Chapter 6

Usual functions

Contents

6.1 An overview of inverse function

Let I be an interval of $\mathbb{R}$, f a function defined on I and $J = f(I)$. Our interest lies in the existence of the inverse function of f, i.e the existence of a function f^{-1} from J into I such that:

$$\forall x \in I, f^{-1}(f(x)) = x \quad \text{and} \quad \forall y \in J, f(f^{-1}(y)) = y$$

Proposition 6.1: Existence of an inverse function

Let I be an interval and f a function defined on I. If f is continuous and strictly monotone on I then f is a bijection from I to $J = f(I)$ and admits a reciprocal function f^{-1} from J to I which has the following properties:

1. f^{-1} is continuous on J.
2. f^{-1} is strictly monotonic on J and has the same direction of monotonicity as f.
3. f^{-1} is bijective.

Remark 6.1 *The graphical representations of f and f^{-1} are symmetrical with respect to the line with equation $y = x$.*

Example 6.1

Let f be a function defined by:

$$f: \begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \ln(x) \end{array}$$

We have:

x	0	$+\infty$
$f(x)$	$\|\|$ $-\infty$ $\nearrow$	$+\infty$

Set $I = \mathbb{R}_+^*$, then $J = f(I) =]-\infty, +\infty[= \mathbb{R}$
From the table of variations of f we have:

1. f is continuous on I
2. f is strictly increasing on I

then f admits an inverse function f^{-1} denoted by e^x or $\exp(x)$ defined by:

$$f^{-1}: \begin{array}{l} \mathbb{R} \rightarrow]0, +\infty[\\ x \mapsto f^{-1}(x) = e^x \end{array}$$

Proposition 6.2: (Differentiability at a point)

Let $f : I \rightarrow J$ be a bijective and differentiable function at $x_0 \in I$.
If we have $f'(x_0) \neq 0$ then f^{-1} is differentiable at $y_0 = f(x_0)$ and moreover:

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Proposition 6.3: (Differentiability on an interval)

Let $f : I \rightarrow J$ be a bijective and differentiable function on I (with I is an open interval).
If we have: $\forall x \in I;\ f'(x) \neq 0$, then f^{-1} is differentiable on J and moreover:

$$\forall y \in J;\ (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Example 6.2

Let $f(x) = \ln(x)$ and $I = \mathbb{R}_+^*$, then $J = f(I) = \mathbb{R}$. From the previous example, f is bijective from I into J and admits an inverse function $f^{-1}(x) = e^x$.

We have: for all $x \in \mathbb{R}_+^*$, $f(x)$ is differentiable and moreover $f'(x) = \dfrac{1}{x} \neq 0$. According to proposition (5.3) f^{-1} is differentiable on $J = \mathbb{R}$ and

$$\forall y \in \mathbb{R};\ (f^{-1})'(y) = (e^y)' = \frac{1}{f'(f^{-1}(y))} = \frac{1}{\frac{1}{e^y}} = e^y$$

Remark 6.2 *In the previous formula, we can replace y by x and write:*

$$\forall x \in \mathbb{R};\ (e^x)' = e^x$$

6.2 Logarithmic Functions

6.2.1 The neperian logarithm function

Definition 6.1

The function that satisfies the following two conditions is called the neperian logarithm function and is denoted by ln:

1. $\forall x \in \mathbb{R}_+^*;\ (\ln(x))' = \dfrac{1}{x}$
2. $\ln(1) = 0$

Remark 6.3 *(Properties of derivatives)*

1. *According to the previous definition, the function $\ln(x)$ is differentiable on $\mathbb{R}_+^*$ and $\forall x \in \mathbb{R}_+^*;\ (\ln(x))' = \dfrac{1}{x}$.*

2. *The function $\ln(|x|)$ is differentiable on $\mathbb{R}^*$ and $\forall x \in \mathbb{R}^*;\ (\ln(|x|))' = \dfrac{1}{x}$*

3. *Let g be a function differentiable and non-zero on I then the function $\ln(|g(x)|$ is differentiable on I and its derivative: $(\ln(|g(x)|)' = \dfrac{g'(x)}{g(x)}$*

Proposition 6.4: (Limits and classical inequalities)

1. $\lim_{x \to +\infty} \ln(x) = +\infty$
2. $\lim_{x \to 0^+} \ln(x) = -\infty$
3. $\lim_{x \to +\infty} \frac{\ln(x)}{x} = 0$
4. $\lim_{x \to +\infty} \frac{\ln(x)}{x^\alpha} = 0$ (with $\alpha \in \mathbb{R}_+^*$).
5. $\lim_{x \to 0^+} x \ln(x) = 0$
6. $\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$
7. $\forall x \in]-1, +\infty[;\ \ln(x+1) \leq x$

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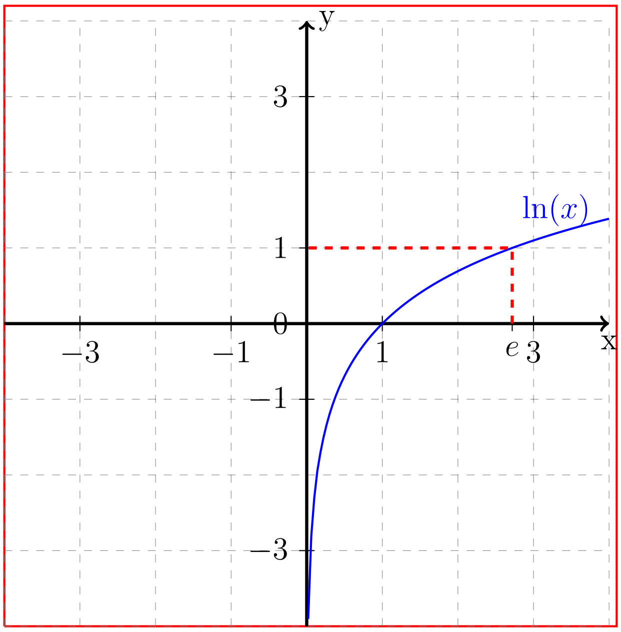

Figure 6.1 – *Graphical representation of the function* $\ln(x)$

Proposition 6.5: (Algebraic properties of the function $\ln(x)$)

For all $x,y \in \mathbb{R}_+^*$ and $\alpha \in \mathbb{Q}$, we have the following properties:

1. $\ln(x \times y) = \ln(x) + \ln(y)$
2. $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
3. $\ln\left(\frac{1}{x}\right) = -\ln(x)$
4. $\ln(x^\alpha) = \alpha \ln(x)$

6.2.2 The logarithmic function with base a

Definition 6.2

Let $a \in]0,1[\cup]1,+\infty[$.
We call the logarithm function with base a and denote $\log_a$, the function defined by:

$$\forall x \in]0,+\infty[;\ \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

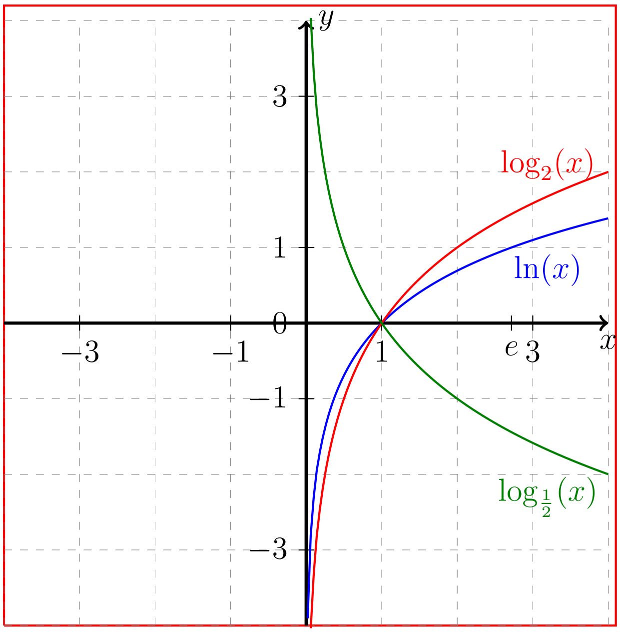

Figure 6.2 – *Graphical representation of the logarithmic functions and logarithms with base a for $a = \frac{1}{2}$, $a = 2$*

