be a partition of $[a,b]$, we have:

$$\forall k \in \{1,\ldots,n\};\ [x_k,x_{k-1}] \subset [a,b] \implies m = \inf_{x\in[a,b]} f(x) \leq \inf_{x\in[x_k,x_{k-1}]} f(x) = m_k$$

$$\implies \forall k \in \{1,\ldots,n\};\ m(x_k - x_{k-1}) \leq m_k(x_k - x_{k-1})$$

   Therefore,

$$\sum_{k=1}^{k=n} m(x_k - x_{k-1}) \leq \sum_{k=1}^{k=n} m_k(x_k - x_{k-1})$$

$$\implies m\sum_{k=1}^{k=n}(x_k - x_{k-1}) \leq L(f,P) \implies m(b-a) \leq L(f,P)$$

   (Since $\displaystyle\sum_{k=1}^{k=n}(x_k - x_{k-1}) = b - a$)

   (b) From (1) we have: $L(f,P) \leq U(f,P)$

   (c) We have: $U(f,P) \leq M(b-a)$ the proof is similar to (a).

Notations: Let $f : [a,b] \to \mathbb{R}$ be a bounded function

1. The set of all partitions of $[a,b]$ is denoted by $S_{[a,b]}$.

2. We denote by $U_{[a,b]}(f)$ the set consisting of all upper Darboux sums associated to f obtained with all possible partitions of $[a,b]$ i.e.:

$$U_{[a,b]}(f) = \{U(f,P)/P \in S_{[a,b]}\}$$

3. We denote by $L_{[a,b]}(f)$ the set consisting of all lower Darboux sums associated to f obtained with all possible partitions of $[a,b]$ i.e.:

$$L_{[a,b]}(f) = \{L(f,P)/P \in S_{[a,b]}\}$$

Proposition 2.2

If f is a bounded function on $[a,b]$ then:

$$\sup\left(L_{[a,b]}(f)\right) \leq \inf\left(U_{[a,b]}(f)\right)$$

Proof

1. The sets $L_{[a,b]}(f)$ and $U_{[a,b]}(f)$ are non-empty.
2. We have:
$$\forall P,P' \in S_{[a,b]};\ L(f,P) \leq U(f,P')$$
$\implies$ The elements of $U_{[a,b]}(f)$ are upper bounds of $L_{[a,b]}(f)$
$$\implies \forall U(f,P') \in U_{[a,b]}(f);\ \sup\left(L_{[a,b]}(f)\right) \leq U(f,P')$$
So $\sup\left(L_{[a,b]}(f)\right)$ is a lower bound of $U_{[a,b]}(f)$
$$\implies \sup\left(L_{[a,b]}(f)\right) \leq \inf\left(U_{[a,b]}(f)\right)$$

2.3 Integrable functions

2.3.1 Lower and upper integral of f on $[a,b]$.

Definition 2.3

Let $f : [a,b] \to \mathbb{R}$ be a bounded function

- We denote by $U_a^b(f)$ the upper integral of f on $[a,b]$, defined by:
$$U_a^b(f) := \inf\left(U_{[a,b]}(f)\right)$$
- We denote by $L_a^b(f)$ the lower integral of f on $[a,b]$, defined by:
$$L_a^b(f) := \sup\left(L_{[a,b]}(f)\right)$$

2.3.2 Riemann integral

Definition 2.4

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. We say that f is Riemann integrable on $[a,b]$ if:
$$U_a^b(f) = L_a^b(f)$$
in this case the common value of $U_a^b(f)$ and $L_a^b(f)$ is called the definite integral (Riemann integral) of f on $[a,b]$ and is denoted by
$$\int_a^b f(x)\,\mathrm{d}x$$
We denote by $\mathcal{R}([a,b])$ the set of integrable functions on $[a,b]$.

Remarks:

1. a and b are called the bounds of the integral.

2. The number $\int_a^b f(x)\,\mathrm{d}x$ does not depend on x , it depends on a and b i.e. we can replace x by any other letter y,t,u....

