2023/2024 — Analysis 1

Exercise Sheet 5
Differentiable functions

Department: Common Core in Mathematics and Computer Science Batna 2-University.

Exercise 1

By calculating the right and left derivatives of the following functions, determine which one is differentiable at a:

1. $f_1(x) = x^2 + |x+1|,\ a = 1, -1$
2. $f_2(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & si\ x \in \mathbb{R}^* \\ 0, & si\ x = 0 \end{cases} \quad , a = 0$

Exercise 2

Compute the derivatives of the following functions and precise their domains of definition.

1. $\sqrt[4]{x^3}$
2. $\dfrac{x}{x^3+1}$
3. $\dfrac{(1+\sqrt{x})^3}{(x+1)^2}$
4. $x\sqrt[n]{x},\ n \in \mathbb{N}^*$
5. $x \ln|x+1|$
6. $x^2 e^{1/x}$
7. $\sin(\cos(5x))$
8. $a^x\ , a \in \mathbb{R}^{+*}$
9. $(x + \ln x)^n, \quad n \in \mathbb{N}^*$
10. $x^3 \ln(x)$
11. $x^2 e^x$

Exercise 3

Study the differentiability on $\mathbb{R}$ of the following functions:

1. $f(x) = x|x|$
2. $g(x) = \dfrac{1}{2+|x|}$
3. $h(x) = \begin{cases} x^2 \cos\dfrac{1}{x}, & si\ x \neq 0 \\ 0, & si\ x = 0 \end{cases}$

Exercise 4

Compute the nth derivative of the following functions

1. $x\sqrt{x}$
2. $\ln(x)$
3. e^{ax}
4. $\dfrac{1}{1-x}$

Exercise 5

Let a and b be two real numbers and f be a function defined on $[0, +\infty[$ by

$$f(x) = \begin{cases} 2\sqrt{x}, & si\ 0 \leq x \leq 1 \\ ax + b, & si\ x > 1 \end{cases}$$

Find a and b so that f is differentiable on $]0, +\infty[$

Exercise 6

Show that:

1. $\forall x \in]0, \pi[\ : x\cos(x) - \sin(x) < 0$
2. $\forall x \in]0, \dfrac{\pi}{2}[\ : \dfrac{2x}{\pi} < \sin(x) < x$

Exercise 7

In which of the following functions Rolle's theorem is applicable?

1. $x^2 - 2$, on $[-2, 2]$
2. $|x - 2|$, on $[1, 3]$
3. $\sqrt{1 - x^2}$, on $[-1, 1]$
4. $\tan(x)$, on $[\frac{\pi}{4}, \frac{\pi}{3}]$

Exercise 8

Let f be a function defined by

$$f(x) = e^{x^2} \cos(x)$$

Show that for all $a > 0$, the equation $f'(x) = 0$ has at least one solution on $[-a, a]$.

Exercise 9

1. apply the Mean value Theorem for the function $f : x \to x - x^3$ on the segment $[-2, 1]$ and compute the value $c \in]-2, 1[$ appearing in this formula.
2. apply the Mean value Theorem for the function $f : x \to x^2$ on the segment $[a, b]$ and compute the value $c \in]a, b[$ appearing in this formula.

Exercise 10

1. Using the Mean value Theorem, show that: $\frac{1}{1+x} < \ln(1 + x) - \ln(x) < \frac{1}{x}$
2. Compute $\lim_{x \to +\infty} x[\ln(1 + x) - \ln(x)]$
3. Deduce that: $\lim_{x \to +\infty} (1 + \frac{1}{x})^x = e$
4. Compute: $\lim_{x \to -\infty} (1 + \frac{1}{x})^x$.