Remark 6.4 *(Properties of the function $\log_a$)*

1. *We have: $\ln(x) = \log_e(x)$ i.e., the neperian logarithm function is the logarithm function with base e.*
2. *The logarithm function with base a verifies relations analogous to those stated for the neperian logarithm function.*

6.3 Exponential Functions

6.3.1 The exponential function

Definition 6.3

The inverse function of the function $\ln(x)$ is called the exponential function and is denoted by: $\exp(x)$ or e^x, and satisfies the following properties:

1. $\forall x \in]0,+\infty[;\ x = e^{\ln(x)}$
2. $\forall y \in \mathbb{R};\ y = \ln(e^y)$

Proposition 6.6

1. The function e^x is continuous and strictly increasing on $\mathbb{R}$.
2. The function e^x is differentiable on $\mathbb{R}$ and we have: $\forall x \in \mathbb{R};\ (e^x)' = e^x$
3. If u is differentiable on I then: the function $e^{u(x)}$ is differentiable on I and its derivative defined by: $\forall x \in I;\ (e^{u(x)})' = u'(x).e^{u(x)}$

Proposition 6.7: (Limits and inequalities)

1. $\lim\limits_{x\to-\infty} e^x = 0$
2. $\lim\limits_{x\to+\infty} e^x = +\infty$
3. $\lim\limits_{x\to+\infty} xe^{-x} = 0,\ \lim\limits_{x\to+\infty} \frac{x^\alpha}{e^x} = 0,\ \lim\limits_{x\to+\infty} \frac{e^x}{x^\alpha} = +\infty$ (with $\alpha \in \mathbb{R}$)
4. $\lim\limits_{x\to0} \frac{e^x - 1}{x} = 1$
5. $\forall x \in \mathbb{R};\ e^x \geq 1 + x$

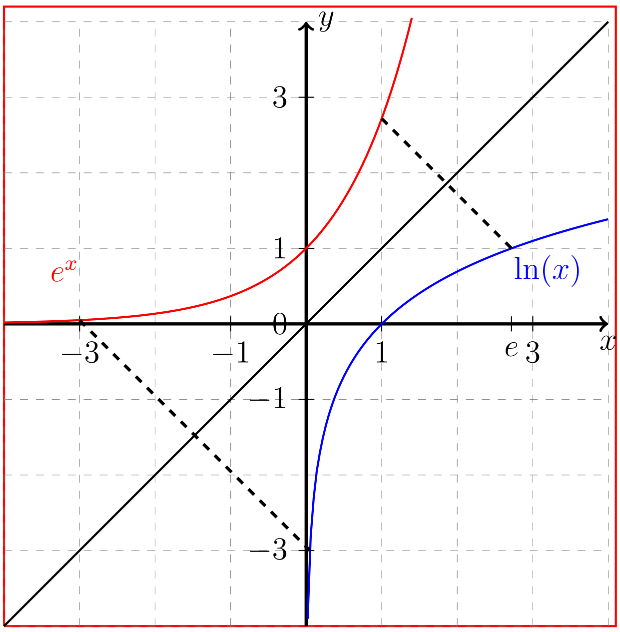

Figure 6.3 – *Graphical representation of the function e^x*

Proposition 6.8: (Algebraic properties of the function e^x)

For all $x,y \in \mathbb{R}$ and $\alpha \in \mathbb{Q}$, we have:

1. $e^{x+y} = e^x \times e^y$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $e^{x-y} = \dfrac{e^x}{e^y}$
4. $e^{\alpha x} = (e^x)^\alpha$

6.3.2 The exponential function with base a

Definition 6.4

Let $a \in]0,1[\cup]1,\infty[$.
The inverse function of the function $\log_a(x)$ is called the exponential function with base a and is denoted a^x:

1. $\forall x \in \mathbb{R};\ a^x = e^{x\ln(a)}$
2. $\forall x \in \mathbb{R};\ \log_a(a^x) = \log_a(e^{x\ln(a)}) = \dfrac{\ln(e^{x\ln(a)})}{\ln(a)} = x$

Remark 6.5 *The function a^x is differentiable on $\mathbb{R}$ and we have:*

$$\forall x \in \mathbb{R};\ (a^x)' = \ln(a)a^x$$

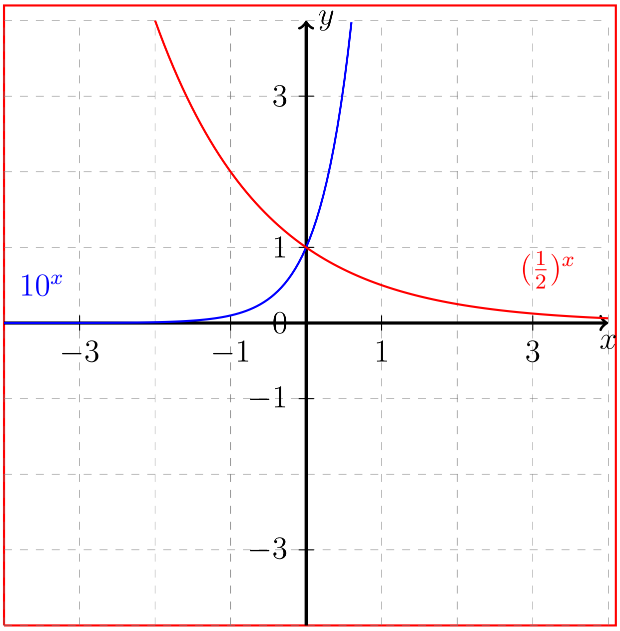

Figure 6.4 – *Graphical representation of functions* 10^x *et* $\left(\frac{1}{2}\right)^x$

Remark 6.6 *The exponential function with base a verifies similar properties to those of the exponential function.*

6.4 Power functions

Definition 6.5

Let $\alpha \in \mathbb{R}$, we name power function of exponent α, the function defined by:

$$\forall x \in]0, +\infty[;\ x^\alpha = e^{\alpha \ln(x)}$$

Remark 6.7 *If* $n \in \mathbb{N}^*$, *we have :*

$$e^{n\ln(x)} = e^{\sum_{k=1}^{n} \ln(x)} = \prod_{k=1}^{k=n} e^{\ln(x)} = \prod_{k=1}^{k=n} x = \underbrace{x \times x \times .... \times x}_{nfois} = x^n$$

Proposition 6.9

1. For $\alpha \in \mathbb{R}^*$, the power function with exponent α is a continuous function on $]0,+\infty[$ and strictly monotonic (strictly increasing if $\alpha > 0$ and strictly decreasing if $\alpha < 0$).
2. It is differentiable on $]0, +\infty[$ with derivative : $(x^\alpha)' = \alpha x^{\alpha-1}$, $\forall x \in]0, +\infty[$
3. We have:

$$\lim_{x\to+\infty} x^\alpha = \begin{cases} 0, & si\ \alpha < 0 \\ 1, & si\ \alpha = 0 \\ +\infty, & si\ \alpha > 0 \end{cases} \quad \text{and} \quad \lim_{x\to 0^+} x^\alpha = \begin{cases} +\infty, & si\ \alpha < 0 \\ 1, & si\ \alpha = 0 \\ 0, & si\ \alpha > 0 \end{cases}$$

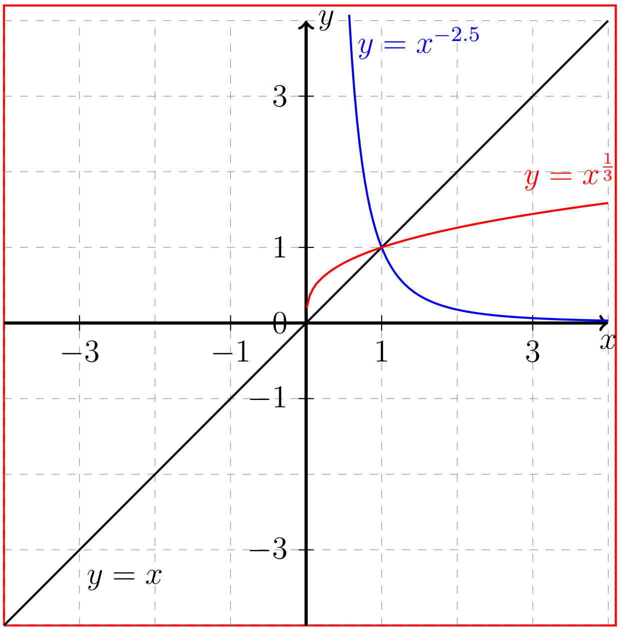

Figure 6.5 – *Graphical representation of functions* x^α, *with* $\alpha = -2.5,1,\dfrac{1}{3}$

Proposition 6.10

For $x \in \mathbb{R}_+^*$ and $\alpha,\beta \in \mathbb{R}$ we have the following relationships:

1. $x^{\alpha+\beta} = x^\alpha x^\beta$.
2. $x^{-\alpha} = \dfrac{1}{x^\alpha}$.
3. $x^{\alpha-\beta} = \dfrac{x^\alpha}{x^\beta}$.
4. $x^{\alpha\beta} = (x^\alpha)^\beta = (x^\beta)^\alpha$.