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(y)\,\mathrm{d}y = \int_a^b f(t)\,\mathrm{d}t$$

Example 2.2

Let f be a function defined by:

$$\forall x \in \mathbb{R};\ f(x) = c \quad \text{with } c \in \mathbb{R}$$

1. Show that f is Riemann integrable on $[a,b]$ with $a,b \in \mathbb{R}$.
2. Determine $\int_a^b f(x)\,\mathrm{d}x$

Solution:

1. Let $P = \{x_0,x_1,...,x_n\} \in S_{[a,b]}$ (a partition of $[a,b]$), we have:

$$\forall k \in \{1,2,...,n\};\ \begin{cases} m_k = \inf\limits_{x\in[x_{k-1},x_k]} f(x) = c \\ \quad\text{and} \\ M_k = \sup\limits_{x\in[x_{k-1},x_k]} f(x) = c \end{cases}$$

$$\Longrightarrow \begin{cases} U(f,P) = \sum\limits_{k=1}^{k=n} M_k(x_k - x_{k-1}) = c\sum\limits_{k=1}^{k=n}(x_k - x_{k-1}) = c(b-a) \\ \quad\text{and} \\ L(f,P) = \sum\limits_{k=1}^{k=n} m_k(x_k - x_{k-1}) = c\sum\limits_{k=1}^{k=n}(x_k - x_{k-1}) = c(b-a) \end{cases}$$

$$\Longrightarrow \begin{cases} \forall U(f,P) \in U_{[a,b]}(f);\ U(f,b) = c(b-a) \\ \quad\text{and} \\ \forall L(f,P) \in L_{[a,b]}(f);\ L(f,b) = c(b-a) \end{cases}$$

$$\Longrightarrow \begin{cases} U_a^b(f) = \inf\left(U_{[a,b]}(f)\right) = c(b-a) \\ \quad\text{and} \\ L_a^b(f) = \sup\left(L_{[a,b]}(f)\right) = c(b-a) \end{cases}$$

$$\Longrightarrow U_a^b(f) = L_a^b(f)$$

so f is Riemann integrable on $[a,b]$.

2. We have f is integrable on $[a,b]$ so

$$U_a^b(f) = L_a^b(f) = \int_a^b f(x)\,\mathrm{d}x = \int_a^b c\,\mathrm{d}x = c(b-a)$$

Example 2.3

Let f be a function defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

Show that f is not Riemann integrable on any interval $[a,b]$.

Solution:

Let $P = x_0,x_1,...,x_n \in S_{[a,b]}$ (a partition of $[a,b]$), we have:

$$\forall k \in \{1,2,...,n\}; \begin{cases} m_k = \inf\limits_{x\in[x_{k-1},x_k]} f(x) = 0 \\ \quad \text{and} \\ M_k = \sup\limits_{x\in[x_{k-1},x_k]} f(x) = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} U(f,P) = \sum\limits_{k=1}^{k=n} M_k(x_k - x_{k-1}) = \sum\limits_{k=1}^{k=n}(x_k - x_{k-1}) = (b-a) \\ \quad \text{and} \\ L(f,P) = \sum\limits_{k=1}^{k=n} m_k(x_k - x_{k-1}) = 0\sum\limits_{k=1}^{k=n}(x_k - x_{k-1}) = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \forall U(f,P) \in U_{[a,b]}(f);\ U(f,b) = (b-a) \\ \quad \text{and} \\ \forall L(f,P) \in L_{[a,b]}(f);\ L(f,b) = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} U_a^b(f) = \inf\left(U_{[a,b]}(f)\right) = (b-a) \\ \quad \text{and} \\ L_a^b(f) = \sup\left(L_{[a,b]}(f)\right) = 0 \end{cases}$$

$$\Longrightarrow U_a^b(f) \neq L_a^b(f)$$

so f is not Riemann integrable on $[a,b]$.

