State Space Approach

Because trend in engineering systems is toward **greater complexity** due, mainly, to the requirement of : * Complex task and ** great accuracy.

⇒ complex systems may be :
- non-linear and/or
- m.i.m.o and/or
- time-varying

Adding to that :
- Meeting increasingly stringent requirements of control systems,
- Increase in system complexity and
- Easy access to large-scale computers

Modern Control Theory (≃ 1960)
based on
concept of state
the concept itself is not new

* Control Theory : Modern Vs. Conventional
Modern ↓ is essentially a time-domain approach
Conventional ↓ Complex Frequency " "

Terminology :

Definir :

- State
- State variables
- " Vector
- " Space
- " " Eqs.

State:

is the smallest set of variables (called state variables) such that their knowledge at $t=t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.
Thus, the state of a dynamic system, at time t, is uniquely determined by the state, at time t_0 and the input for $t \geq t_0$ and it is independent of the state and input before t_0.
Note that, in dealing with linear time-invariant systems, we usually choose the reference time $t_0 = 0$

State Variables:

are the variables making up the smallest set of variables that determine the state of the dynamic system.
If, at least, n variables $x_1, x_2, \dots, x_n$ are needed to completely describe the behavior of a dynamic system (so that once the input is given for $t \geq t_0$ and the initial state, at $t = t_0$, is specified, the future state of the system is completely determined), then such n variables are a set of state variables.

Note that state variables need not to be physically measurable or observable

$\Rightarrow$ Variables not representing physical quantities and those that are neither measurable nor observable can be chosen as state variables $\rightarrow$ An advantage of S.S. methods

Practically speaking, however, it is convenient to choose easily measurable quantities for the state variables, if this is possible at all, because optimal control laws will require the feedback of all state variables with suitable weighting.

State Vector:

is a vector, whose components are the state variables, that determines uniquely the system state $\mathbf{x}(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $u(t)$, for $t \geq t_0$, is specified.

State Space:

the n-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, $\dots$, x_n axis is called state space

$\Rightarrow$

Any state can be represented by a point in the state space

State space Equations:

In s.s representation we are concerned with 3 types of variables, involved in the modeling of dynamic system. These are:

Input Variables (i.v)
Output " (o.v)
State " (s.v)

S.S Representation is not **unique** except that the N^re of s.v is the same for any representation of the given system.

Consider the above system in which the output $y(t)$, for $t \geq t_1$ depends on the value $y(t_1)$ and the input $u(t)$, for $t \geq t_1$

The dynamic system must involve elements that memorize the values of the input for $t \geq t_1$.

Since integrators, in a continuous-time control system serve as memory devices, their outputs can be considered as the variables that define the internal state of the system → state variables

⇒ The number of s.v to completely define the dynamics of the system is equal to the number of integrators involved in the system.

Assume that a m.i.m.o system involve n integrators, also there are r inputs $u_1(t), u_2(t), \dots, u_r(t)$ and m outputs $y_1(t), y_2(t), \dots, y_m(t)$.

Define n outputs of the integrators as state variables $x_1(t), x_2(t), \dots, x_n(t)$.

Then the system may be described by:

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\
\dot{x}_2(t) &= f_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\
&\vdots \\
\dot{x}_n(t) &= f_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t)
\end{aligned}
$$

and the system outputs may be given by:

$$
\begin{aligned}
y_1(t) &= g_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\
y_2(t) &= g_2(\quad \text{"} \quad) \\
&\vdots \\
y_m(t) &= g_m(\quad \text{"} \quad)
\end{aligned}
$$

If we define

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \mathbf{f}(\mathbf{x},\mathbf{u},t) = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots u_r; t) \\ f_2(\qquad \text{"} \qquad) \\ \vdots \\ f_n(\qquad \text{"} \qquad) \end{bmatrix}$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix}; \quad \mathbf{g}(\mathbf{x},\mathbf{u},t) = \begin{bmatrix} g_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots u_r; t) \\ g_2(\qquad \text{"} \qquad) \\ \vdots \\ g_m(\qquad \text{"} \qquad) \end{bmatrix}$$

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

The above eqs become

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \quad \text{... state eq. ... (1)}$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \quad \text{... output eq. ... (2)}$$

If vector functions $\mathbf{f}$ and/or $\mathbf{g}$ involve time t explicitly, then the system is called a time-varying system.

If eqs (1) and (2) are linearized about the operating state, then we have the following linearized state eq. and output eq.