6.5 Circular (or trigonometric) functions

6.5.1 Recalls on the functions $cos(x)$ and $sin(x)$.

Proposition 6.11

The functions $\begin{cases} x \longmapsto \cos(x) \\ \text{and} \\ x \longmapsto \sin(x) \end{cases}$ are defined on $\mathbb{R}$ and satisfy the following properties:

1. $\forall x \in \mathbb{R};\ |\cos(x)| \leq 1 \wedge |\sin(x)| \leq 1$
2. $\cos(x)$ and $\sin(x)$ are 2π-periodic i.e.:

$$\forall x \in \mathbb{R}; \begin{cases} \cos(x+2\pi) = \cos(x) \\ \text{and} \\ \sin(x+2\pi) = sin(x) \end{cases}$$

3. The function $\cos(x)$ is even and the function $\sin(x)$ is odd, i.e.:

$$\forall x \in \mathbb{R}; \begin{cases} \cos(-x) = \cos(x) \\ \text{and} \\ \sin(-x) = -\sin(x) \end{cases}$$

4. The functions $\cos(x)$ and $\sin(x)$ belong to $C^\infty(\mathbb{R})$ and we have:

a $\forall x \in \mathbb{R}; \begin{cases} (\cos(x))' = -\sin(x) \\ \text{and} \\ (\sin(x))' = \cos(x) \end{cases}$

b $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}; \begin{cases} \cos^{(n)}(x) = \cos(x + n\frac{\pi}{2}) \\ \text{and} \\ \sin^{(n)}(x) = \sin(x + n\frac{\pi}{2}) \end{cases}$

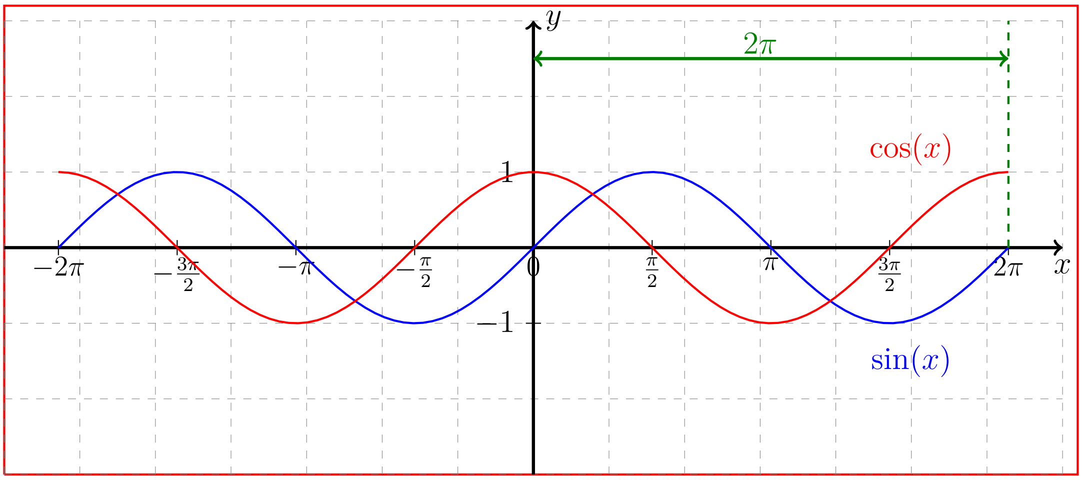

Figure 6.6 – *Graphical representation of functions* $\sin(x)$ *and* $\cos(x)$

Proposition 6.12: (Formules d'addition)

For all $(x,y) \in \mathbb{R}^2$, we have the following formulas:

- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
- $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
- $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$
- $\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

6.5.2 Recall about the function $\tan(x)$

Definition 6.6

The tangent function is one of the main trigonometric functions and defined by:

$$\begin{array}{rrcl} \tan: & \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \tan(x) = \dfrac{\sin(x)}{\cos(x)} \end{array}$$

Proposition 6.13

The function $\tan(x)$ is differentiable on $\mathbb{R}\backslash\{\frac{\pi}{2} + k\pi/k \in \mathbb{Z}\}$ and we have:

$$\forall x \in \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi/k \in \mathbb{Z}\};\ (\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$$

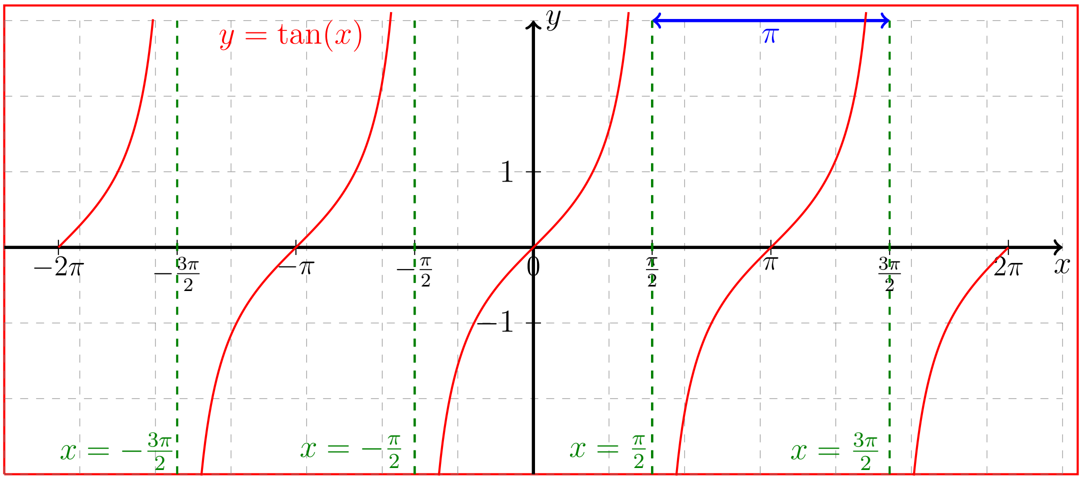

Figure 6.7 – *Graphical representation of the function* $\tan(x)$

Proposition 6.14

The function $\tan(x)$ checks the following properties:

1. The function $\tan(x)$ is π-periodic i.e :

$$\forall x \in \mathbb{R} \backslash \{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\}; \ \tan(x+\pi) = \tan(x)$$

2. For any x,$y \in \mathbb{R} \backslash \{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\}$ we have:

$$\begin{cases} \tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \\ \quad \text{and} \\ \tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)} \end{cases}$$

3. $\forall x \in \mathbb{R} \backslash \{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\}; \ \tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}$

Proposition 6.15: (Some usual limits)

1. $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$
2. $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$
3. $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$
4. $\lim_{x\to -\frac{\pi}{2}} \tan(x) = -\infty$
5. $\lim_{x\to +\frac{\pi}{2}} \tan(x) = +\infty$
6. $\lim_{x\to 0} \frac{\tan(x)}{x} = 1$

6.6 Hyperbolic Functions

6.6.1 Hyperbolic cosine, sine and tangent functions

Any function f defined on $\mathbb{R}$ can be uniquely decomposed into a sum of two functions f_{ev} and f_{od} where f_{ev} is an even function and f_{od} is an odd function. This means for every $x \in \mathbb{R}$ we can write