Theorem 2.1

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. For f to be integrable on $[a,b]$ it is necessary, and sufficient, that

$$\forall\varepsilon > 0,\exists P \in S_{[a,b]};\ U(f,d) - L(f,d) < \varepsilon$$

Proof

1. We assume that f is integrable on $[a,b]$ so

$$U_a^b(f) = L_a^b(f) = \int_a^b f(x)\,\mathrm{d}x$$

$$\text{such that:} \begin{cases} U_a^b(f) = \inf\{U(f,P)/P \in S_{[a,b]}\} \\ L_a^b(f) = \sup\{L(f,P)/P \in S_{[a,b]}\} \end{cases}$$

So

$$\forall \varepsilon > 0; \begin{cases} \exists P_1 \in S_{[a,b]};\ U(f,P_1) < U_a^b(f) + \frac{\varepsilon}{2} \\ \exists P_2 \in S_{[a,b]};\ L_a^b(f) - \frac{\varepsilon}{2} < L(f,P_2) \end{cases}$$

Let's put $P = P_1 \cup P_2$, then $P_1 \subset d$ and $P_2 \subset d$, from the properties of Darboux sums we have:

$$\begin{cases} U(f,P) \leq U(f,P_1) \\ \qquad \wedge \\ L(f,P_2) \leq L(f,P) \end{cases}$$

$$\Longrightarrow \forall \varepsilon > 0, \exists d \in S_{[a,b]}; \begin{cases} U(f,P) < U_a^b(f) + \frac{\varepsilon}{2} \\ L_a^b(f) - \frac{\varepsilon}{2} < L(f,P) \end{cases}$$

By summing, we obtain:

$$\forall \varepsilon > 0, \exists P \in S_{[a,b]};\ U(f,P) - L(f,P) < \varepsilon$$

Therefore, $(f \in \mathcal{R}([a,b]) \Longrightarrow \forall \varepsilon > 0, \exists P \in S_{[a,b]};\ U(f,P) - L(f,P) < \varepsilon)$ is true.

2. From proposition (2.2) we have: $L_a^b(f) \leq U_a^b(f)$.
We Assume that $\forall \varepsilon > 0, \exists P \in S_{[a,b]};\ U(f,P) - L(f,P) < \varepsilon$

$$\Longrightarrow \forall \varepsilon > 0, \exists P \in S_{[a,b]};\ U(f,P) - \varepsilon < L(f,P) \leq U(f,P)$$

So $U(f,P) = \sup\left(L_{[a,b]}(f)\right) = L_a^b(f)$.
On the other hand, we have:

$$\forall \varepsilon > 0, \exists P \in S_{[a,b]};\ L(f,p) \leq U(f,P) < L(f,P) + \varepsilon$$

So $L(f,P) = \inf\left(U_{[a,b]}(f)\right) = U_a^b(f) \Longrightarrow U_a^b \leq L_a^b$.
Finally, we obtain $(U_a^b \leq L_a^b) \wedge (L_a^b \leq U_a^b) \Longrightarrow U_a^b = L_a^b$, So $f \in \mathcal{R}([a,b])$. So the inverse implication is true.

2.3.3 Riemann sums

Definition 2.5

Let $f : [a,b] \to \mathbb{R}$ be a bounded function, $P = \{x_0,x_1,...,x_n\}$ a partition of $[a,b]$ and $\alpha_1,\alpha_2,...,\alpha_n$ are real numbers such that: $\forall i = \overline{1,n};\ \alpha_i \in [x_{i-1},x_i]$
Define:

$$S(f,P) = \sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1})$$

The number $S(f,P)$ is called the Riemann sum corresponds to the partition P and the point system $\{\alpha_1,\alpha_2,...,\alpha_n\}$