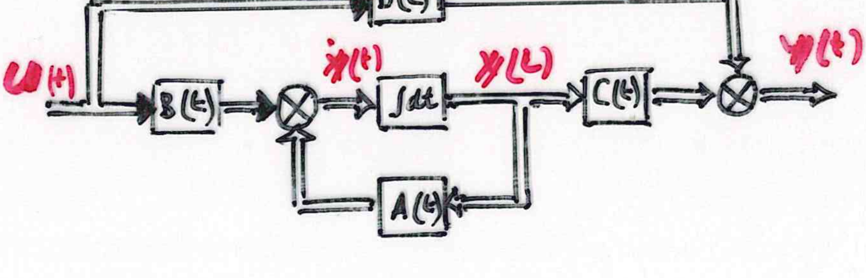
$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t)$$

where

$A(t)$ ... state matrix $\qquad$ $B(t)$ ... input matrix

$C(t)$ ... output " $\qquad$ $D(t)$ ... Direct Transmission matrix

Block-diagram representation is as follows:



If $\mathbb{f}$ and $\mathbb{g}$ do not involve time t explicitly, then the system is called a time-invariant system. Eqs ① and ② can be simplified to

$$\dot{\mathbb{x}}(t) = \mathbb{f}(\mathbb{x}, \mathbb{u})$$
$$\mathbb{y}(t) = \mathbb{g}(\mathbb{x}, \mathbb{u})$$

and then linearized about the operating state as follows:

$$\dot{\mathbb{x}}(t) = A\,\mathbb{x}(t) + B\,\mathbb{u}(t)$$
$$\mathbb{y}(t) = C\,\mathbb{x}(t) + D\,\mathbb{u}(t)$$

Correlation between TFs & S.S Eqs:

S.S. eqs $\longrightarrow$ TF

TF is given as $G(s) = \dfrac{Y(s)}{U(s)}$ ...⓪ for i.c. = 0

S.S eqs

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Big|\ \mathcal{L} \quad \Rightarrow$$

$$sX - X(0) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s) \quad \text{...} ①$$

Assuming
$X(0) = 0$ as i.c. = 0

$$sX - AX(s) = BU(s)$$

or

$$(sI - A)X(s) = BU(s)$$
$$X(s) = (sI - A)^{-1}BU(s) \quad \text{--} ②$$

② in ①

$$Y(s) = [C(sI - A)^{-1}B + D]\,U(s) \quad \text{--} ③$$

Comparing ⓪ to ③ $\Rightarrow$

$$G(s) = C(sI - A)^{-1}B + D$$

Transfer Matrix: m.i.m.o

Define

$$\mathbb{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \quad \mathbb{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix} \qquad \mathbb{Y}(s) = \underbrace{G(s)}_{\text{Transfer Matrix } (m \times r)}\,\mathbb{U}(s)$$

State-Space Representation of Dynamic Systems

system dynamic: O. diff. eqs. in which t is the independent variable

n^{th}-order diff. eq.
↓ S.S. representation
1^{st}-order vector-matrix diff. eq.
(State Eq.)

Forcing Function:
1) does not involve derivative terms
2) " involve " "

1) u $\quad \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \dots + a_{n-1}\dot{y} + a_n y = u$

La connaissance des $y(0), \dot{y}(0), \dots, \overset{(n-1)}{y}(0)$ ensemble avec la sortie $u(t)$ pour $t \geq 0$ détermine complètement le comportement futur du système

Définissons (choix à discussion)

$$x_1 = y \qquad \dot{x}_1 = x_2$$
$$x_2 = \dot{y} \qquad \dot{x}_2 = x_3$$
$$\vdots \qquad \vdots$$
$$x_n = \overset{(n-1)}{y} \qquad \dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_n x_1 - \dots - a_n x_n + u$$

alors

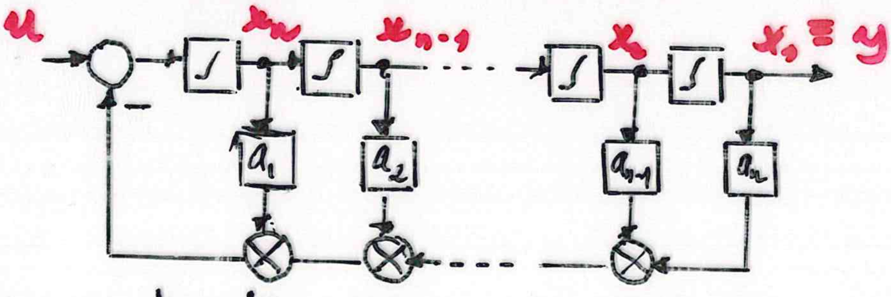
$$\dot{X} = AX + BU \qquad \text{state eq.}$$

où

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \dots & & -a_1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$y = [1, 0, \dots, 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow y = CX \quad \text{où } C = [1, 0, 0, \dots, 0]$$

output eq.