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

and we choose

$$\begin{cases} f_p(x) = \dfrac{f(x)+f(-x)}{2} \\ \quad et \\ f_i(x) = \dfrac{f(x)-f(-x)}{2} \end{cases}$$

Remark 6.8 *We can easily check that this decomposition is unique, and f_{ev} is an even function and f_{od} is an odd function.*

Definition 6.7: (Hyperbolic cosine)

We call the hyperbolic cosine function and denoted (ch or cosh), the even part of the exponential function defined by:

$$\begin{array}{rccl} \text{ch}: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \text{ch}(x) = \dfrac{e^x + e^{-x}}{2} \end{array}$$

Definition 6.8: (Hyperbolic sine)

The hyperbolic sine function, denoted by (sh or sinh), is the odd part of the exponential function defined by:

$$\begin{array}{rccl} \text{sh}: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \text{sh}(x) = \dfrac{e^x - e^{-x}}{2} \end{array}$$

Definition 6.9: (Hyperbolic tangent)

The hyperbolic tangent function, denoted by (th or tanh), is the quotient of the hyperbolic sine function with the hyperbolic cosine function and defined by:

$$\begin{array}{rccl} \text{th}: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \text{th}(x) = \dfrac{\text{sh}(x)}{\text{ch}(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \end{array}$$

Proposition 6.16

- The function $\text{ch}(x)$ is a function defined on $\mathbb{R}$, continuous and even.
- The function $\text{sh}(x)$ is a function defined on $\mathbb{R}$, continuous and odd.
- The function $\text{th}(x)$ is a function defined on $\mathbb{R}$, continuous and odd.
- The functions $\text{ch}(x)$, $\text{sh}(x)$ and $\text{th}(x)$ are differentiable on $\mathbb{R}$ and their derivatives are defined by:

$$\forall x \in \mathbb{R}; \begin{cases} (\text{ch}(x))' = \text{sh}(x) \\ (\text{sh}(x))' = \text{ch}(x) \\ (\text{th}(x))' = \dfrac{1}{\text{ch}(x)^2} = 1 - \text{th}(x)^2 \end{cases}$$

Proof

These properties can be verified using the properties of the e^x function. In our proof, we're interested with the function $\text{th}(x)$.
We have:

$$\forall x \in \mathbb{R};\ \text{th}(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

- The continuity: The functions $(e^x - e^{-x})$ and $(e^x + e^{-x})$ are continuous on $\mathbb{R}$, with $e^x + e^{-x} \neq 0$ then the quotient function $\frac{e^x - e^{-x}}{e^x + e^{-x}}$ is continuous on $\mathbb{R} \implies \text{th}(x)$ is continuous on $\mathbb{R}$
- The parity: We have:

$$\forall x \in \mathbb{R};\ \text{th}(-x) = \frac{e^{-x} - e^x}{e^{-x} + e^x} = -\frac{e^x - e^{-x}}{e^x + e^{-x}} = -\text{th}(x)$$

  So $\text{th}(x)$ is odd.

- The differentiability: The functions $(e^x - e^{-x})$ et $(e^x + e^{-x})$ are differentiable on $\mathbb{R}$, with $e^x + e^{-x} \neq 0$ then the quotient function $\frac{e^x - e^{-x}}{e^x + e^{-x}}$ is differentiable on $\mathbb{R} \implies \text{th}(x)$ is differentiable on $\mathbb{R}$ and we have:

$$\forall x \in \mathbb{R};\ (\text{th}(x))' = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)' = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$\iff \text{th}(x)' = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = 1 - \text{th}(x)^2$$

  also $\text{th}(x)' = \frac{4}{(e^x + e^{-x})^2} = \frac{1}{\text{ch}(x)^2}$.

Remark 6.9 *The functions $ch(x)$,$sh(x)$ and $th(x)$ have the following properties:*

1. $\text{ch}(0) = 1$, $\text{sh}(0) = 0$ *and* $\text{th}(0) = 0$.
2. $\lim_{x \to -\infty} sh(x) = -\infty$, $\lim_{x \to -\infty} ch(x) = +\infty$ *and* $\lim_{x \to -\infty} th(x) = -1$
3. $\lim_{x \to +\infty} sh(x) = +\infty$, $\lim_{x \to +\infty} ch(x) = +\infty$ *and* $\lim_{x \to +\infty} th(x) = 1$

Therefore, the above results can be grouped together in tabular form.

x	$-\infty$	0	$+\infty$
$\mathrm{sh}(x)' = \mathrm{ch}(x)$		$+$	
$\mathrm{sh}(x)$	$-\infty$ ↗	0 ↗	$+\infty$

(a) *Function sh(x)*

x	$-\infty$	0	$+\infty$
$\mathrm{ch}(x)' = \mathrm{sh}(x)$	$-$	0	$+$
$\mathrm{ch}(x)$	$+\infty$ ↘	1 ↗	$+\infty$

(b) *Function ch(x)*

Figure 6.8 – *Functions sh(x) and ch(x)*

x	$-\infty$	0	$+\infty$
$\mathrm{th}(x)' = \dfrac{1}{\mathrm{ch}(x)^2}$		$+$	
$\mathrm{th}(x)$	-1 ↗	0 ↗	1

Figure 6.9 – *Function th(x)*

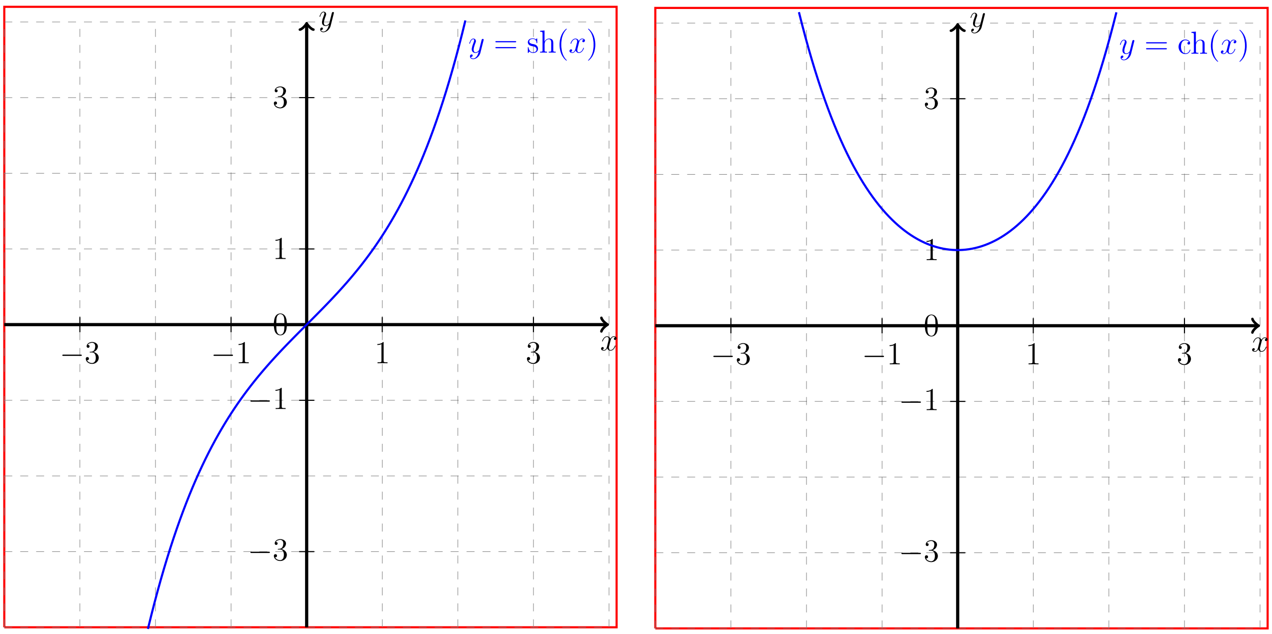

Figure 6.10 – *Graphical representation of functions sh(x) et ch(x)*

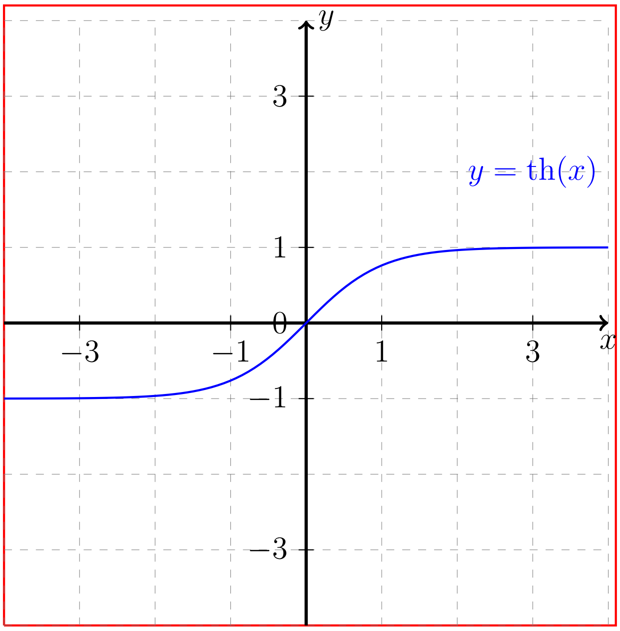

Figure 6.11 – *Graphical representation of the function th(x)*

Proposition 6.17

For every real x, we have:

- $\mathrm{ch}(x) + \mathrm{sh}(x) = e^x$
- $\mathrm{ch}(x) - \mathrm{sh}(x) = e^{-x}$
- $\mathrm{ch}(x)^2 - \mathrm{sh}(x)^2 = 1$

Proposition 6.18: (Addition formulas)

For all $(x,y) \in \mathbb{R}^2$, we have the following formulas:

- $\mathrm{ch}(x+y) = \mathrm{ch}(x)\mathrm{ch}(y) + \mathrm{sh}(x)\mathrm{sh}(y)$
- $\mathrm{ch}(x-y) = \mathrm{ch}(x)\mathrm{ch}(y) - \mathrm{sh}(x)\mathrm{sh}(y)$
- $\mathrm{sh}(x+y) = \mathrm{sh}(x)\mathrm{ch}(y) + \mathrm{ch}(x)\mathrm{sh}(y)$
- $\mathrm{sh}(x-y) = \mathrm{sh}(x)\mathrm{ch}(y) - \mathrm{ch}(x)\mathrm{sh}(y)$
- $\mathrm{th}(x+y) = \dfrac{\mathrm{th}(x) + \mathrm{th}(y)}{1 + \mathrm{th}(x)\mathrm{th}(y)}$
- $\mathrm{th}(x-y) = \dfrac{\mathrm{th}(x) - \mathrm{th}(y)}{1 - \mathrm{th}(x)\mathrm{th}(y)}$

Proof

We prove these formulas by using the expressions of hyperbolic functions with the exponential function. We have:

$$\begin{aligned}\operatorname{ch}(x)\operatorname{ch}(y)+\operatorname{sh}(x)\operatorname{sh}(y) &= \frac{1}{4}\left((e^x+e^{-x})(e^y+e^{-y})+(e^x-e^{-x})(e^y-e^{-y})\right)\\ &= \frac{1}{4}\left(2e^xe^y+2e^{-x}e^{-y}\right)\\ &= \frac{1}{2}\left(e^{(x+y)}+e^{-(x+y)}\right)\\ &= \operatorname{ch}(x+y).\end{aligned}$$

The other relations are shown using the same method.

Proposition 6.19: (Some usual limits of hyperbolic functions)

1. $\lim\limits_{x\to+\infty} \frac{\operatorname{ch}(x)}{e^x} = \frac{1}{2}$
2. $\lim\limits_{x\to+\infty} \frac{\operatorname{sh}(x)}{e^x} = \frac{1}{2}$
3. $\lim\limits_{x\to 0} \frac{\operatorname{sh}(x)}{x} = 1$
4. $\lim\limits_{x\to 0} \frac{\operatorname{ch}(x)-1}{x^2} = \frac{1}{2}$

6.7 Inverse Trigonometric Functions

6.7.1 The function arc-sinus

According to the variation table below, we have:
The function $\sin(x)$ is continuous and strictly increasing on $[-\frac{\pi}{2},\frac{\pi}{2}]$, then the function $\sin(x)$ represents a bijection from $[-\frac{\pi}{2},\frac{\pi}{2}]$ to $[-1,1]$.

x	$-\frac{\pi}{2}$	0	$+\frac{\pi}{2}$
$\sin(x)' = \cos(x)$		$+$	
$\sin(x)$	-1 ↗	0 ↗	1

Figure 6.12 – *Function* $\sin(x)$

Definition 6.10

The inverse function of the restriction of $\sin(x)$ on $[-\frac{\pi}{2},\frac{\pi}{2}]$ is called the arcsine function and is denoted by $\arcsin(x)$ or $\sin^{-1}(x)$:

$$\begin{array}{rccc} \arcsin: & [-1,1] & \longrightarrow & [-\frac{\pi}{2},\frac{\pi}{2}] \\ & x & \longmapsto & \arcsin(x) \end{array}$$

Proposition 6.20

The function $arcsin(x)$ has the following properties:

1. The function $\arcsin(x)$ is continuous and strictly increasing on $[-1,1]$. (According to the inverse function theorem)
2. $\forall x \in [-\frac{\pi}{2},\frac{\pi}{2}]$; $\arcsin(\sin(x)) = x$.
3. $\forall y \in [-1,1]$; $\sin(\arcsin(y)) = y$.
4. $\forall x \in [-\frac{\pi}{2},\frac{\pi}{2}], \forall y \in [-1,1]$; $(\sin(x) = y \Longleftrightarrow x = \arcsin(y))$.
5. The function $arcsin(x)$ is odd.

Proof

Let's prove property (5).

1. The function $arcsin(x)$ is defined on $[-1,1]$, so in this case the domain of definition is symmetric about 0.
2. Let $x \in [-1,1]$ and:

$$\arcsin(-x) = y \tag{6.1}$$

$$\Leftrightarrow -x = \sin(y) \Leftrightarrow x = -\sin(y) \Leftrightarrow x = \sin(-y) \text{ (Since } \sin(x) \text{ is odd)}$$

We have: $y \in [-\frac{\pi}{2},\frac{\pi}{2}] \implies -y \in [-\frac{\pi}{2},\frac{\pi}{2}]$
So we obtain: $\arcsin(x) = -y \Leftrightarrow -\arcsin(x) = y$
From equation (6.1) we get: $\arcsin(-x) = -\arcsin(x)$

$\implies$ The function $\arcsin(x)$ is odd.

Remark 6.10 *The following table contains some usual values for the function* $\arcsin(x)$

$\sin(0) = 0$	$\arcsin(0) = 0$
$\sin(\frac{\pi}{6}) = \frac{1}{2}$	$\arcsin(\frac{1}{2}) = \frac{\pi}{6}$
$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$	$\arcsin(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$
$\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$	$\arcsin(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$
$\sin(\frac{\pi}{2}) = 1$	$\arcsin(1) = \frac{\pi}{2}$

Proposition 6.21

The arcsine function is differentiable on $]-1,1[$ and verifies:

$$\forall x \in]-1,1[;\ (\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$$

Proof

The function $\sin(x)$ has the following two properties:

1. $\sin(x)$ is differentiable on $]-\frac{\pi}{2},\frac{\pi}{2}[$.
2. $\forall x \in]-\frac{\pi}{2},\frac{\pi}{2}[;\ (\sin(x))' = \cos(x) \neq 0$