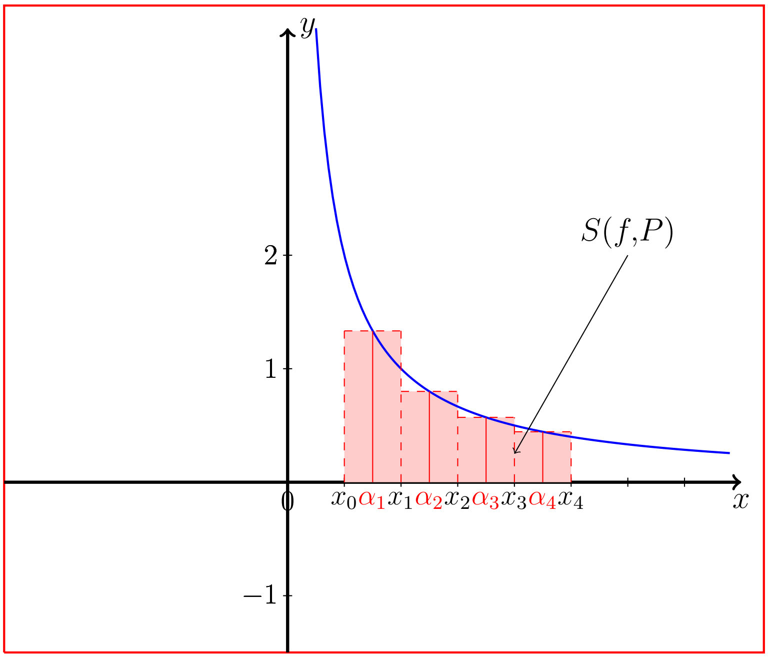


Figure 2.4 – *Riemann sum of* $f(x) = \frac{1}{x}$ *corresponds to the partition* $P = \{x_0,x_1,x_3,x_4\}$ *and the point system* $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$

Theorem 2.2

Let f be an integrable function on $[a,b]$ then:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{h\to 0} S(f,P)$$

Remark 2.2 *$P = \{x_0,x_1,...,x_n\}$ is a partition of $[a,b]$ with a norm h.*

Example 2.4

Assuming $P = \{x_0,x_1,...,x_n\}$ the uniform partition of $[a,b]$ defined by:

$$x_k = a + k\left(\frac{b-a}{n}\right) / k \in \{0,1,...,n\} \text{ with } h = \frac{b-a}{n}$$

then:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to+\infty}\left(\frac{b-a}{n}\right)\sum_{k=1}^{n} f\left(a + k\left(\frac{b-a}{n}\right)\right)$$

2.3.4 Examples of integrable functions

Theorem 2.3

Any monotonic function f on $[a,b]$ is integrable on $[a,b]$.

Proof

In our proof, we assume that f is increasing (the same technique is used if f is decreasing).
Let $P = x_0,x_1,...,x_n$ be a partition of $[a,b]$ with its norm h.
We have f is then increasing:

$$\forall k \in \{1,2...,n\};\ \begin{cases} m_k = \inf\limits_{x\in[x_{k-1},x_k]} f(x) = f(x_{k-1}) \\ M_k = \sup\limits_{x\in[x_{k-1},x_k]} f(x) = f(x_k) \end{cases}$$

$$\implies U(f,d) - L(f,d) = \sum_{k=1}^{n}(f(x_k - f(x_{k-1})(x_k - x_{k-1}) \leq h\sum_{k=1}^{n}(f(x_k - f(x_{k-1})$$
$$\leq h(f(b) - f(a))$$

Let $\varepsilon > 0$, if we choose a partition with a norm $h < \dfrac{\varepsilon}{f(b)-f(a)}$, we get:

$$\forall\varepsilon > 0, \exists P \in S_{[a,b]} \implies U(f,P) - L(f,P) < \varepsilon$$

Therefore f is integrable on $[a,b]$.

Theorem 2.4

Any continuous function on $[a,b]$ is integrable on $[a,b]$.

Example 2.5

1. Show that $f(x) = e^x$ is integrable on $[a,b]$.
2. Using the Riemann sum, show that:

$$\int_0^1 e^x \,\mathrm{d}x = e - 1$$

Solution:

1. According to the previous theorem $f(x) = e^x$ is continuous on $[0,1]$, so f is integrable on $[0,1]$

2. If we choose the following uniform partition of $[0,1]$:

$$P = \{x_0, x_1, \ldots, x_n\} \text{ such that: } x_k = \frac{k}{n} / k = \overline{1,n} \text{ and } h = \frac{1}{n}$$