2) $b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \dots + b_{n-1}\dot{u}$

4) $b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \dots + b_{n-1} \dot{u}$

si
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \dots - a_1 x_n + \underbrace{b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \dots + b_{n-1} \dot{u}}_{\text{Forcing F.}}$$
et
$$y = x_1$$

Ce choix peut ne pas aboutir à une

Solution Unique

le problème peut être résolu en adoptant un choix permettant d'éliminer les dérivatives de u au sein de l'éq. d'état.

L'une des méthodes possible est le choix des variables d'état comme :

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u \qquad = \dot{x}_1 - \beta_1 u$$
$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u \qquad = \dot{x}_2 - \beta_2 u$$
$$\vdots$$
$$x_n = \overset{(n-1)}{y} - \beta_0 \overset{(n-1)}{u} - \beta_1 \overset{(n-2)}{u} - \dots - \beta_{n-2} \dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u$$

où

$$\beta_0 = b_0$$
$$\beta_1 = b_1 - a_1 \beta_0$$
$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0$$
$$\beta_3 = b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0$$
$$\beta_n = b_n - a_1 \beta_{n-1} - \dots - a_{n-1} \beta_1 - a_n \beta_0$$

Un tel choix permet de garantir l'existence d'une solution unique de l'éq. d'état. à noter que ceci n'est pas le seul choix

Variables d'état en eq.

$$\dot{x}_1 = x_2 + \beta_1 u$$
$$\dot{x}_2 = x_3 + \beta_2 u$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n + \beta_{n-1} u$$
$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \dots - a_1 x_n + \beta_n u$$!

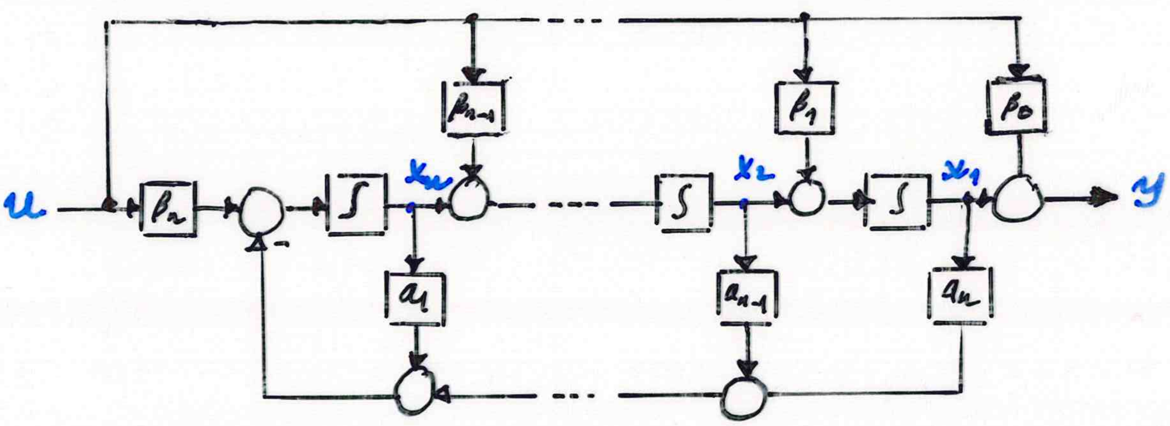
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix} + \begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_{n-1}\\ \beta_n\end{bmatrix} u$$

$$y = [1 \quad 0 \quad \cdots \quad 0]\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} + \beta_0 u$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Noter que les derivatives du u ont un effet uniquement sur les elements de B

Block diagram realization :



$\equiv$

$$FT \equiv \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$\exists$ plusieurs autres repr. dans s.s telles que :

controllable canonical form
observable "
Jordan "
Diagonal "

Ex

$$\frac{Y(s)}{U(s)} = \frac{160(s+4)}{s^3 + 18 s^2 + 192 s + 640}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -640 & -192 & -18\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 160\\ -2240\end{bmatrix}$$

$$y = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Nonuniqueness of set of state variables

Provided that, for every set of values $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n$, there corresponds a unique set of values $x_1, x_2, \dots, x_n$ and vice versa. Thus, if $\underline{x}$ is a state vector, then $\hat{\underline{x}}$, where

$$\hat{\underline{x}} = P\,\underline{x}$$

is also a state vector, provided the matrix P is nonsingular

Different state vectors convey the same information about the system behaviour.