$\Longrightarrow$ (from proposition (5.3)), the function $\arcsin(x)$ is differentiable on $]-1,1[$ and we have:

$$\forall x \in]-1,1[;\ (\arcsin(x))' = \frac{1}{\cos(\arcsin(x))} \tag{6.2}$$

Let $x \in]-1,1[$, and $y = \arcsin(x)$

$$\Longrightarrow\ y \in]-\frac{\pi}{2},\frac{\pi}{2}[\wedge \cos(y) > 0$$

Based on the relationship $\cos^2(y) + \sin^2(y) = 1$, we deduce that: $\cos(y) = \sqrt{1-\sin^2(y)}$. Since for all $x \in]-1,1[$ we have: $\sin(\arcsin(x)) = x$

$$\Longrightarrow\ \cos(\arcsin(x)) = \sqrt{1-x^2}$$

From equation (6.2) we obtain:

$$\forall x \in]-1,1[;\ (arcsin(x))' = \frac{1}{\sqrt{1-x^2}}.$$

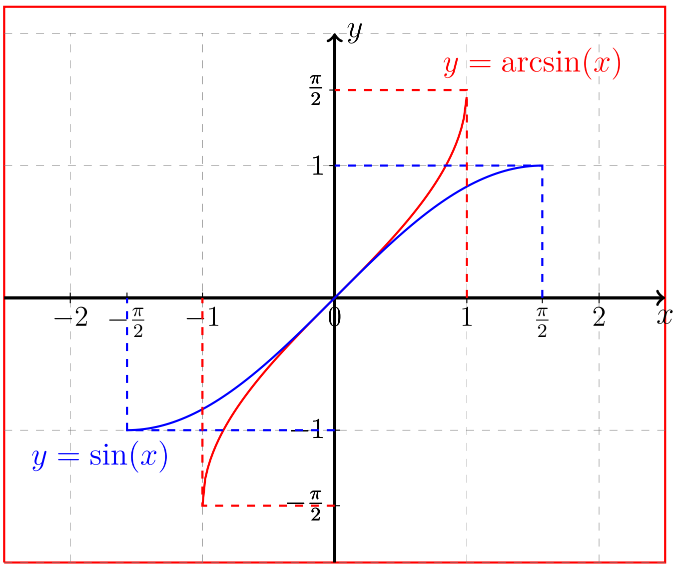

Figure 6.13 – *Graphical representation of the function* $\arcsin(x)$

6.7.2 The Arccosine Function

In the variation table below, we have:
The function $\cos(x)$ is continuous and strictly decreasing on $[0,\pi]$, so the function $cos(x)$ makes a bijection from $[0,\pi]$ into $[-1,1]$.

x	0 $\qquad\qquad$ π
$(\cos(x))' = -\sin(x)$	$-$
$\cos(x)$	$1 \searrow -1$

Figure 6.14 – *The function* $\cos(x)$

Definition 6.11

The inverse function of the restriction of $\cos(x)$ on $[0,\pi]$ is called the arccosine function and is denoted by $\arccos(x)$ or $\cos^{-1}(x)$:

$$\begin{array}{rccl} \arccos: & [-1,1] & \longrightarrow & [0,\pi] \\ & x & \longmapsto & \arccos(x) \end{array}$$

Proposition 6.22

The function $\arccos(x)$ has the following properties:

1. The function $\arccos(x)$ is continuous and strictly decreasing on $[-1,1]$. (From the inverse function theorem)
2. $\forall x \in [0,\pi];\ \arccos(\cos(x) = x$.
3. $\forall y \in [-1,1];\ \cos(\arccos(y) = y$.
4. $\forall x \in [0,\pi], \forall y \in [-1,1];\ (\cos(x) = y \iff x = \arccos(y))$.
5. The function $\arccos(x)$ is neither even nor odd.

Remark 6.11 *The table below shows some usual values for the function* $\arccos(x)$.

$\cos(0) = 1$	$\arccos(1) = 0$
$\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$	$\arccos(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$
$\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$	$\arccos(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$
$\cos(\frac{\pi}{3}) = \frac{1}{2}$	$\arccos(\frac{1}{2}) = \frac{\pi}{3}$
$\cos(\frac{\pi}{2}) = 0$	$\arccos(0) = \frac{\pi}{2}$

Proposition 6.23

The arccosine function is differentiable on $]-1{,}1[$ and verifies:

$$\forall x \in]-1{,}1[;\ (\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}$$

Proof

We have the function $\cos(x)$ satisfying the following two properties:

1. $\cos(x)$ is differentiable on $]0,\pi[$.
2. $\forall x \in]0,\pi[;\ (\cos(x))' = -\sin(x) \neq 0$

$\Longrightarrow$ (from proposition (6.3)), the function $\arccos(x)$ is differentiable on $]-1{,}1[$ and we have:

$$\forall x \in]-1{,}1[;\ (\arccos(x))' = \frac{1}{-\sin(\arccos(x))} \tag{6.3}$$

Let $x \in]-1{,}1[$, and $y = \arccos(x)$

$$\Longrightarrow\ y \in]0,\pi[\wedge \sin(y) > 0$$

Using the relationship $\cos^2(y) + \sin^2(y) = 1$, we deduce that $\sin(y) = \sqrt{1-\cos^2(y)}$. Since for any $x \in]-1{,}1[$ we have: $\cos(\arccos(x)) = x$, then we get:

$$\sin(\arccos(x)) = \sqrt{1-x^2}$$

From equation (6.3) we obtain:

$$\forall x \in]-1{,}1[;\ (arccos(x))' = -\frac{1}{\sqrt{1-x^2}}.$$

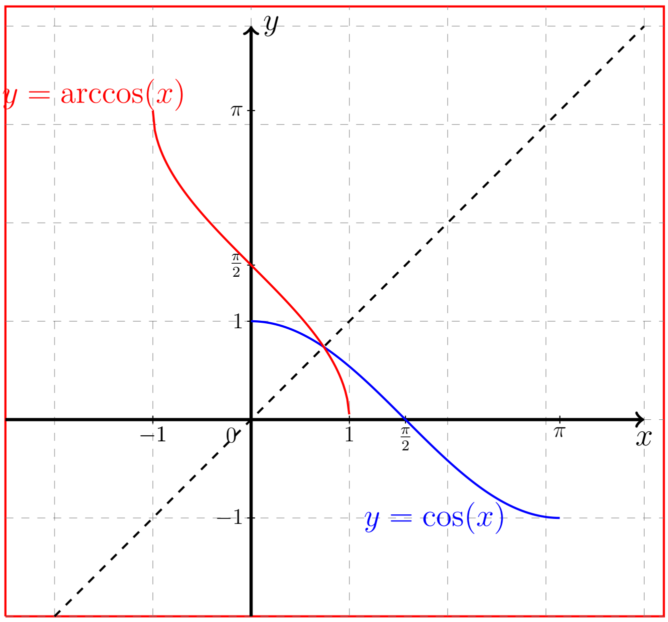

Figure 6.15 – *Graphical representation of the function* $\arccos(x)$

6.7.3 The Arctangent function

The function $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ is defined on $D = \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. It is continuous and differentiable on its domain of definition and for all $x \in D$ we have:

$$(\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$$

Consider the restriction of the function $\tan(x)$ on the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$, from the table of variation below we have: the function $\tan(x)$ is continuous and strictly increasing on $]-\frac{\pi}{2}, \frac{\pi}{2}[$, then the function $\tan(x)$ makes a bijection from $]-\frac{\pi}{2}, \frac{\pi}{2}[$ into $\mathbb{R}$.

x	$-\frac{\pi}{2}$		$\frac{\pi}{2}$
$(\tan(x))' = \dfrac{1}{\cos^2}$	‖	$+$	‖
$\tan(x)$	$-\infty$	↗	$+\infty$

Figure 6.16 – *The function* $\tan(x)$

Definition 6.12

We call the arctangent function $\arctan(x)$ or $\tan^{-1}(x)$ the inverse of the tangent function on $]-\frac{\pi}{2}, \frac{\pi}{2}[$ defined by:

$$\begin{array}{rccl} \arctan : &]-\infty, +\infty[& \longrightarrow &]-\frac{\pi}{2}, \frac{\pi}{2}[\\ & x & \longmapsto & \arctan(x) \end{array}$$

Proposition 6.24

The function $\arctan(x)$ has the following properties:

1. The function $\arctan(x)$ is continuous and strictly increasing on $\mathbb{R}$, with values in $]-\frac{\pi}{2}, \frac{\pi}{2}[$
2. $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[;\ \arctan(\tan(x)) = x$
3. $\forall y \in \mathbb{R};\ \tan(\arctan(y)) = y$.
4. $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[, \forall y \in \mathbb{R};\ \tan(x) = y \iff x = \arctan(y)$
5. The function *arctan*(x) is odd.

Remark 6.12 *The table below shows some usual values for the function* $\arctan(x)$.