According to the Riemann sum corresponds to P we have:

$$\int_0^1 f(x)\,\mathrm{d}x = \lim_{n\to+\infty} \left(\frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}) \right)$$

$$\Longrightarrow \int_0^1 e^x \,\mathrm{d}x = \lim_{n\to+\infty} \left(\frac{1}{n} \sum_{k=1}^{n} e^{\frac{k}{n}} \right) = \lim_{n\to+\infty} \left(\frac{1}{n} \left(e^{\frac{1}{n}} + e^{\frac{2}{n}} + \ldots. + e^{\frac{n}{n}} \right) \right)$$

$$= \lim_{n\to+\infty} \frac{1}{n} \left(e^{\frac{1}{n}} \frac{e^{\frac{n}{n}} - 1}{e^{\frac{1}{n}} - 1} \right)$$

$$= \lim_{n\to+\infty} \left(e^{\frac{1}{n}} \frac{e^{\frac{n}{n}} - 1}{\frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}}} \right) = e - 1$$

2.4 Properties of definite integrals

Proposition 2.3

Let f and g be two functions integrable on the interval $[a,b]$, then we have:

1. $\int_a^a f(x)\,\mathrm{d}x = 0$

2. $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$

3. $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$

4. If we have: $\forall x \in [a,b];\ f(x) \geq 0$ then:
$$\int_a^b f(x)\,\mathrm{d}x \geq 0$$

5. If we have: $\forall x \in [a,b];\ f(x) \leq g(x)$ then:
$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$$

6. For all real number $c \in]a,b[$ the function f is intégrable on $[a,c]$ and $[c,b]$, and in addition:
$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \quad \text{(Chasles relation)}$$

7. For all real numbers $\alpha,\beta \in \mathbb{R}$ the function $\alpha f + \beta g$ is intégrable on $[a,b]$ and
$$\int_a^b (\alpha f(x) + \beta g(x))\,\mathrm{d}x = \alpha\int_a^b f(x)\,\mathrm{d}x + \beta\int_a^b g(x)\,\mathrm{d}x$$

8. $\left(\int_a^b f(x)g(x)\,\mathrm{d}x\right)^2 \leq \left(\int_a^b f^2(x)\,\mathrm{d}x\right)\left(\int_a^b g^2(x)\,\mathrm{d}x\right)$ (Cauchy-Schwarz inequality)

2.4.1 Theorem of the Mean for Integrals

Theorem 2.5: Mean Theorem

Let f and g be two integrable functions on $[a,b]$, with g having a constant sign in $[a,b]$ (i.e. $g \geq 0$ or $g \leq 0$ on $[a,b]$), then there exists a number $\mu \in [m,M]$ such that:

$$\int_a^b f(x)g(x)\,\mathrm{d}x = \mu \int_a^b g(x)\,\mathrm{d}x$$

With $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$.
Moreover, if f is continuous, there exists $\xi \in [a,b]$ such that: $\mu = f(\xi)$.

Remark 2.3 *If $g = 1$, then :*

$$\exists\mu \in [m,M];\ \int_a^b f(x)\,\mathrm{d}x = \mu(b-a)$$

with: $m = \inf_{x\in[a,b]} f(x)$ *and* $M = \sup_{x\in[a,b]} f(x)$.

Example 2.6

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(x)\,\mathrm{d}x$. Show that there exists a number $\mu \in [0,1]$ such that:

$$I = \mu \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,\mathrm{d}x$$

Solution:
let's put $f(x) = g(x) = \cos(x)$, we have:

$$\begin{cases} \forall x \in [-\frac{\pi}{2},\frac{\pi}{2}];\ \cos(x) \geq 0 \\ \qquad \text{and} \\ m = \inf_{x\in[-\frac{\pi}{2},\frac{\pi}{2}]} f(x) = 0 \text{ and } M = \sup_{x\in[-\frac{\pi}{2},\frac{\pi}{2}]} f(x) = 1 \end{cases}$$

According to the mean theorem

$$\exists\mu \in [m,M];\ I = \mu \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,\mathrm{d}x \implies \exists\mu \in [0,1];\ I = \mu \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,\mathrm{d}x$$

2.5 Antiderivative and definite integral as a function of its upper bound

Definition 2.6

Let f be an integrable function on $[a,b]$, we define:

$$\Phi(x) = \int_a^x f(t)dt$$

We call $\Phi(x)$ the integral of f defined as a function of its upper bound.

