14/01/2024 08 :45-10 :15	**Correction** Analysis 1	Department : Common Core in Mathematics and Computer Science Batna 2-University.

Corrigé 01 (06 points)

1 By induction proof

For $n = 0$, we find $0 \leq v_0 = \frac{3}{2} \leq 2$, so the inequality is true for $n = 0$.

Assume that the inequality is true for a certain rank n (any rank) and show that the inequality is true for the next rank, i.e. for rank $n + 1$.

$$1 \leq v_n \leq 2 \Rightarrow 3 \leq 3v_n \leq 6 \Rightarrow 1 \leq -2 + 3v_n \leq 4 \Rightarrow$$
$$1 \leq v_{n+1} \leq \sqrt{4} = 2 \Rightarrow 1 \leq v_{n+1} \leq 2$$

Conclusion : $\forall n \in \mathbb{N}; 1 \leq v_n \leq 2$

2

$$\begin{aligned} v_{n+1} - v_n &= \frac{(v_{n+1} - v_n)(v_{n+1} + v_n)}{v_{n+1} + v_n} = \frac{v_{n+1}^2 - v_n^2}{v_{n+1} + v_n} = \frac{-2 + 3v_n - v_n^2}{v_{n+1} + v_n} \\ &= -\frac{2 - 3v_n + v_n^2}{v_{n+1} + v_n} = -\frac{(v_n - 1)(v_n - 2)}{v_{n+1} + v_n} \end{aligned}$$

According to the previous question,

$$\begin{cases} v_n \geq 1 \\ v_n \leq 2 \end{cases} \Rightarrow \begin{cases} v_n + v_{n+1} > 0 \\ v_n - 1 \geq 0 \\ v_n - 2 \leq 0 \end{cases} \Rightarrow -\frac{(v_n - 1)(v_n - 2)}{v_{n+1} + v_n} \geq 0$$

So v_n is increasing.

Since the sequence v_n is increasing and bounded from above, then it converges and $\lim\limits_{n \to +\infty} v_n = \lim\limits_{n \to +\infty} v_{n+1} = \ell$.

$$\lim_{n \to +\infty} v_{n+1} = \sqrt{-2 + 3 \lim_{n \to +\infty} v_n} \Rightarrow \ell^2 - 3\ell + 2 = 0 \Rightarrow (\ell - 1)(\ell - 2) = 0$$
$$\Rightarrow \ell = 1 \quad \vee \quad \ell = 2 \text{ and since } 1 < v_0 = \frac{3}{2} \Rightarrow \ell = 2$$

Corrigé 02 (10 points)

1

$$f(x) = \begin{cases} e^x + 2, & si\ x \geq 0 \\ a\cos(x) + bx + 1, & si\ x < 0 \end{cases}$$

It is clear that $f(0) = 3$ and $e^x + 2$ is differentiable on $]0, +\infty[$, with $(2 + e^x)' = e^x$

(a) The same on $]-\infty, 0[$ with $(a\cos(x) + bx + 1)' = -3\sin(x) + b$

(b) The only point that remains is $a = 0$, for which we need to compute the right and left derivatives because the function f changes its form at 0.

$$\begin{aligned}
\lim_{x\to 0^+}\frac{f(x)-f(0)}{x} &= \lim_{x\to 0^+}\frac{e^x+2-3}{x} = \lim_{x\to 0^+}\frac{e^x-1}{x}\\
&= 1.\\
&\text{On the other hand, since f is continuous at } a=0\\
, &\lim_{x\to 0^+} f(x) = f(0) = 3 = \lim_{x\to 0^-} f(x) = a+1 \Rightarrow a=2\\
\lim_{x\to 0^-}\frac{f(x)-f(0)}{x} &= \lim_{x\to 0^-}\frac{a\cos(x)+bx+1-(a+1)}{x}\\
&= \lim_{x\to 0^-}\frac{a(\cos(x)-1)+bx}{x} = \lim_{x\to 0^-}\frac{a(\cos(x)-1)}{x}+b\\
&= -a\sin(0)+b\\
&= b=1
\end{aligned}$$

From the above we find : $a=2\wedge b=1$

2 We have :

$$\begin{aligned}
t>0 &\Rightarrow 2t+2>2\\
&\Rightarrow 0<\tfrac{1}{2t+2}<\tfrac{1}{2}\\
&\Rightarrow h(t)\in]0,\tfrac{\pi}{2}[
\end{aligned}$$

$$\begin{aligned}
f'(t) &= (\tan h(t))'\, e^{\tan h(t)}\\
&= h'(t)\tan' h(t) e^{\tan h(t)}\\
&= \frac{-2\pi}{(2+2t)^2}\left(1+\tan^2 h(t)\right) e^{\tan h(t)}
\end{aligned}$$

Corrigé 03 (04 points)

1 By applying the Mean Value Theorem on $[1,x], x>1$ to the function $\ln(t)$.
It is clear that $\ln(t)$ is continuous on $[1,x]$ and differentiable on $]1,x[$, hence the existence of $1<c<x$ such that :

$$\begin{aligned}
\ln(x)-\ln(1) &= \frac{x-1}{c}\\
\text{and as } 1<c<x &\Rightarrow 0<\frac{1}{c}<1\\
\text{which implies that} &\\
\ln x &< x-1
\end{aligned}$$

2 By applying the Mean Value Theorem on $[x,1], 0<x<1$ to the function $\ln(t)$.
It is clear that $\ln(t)$ is continuous on $[x,1]$ and differentiable on $]x,1[$, hence the existence of $x<c<1$ such that :

$$\begin{aligned}
\ln(1)-\ln(x) &= \frac{1-x}{c}\\
&\Rightarrow c(-\ln(x)) = 1-x\\
&\Rightarrow 1-x < -\ln(x)\\
&\Rightarrow x-1 > \ln(x)
\end{aligned}$$

From (1) and (2) plus $\ln(1)=1-1$, we find :

$$\forall x\in]0,+\infty[: \ln(x)\le x-1$$