$\tan(0) = 0$	$\arctan(0) = 0$
$\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$	$\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$
$\tan(\frac{\pi}{4}) = 1$	$\arctan(1) = \frac{\sqrt{2}}{2}$
$\tan(\frac{\pi}{3}) = \sqrt{3}$	$\arctan(\sqrt{3}) = \frac{\pi}{3}$

Proposition 6.25

The function $\arctan(x)$ is differentiable on $\mathbb{R}$ and verifies:

$$\forall x \in \mathbb{R};\ (\arctan(x))' = \frac{1}{1+x^2}$$

Proof

The function $\tan(x)$ has the following two properties:

1. The function $\tan(x)$ is differentiable on $]-\frac{\pi}{2},\frac{\pi}{2}[$.
2. $\forall x \in]-\frac{\pi}{2},\frac{\pi}{2}[;\ (\tan(x))' = \dfrac{1}{\cos^2(x)} = 1+\tan^2(x) \neq 0$

From proposition (6.3), the function $\arctan(x)$ is differentiable on $]-\frac{\pi}{2},\frac{\pi}{2}[$ and we have:

$$\forall x \in \mathbb{R};\ (\arctan(x))' = \frac{1}{1+\tan^2(\arctan(x))} = \frac{1}{1+x^2}$$

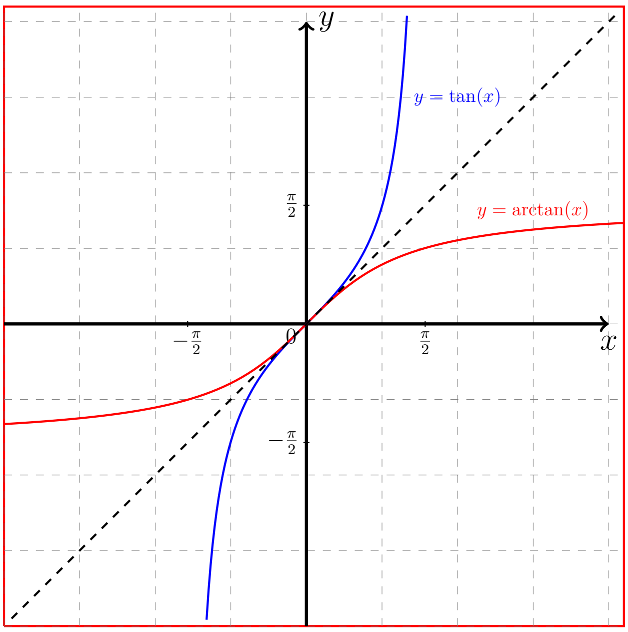

Figure 6.17 – *Graphical representation of the function* $\arctan(x)$

Proposition 6.26: (Some properties)

1. For any $x \in [-1,1]$ we have:
$$\arccos(x) + \arcsin(x) = \frac{\pi}{2}$$
2. For all $x \in \mathbb{R}_-^*$ we have:
$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}$$
3. For every $x \in \mathbb{R}_+^*$ we have:
$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

Proof

We'll show properties (2) and (3).
Set $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$.
Since the functions $\dfrac{1}{x}$ and $\arctan(x)$ are differentiable on $\mathbb{R}^*$), the function f is differentiable on $\mathbb{R}^*$ and we have:

$$f'(x) = \frac{1}{1+x^2} + \left(\frac{1}{x}\right)' \frac{1}{1+\left(\frac{1}{x}\right)^2} = \frac{1}{1+x^2} - \frac{1}{x^2}\left(\frac{x^2}{1+x^2}\right) = 0$$

From this we deduce that f is a constant function on each of the intervals $]-\infty,0[$ and $]0,+\infty[$. On the other hand, we have:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right) = -\frac{\pi}{2}$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right) = \frac{\pi}{2}$$

so f can't be extended by continuity at 0. So we deduce that:

$$\exists C_1, C_2 \in \mathbb{R} \text{ tq:} f(x) = \begin{cases} C_1 & \text{if } x \in]0,+\infty[\\ C_2 & \text{if } x \in]-\infty,0[\end{cases}$$

Since $f(1) = 2\arctan(1) = 2\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{2} = C_1$
and $f(-1) = 2\arctan(-1) = 2\left(-\dfrac{\pi}{4}\right) = -\dfrac{\pi}{2} = C_2$

$$\Longrightarrow f(x) = \begin{cases} \frac{\pi}{2} & \text{if } x \in]0,+\infty[\\ -\frac{\pi}{2} & \text{if } x \in]-\infty,0[\end{cases}$$

So $\forall x \in \mathbb{R}_-^*;\ \arctan(x) + \arctan\left(\dfrac{1}{x}\right) = -\dfrac{\pi}{2}$ and $\forall x \in \mathbb{R}_+^*;\ \arctan(x) + \arctan\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$

6.8 The inverse hyperbolic functions

6.8.1 The inverse of hyperbolic Sine function

From the above table of variation of $\text{sh}(x)$ we have: $\text{sh}(x)$ is continuous and strictly increasing on $\mathbb{R}$. Hence, it realizes a bijection from $\mathbb{R}$ into $\mathbb{R}$.

Definition 6.13

The inverse function of the hyperbolic sine function on $\mathbb{R}$ is denoted $\text{argsh}(x)$ or $\text{sh}^{-1}(x)$.

$$\begin{array}{rccl} \text{argsh}: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \text{argsh}(x) \end{array}$$

Proposition 6.27

The function $\text{argsh}(x)$ has the following properties:

1. The function $\text{argsh}(x)$ is defined on $\mathbb{R}$, it is continuous and strictly increasing on $\mathbb{R}$.
2. $\forall x \in \mathbb{R}$; $\text{argsh}(\text{sh}(x)){=}x$.
3. $\forall y \in \mathbb{R}$; $\text{sh}(\text{argsh}(y)){=}y$.
4. $\forall (x{,}y) \in \mathbb{R}^2$; $y = \text{sh}(x) \iff x = \text{argsh}(y)$.
5. $\text{argsh}(x)$ is odd function.

Proof

We'll show that $\text{argsh}(x)$ is odd.
Let $x \in \mathbb{R}$, and

$$y = \text{argsh}(-x) \tag{6.4}$$

(6.4)$\iff \text{sh}(y) = -x \iff \text{sh}(-y) = x$ (Since $\text{sh}(x)$ is odd)
$\implies -y = \text{argsh}(x) \iff y = -\text{argsh}(x)$.
From (6.4), we get: $\text{argsh}(-x) = -\text{argsh}(x)$.
So, $\forall x \in \mathbb{R}$; $\text{argsh}(-x) = -\text{argsh}(x) \implies \text{argsh}(x)$ is odd.

Proposition 6.28

The function $\text{argsh}(x)$ is differentiable on $\mathbb{R}$ and verifies:

$$\forall x \in \mathbb{R};\ (\text{argsh}(x))' = \frac{1}{\sqrt{1+x^2}}.$$

Proof

The sh(x) function verifies the following two properties:

1. sh(x) is differentiable on $\mathbb{R}$.
2. $\forall x \in \mathbb{R}; (\text{sh}(x))' = \text{ch}(x) = \dfrac{e^x + e^{-x}}{2} \neq 0$

From proposition (6.3), the function argsh(x) is differentiable on $\mathbb{R}$:

$$\forall x \in \mathbb{R};\ (\text{argsh}(x))' = \frac{1}{\text{sh}'(\text{argsh}(x))} = \frac{1}{\text{ch}(\text{argsh}(x))}$$

On the other hand, we have: $\text{ch}(x)^2 - \text{sh}(x)^2 = 1 \implies \text{ch}(x) = \sqrt{1 + \text{sh}^2(x)}$ because ch(x) is positive function.

$$\implies \forall x \in \mathbb{R};\ \text{ch}(\text{argsh}(x)) = \sqrt{1 + (\text{sh}(\text{argsh}(x))^2} = \sqrt{1 + x^2}$$

$$\implies \forall x \in \mathbb{R};\ (\text{argsh}(x))' = \frac{1}{\sqrt{1 + x^2}}$$

.

