2023/2024	**Tutorial=1**	Department: Common Core in Mathematics
Analysis 1	Real Numbers	and Computer Science Batna 2-University.

Exercise 1

Given $x, y, z \in \mathbb{R}$, Prove the following inequalities:

1. $|x + y| \leq |x| + |y|$,
2. $||x| - |y|| \leq |x - y|$
3. $\sqrt{x^2 + y^2} \leq |x| + |y|$
4. $\frac{1}{2}(x^2 + y^2) \geq xy$
5. $xy + xz + yz \leq x^2 + y^2 + z^2$

Exercise 2

Show that:

1. $\sqrt{3}$ is irrational
2. for all $(a, b) \in \mathbb{Q} \times \mathbb{Q}^*$, the numbers $a + b\sqrt{3}$ are irrational.
3. $\frac{\ln 3}{\ln 2}$ is irrational.

Exercise 3

Justify whether the following assertions are true or false :

a. The sum, the product of two rational numbers, the inverse of a non-zero rational number is a rational number.

b. The sum or product of two irrational numbers is an irrational.

c. The sum of a rational number and an irrational number is an irrational.

d. The product of a rational number and an irrational number is an irrational.

Exercise 4

Let $x, y \in \mathbb{R}$, show that:

1. $f(x) = E(x)$ is an increasing function.
2. $E(x) + E(y) \leq E(x + y) \leq E(x) + E(y) + 1$
3. $\forall n \in \mathbb{N}^*$, $E(\frac{E(nx)}{n}) = E(x)$

Exercise 5

For each of the following sets, describe the set of all upper bounds for the set :

1. the set of odd integers;
2. $\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$;
3. $\{r \in \mathbb{Q} : r^3 < 8\}$;

4. $\{\sin x : x \in \mathbb{R}\}$

Exercise 6

For each of the sets in (1),(2),(3) of the preceding exercise, find the least upper bound of the set, if it exists.

Exercise 7

Let A, B be two non-empty bounded parts of $\mathbb{R}$. Note $-A = \{-x,\ x \in A\}$. Show that:

1. $\sup(-A) = -\inf(A)$
2. $\inf(-A) = -\sup(A)$
3. If $A \subset B$, then:
$$\begin{cases} \sup(A) & \leq \sup(B) \\ \inf(B) & \leq \inf(A) \end{cases}$$
4. $\sup(A \cup B) = \max(\sup(A), \sup(B))$
5. $\inf(A \cup B) = \min(\inf(A), \inf(B))$

Exercise 8

Determine (if they exist) $\sup, \inf, \max, \min$ of the following sets :

1. $A = [1, 2] \cap \mathbb{Q}$
2. $B = [1, 2[\cap \mathbb{Q}$
3. $C = \left\{ v_n = \dfrac{1}{n+1},\ n \in \mathbb{N} \right\}$
4. $D = \{x \in \mathbb{R} :\ x^2 \leq 3\}$
5. $E = \{x \in \mathbb{R} :\ |x| > 1\}$
6. $F = \{x \in \mathbb{R} :\ |x^2 - 1| > 1\}$

Exercise 9

Let $a, b \in \mathbb{Q}$ such that $a < b$, Show that:

$$\exists\, c \in \mathbb{Q} :\ a < c < b$$