Proposition 2.4

Let f be an integrable function on $[a,b]$ and $\Phi(x) = \int_a^x f(t)dt$, then:

1. Φ is continuous on $[a,b]$
2. If f is continuous on $[a,b]$, then Φ is differentiable on $[a,b]$ and:

$$\forall x \in [a,b];\ \Phi'(x) = f(x)$$

2.5.1 Newton-Leibnitz theorem

Theorem 2.6

Let Φ be any antiderivative of the continuous function f on $[a,b]$. Then:

$$\int_a^b f(x)\,\mathrm{d}x = \Phi(b) - \Phi(a)$$

Remark: We note by $[\Phi(x)]_a^b$ or $\Phi(x)|_a^b$ for $\Phi(b) - \Phi(a)$. So we get:

$$\int_a^b f(x)\,\mathrm{d}x = [\Phi(x)]_a^b$$

Example 2.7

Compute $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$

According to the previous theorem, we have:

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = [\arctan(x)]_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4}$$

2.6 General integration techniques

2.6.1 Variable change method for definite integral

Theorem 2.7

Let f be a continuous function on $[a,b]$ and $\Phi : [\alpha,\beta] \to [a,b]$ be function $\in C^1([\alpha,\beta])$ such that: $\Phi(\alpha) = a$ and $\Phi(\beta) = b$, then:

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(\Phi(t))\Phi'(t)\,\mathrm{d}t$$

Proof

We put:

$$x = \Phi(t) \implies \mathrm{d}x = \Phi'(t)\mathrm{d}t$$

For the bounds of the integral we have:

$$\begin{cases} x = a \Leftrightarrow \Phi(t) = a \implies t = \alpha \\ x = b \Leftrightarrow \Phi(t) = b \implies t = \beta \end{cases}$$

By replacing in the integral, we obtain:

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(\Phi(t))\Phi'(t)\,\mathrm{d}t$$

Example 2.8

Compute $I = \int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$

Solution: We put:

$$x = \sin(t) \implies \mathrm{d}x = \cos(t)\mathrm{d}t$$

Then:

$$\begin{cases} x = -1 \implies t = -\frac{\pi}{2} \\ x = 1 \implies t = \frac{\pi}{2} \end{cases}$$

Replacing in the integral gives:

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2(t)}\cos(t)\,\mathrm{d}t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos(t)|\cos(t)\,\mathrm{d}t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(t)\,\mathrm{d}t$$

On the other hand, we have: $\cos^2(t) = \frac{1}{2}\cos(2t) + \frac{1}{2}$

$$\implies I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\frac{1}{2}\cos(2t) + \frac{1}{2})\,\mathrm{d}t = \frac{1}{2}\left[\frac{1}{2}\sin(2t) + t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2}$$

2.6.2 Integration by part in a definite integral

Theorem 2.8

Let f and g be two functions differentiable on $[a,b]$ then

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,\mathrm{d}x$$

Example 2.9

Compute $I = \int_0^{\frac{\pi}{3}} x\cos(x)\,\mathrm{d}x$

Solution: We put

$$f(x) = x \longrightarrow f'(x) = 1$$

$$g'(x) = \cos(x) \longrightarrow g(x) = \sin(x)$$

by applying the integration by parts formula, we obtain:

$$I = \int_0^{\frac{\pi}{3}} x\cos(x)\,\mathrm{d}x = [x\sin(x)]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \sin(x)\,\mathrm{d}x$$

$$\Longrightarrow I = [x\sin(x)]_0^{\frac{\pi}{3}} - [\cos(x)]_0^{\frac{\pi}{3}} = \frac{\pi}{2\sqrt{3}} - \frac{1}{2}$$