Proposition 6.29

$$\forall x \in \mathbb{R};\ \text{argsh}(x) = \ln\left(x + \sqrt{1 + x^2}\right)$$

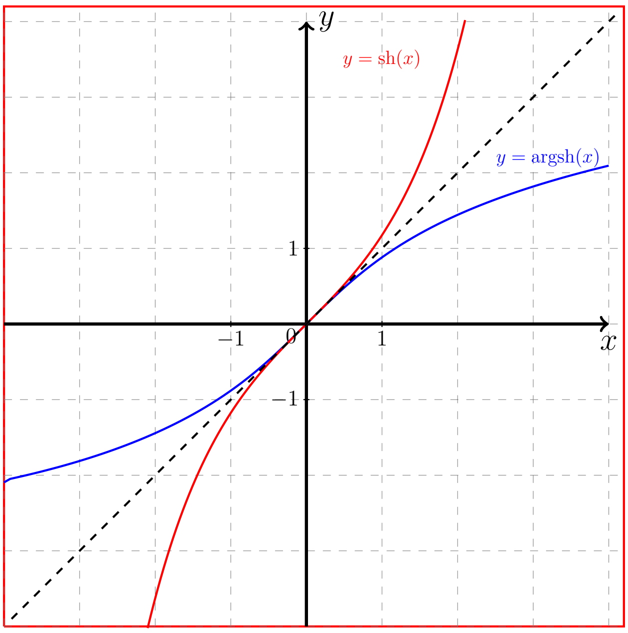

Figure 6.18 – *Graphical representation of the function argsh(x)*

6.8.2 The inverse hyperbolic cosine function

From the table of variation of the function $\text{ch}(x)$ above we have:
$\text{ch}(x)$ is continuous and strictly increasing on $[0,+\infty[$. So it forms a bijection from $[0,+\infty[$ into $[1,+\infty[$.

Definition 6.14

The inverse function of the restriction of $\text{ch}(x)$ on $[0,+\infty[$ is denoted by $\text{argch}(x)$ or $\text{ch}^{-1}(x)$

$$\begin{array}{rccc} \text{argch}: & [1,+\infty[& \longrightarrow & [0,+\infty[\\ & x & \longmapsto & \text{argch}(x) \end{array}$$

Proposition 6.30

The $\text{argch}(x)$ function has the following properties:

1. The function $\text{argch}(x)$ is defined on $[1,+\infty[$, it is continuous and strictly increasing on $[1,+\infty[$.
2. $\forall x \in [0,+\infty[$; $\text{argch}(\text{ch}(x)) = x$.
3. $\forall y \in [1,+\infty[$; $\text{ch}(\text{argch}(y)) = y$.
4. $\forall x \in [0,+\infty[, \forall y \in [1,+\infty[$; $y = \text{ch}(x) \Longleftrightarrow x = \text{argch}(y)$.

Proposition 6.31

The inverse hyperbolic cosine function is differentiable on $]1,+\infty[$ and verifies:

$$\forall x \in]1,+\infty[;\ (\text{argch}(x))' = \frac{1}{\sqrt{x^2-1}}$$

Remark 6.13 *The proof of proposition (6.31) is similar to the proof of proposition (6.28).*

Proposition 6.32

$$\forall x \in]1,+\infty[;\ \text{argch}(x) = \ln\left(x+\sqrt{x^2-1}\right)$$

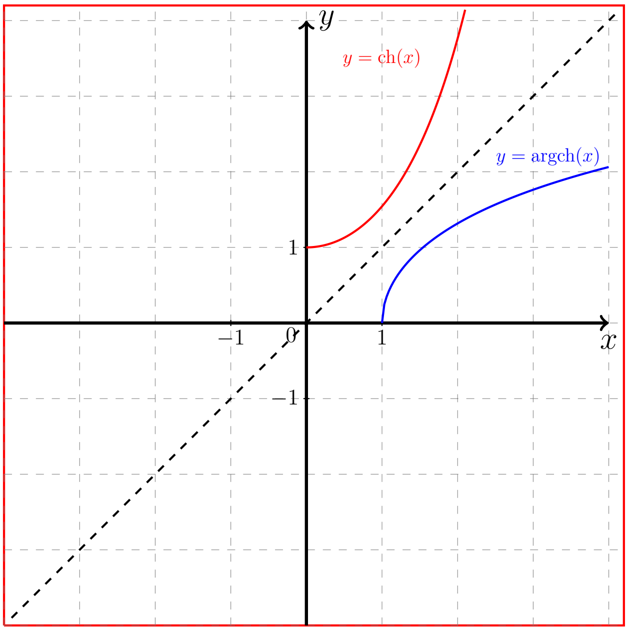

Figure 6.19 – *Graphical representation of the function argch*(x)

6.8.3 The inverse hyperbolic tangent function

From the table of variation of the function $\text{th}(x)$ above we have: $\text{th}(x)$ is continuous and strictly increasing on $\mathbb{R}$. So it makes is a bijection from $\mathbb{R}$ into $]-1,1[$.

Definition 6.15

The inverse function of the function $\text{th}(x)$ on $\mathbb{R}$ is denoted by $\text{argth}(x)$ or $\text{th}^{-1}(x)$

$$\begin{array}{rccc} \text{argth}: &]-1,1[& \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \text{argth}(x) \end{array}$$

Proposition 6.33

The function $\text{argth}(x)$ has the following properties:

1. The function $\text{argth}(x)$ is defined on $]-1,1[$, it is continuous and strictly increasing on $]-1,1[$.
2. $\forall x \in \mathbb{R}$; $\text{argth}(\text{th}(x))=x$.
3. $\forall y \in]-1,1[$; $\text{th}(\text{argth}(y))=y$.
4. $\forall x \in \mathbb{R}, \forall y \in]-1,1[$; $y = \text{th}(x) \Longleftrightarrow x = \text{argth}(y)$.
5. The $\text{argth}(x)$ function is odd.

Proposition 6.34

The function $\operatorname{argth}(x)$ is differentiable on $]-1,1[$ and verifies:

$$\forall x \in]-1,1[;\ (\operatorname{argth}(x))' = \frac{1}{1-x^2}.$$

Proposition 6.35

$$\forall x \in]-1;1[;\ \operatorname{argth}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Proof

Let $x \in]-1;1[$, and $y = \operatorname{argth}(x)$.
We have:

$$\operatorname{th}(x) = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}$$

$$\Longrightarrow\ e^{2y} = \frac{1+\operatorname{th}(y)}{1-\operatorname{th}(y)} = \frac{1+\operatorname{th}(\operatorname{argth}(x))}{1-\operatorname{th}(\operatorname{argth}(x))} = \frac{1+x}{1-x}$$

$$\Longleftrightarrow e^{2y} = \frac{1+x}{1-x} \Longleftrightarrow 2y = \ln\left(\frac{1+x}{1-x}\right) \Longleftrightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\Longrightarrow\ \forall x \in]-1,1[;\ \operatorname{argth}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

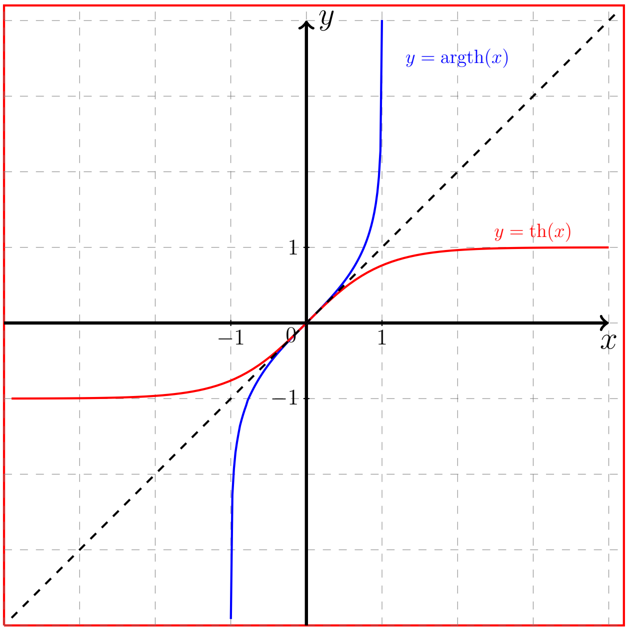

Figure 6.20 – *Graphical representation of the function argth(x)*