CHAPTER 1
VECTOR ANALYSIS FOR ELECTROMAGNETISM

1.1. INTRODUCTION

Electromagnetic field theory is the study of forces between charged particles resulting in energy conversion or signal transmission and reception. These forces vary in magnitude and direction with time and throughout space so that the theory is a heavy user of vector, differential, and integral calculus. This chapter presents a brief review that highlights the essential mathematical tools needed throughout the text. We isolate the mathematical details here so that in later chapters most of our attention can be devoted to the applications of the mathematics rather than to its development.

1.2. REPRESENTATION OF A POINT IN SPACE

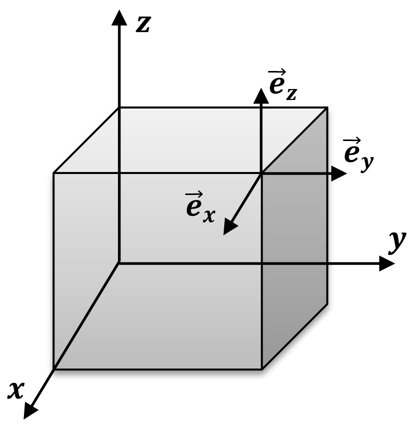
1.2.1 Cartesian Coordinate System

In a three-dimensional space, a point can be located as the intersection of three surfaces. The three surfaces are described by x = constant, y = constant, z = constant. If these three surfaces (in fact, their normal vectors) are **mutually** perpendicular to each other, we call them **orthogonal** coordinate system ***Oxyz***. In Cartesian coordinate system, a point is located by the intersection of the following three surfaces:

- A plane parallel to the ***y-z*** plane (x = constant, normal to the x axis, unit vector $\vec{e}_x$)
- A plane parallel to the ***x-z*** plane (y = constant, normal to the y axis, unit vector $\vec{e}_y$)
- A plane parallel to the ***x-y*** plane (z = constant, normal to the z axis, unit vector $\vec{e}_z$)

This is shown in the figure below.

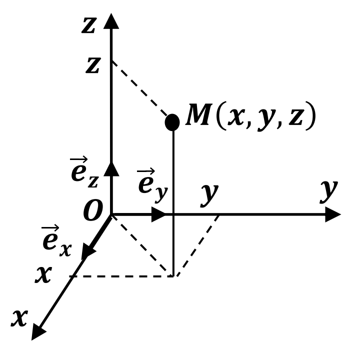
The base vectors $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ meet the following relations:



$$\vec{e}_x \Lambda \vec{e}_y = \vec{e}_z$$
$$\vec{e}_y \Lambda \vec{e}_z = \vec{e}_x$$
$$\vec{e}_z \Lambda \vec{e}_x = \vec{e}_y$$

A position vector is defined as a vector that symbolises either the position or the location of any given point M with respect to any arbitrary reference point like the origin O of an orthogonal coordinate system ***Oxyz***. The direction of the position vector always points from the origin of that vector towards a given point. This position vector is expressed as:

$$\vec{r} = \overrightarrow{OM} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$$



The magnitude or modulus of $\vec{r}$ is calculated by:

$$r^2 = OM^2 = x^2 + y^2 + z^2$$

The change in the position vector from a point M to a point M' is known as the displacement vector. The displacement of an object can also be defined as the vector distance between the initial point and the final point. This displacement vector is given by:

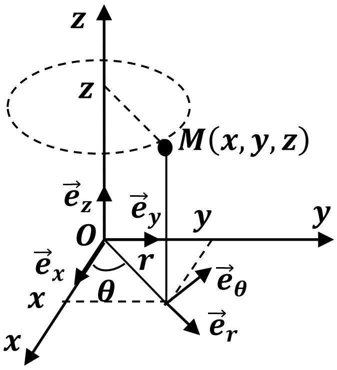
$$d\vec{r} = d\overrightarrow{M} = dx\vec{e}_x + dy\vec{e}_y + dz\vec{e}_z$$

His modulus is such that:

$$(d\vec{r})^2 = \overrightarrow{dM}^2 = (dx)^2 + (dy)^2 + (dz)^2$$

1.2.2. Cylindrical coordinate system

In cylindrical coordinate systems a point M is the intersection of the following three surfaces as shown in the following figure.



- A circular cylindrical surface $r = constant$
- A half-plane containing the z-axis and making angle $\varphi = constant$ with the xz-plane
- A plane parallel to the xy-plane at $z = constant$.

We define a position vector M in the base vectors $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_z$ by its coordinate z and by the polar coordinates r, θ of its project on the Oxy plane. This position vector is expressed as follow:

$$\overrightarrow{OM} = r\vec{e}_r + z\vec{e}_z \begin{cases} x = rcos\theta \\ y = rsin\theta \end{cases}$$

$$\overrightarrow{OM}^2 = r^2 + z^2$$

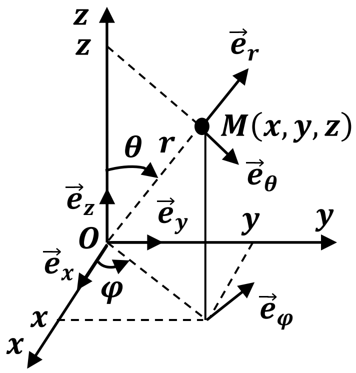
Its displacement vector is given by:

$$\overrightarrow{dM} = dr\vec{e}_r + rd\theta\vec{e}_\theta + dz\vec{e}_z$$

$$\left(d\overrightarrow{M}\right)^2 = dr^2 + (rd\theta)^2 + dz^2$$

1.2.3. Spherical coordinate system

A point M in spherical coordinates is located at the intersection of the following three surfaces:



- A spherical surface centered at the origin with a radius $r = constant$ (sphere of constant r)
- A right circular cone with its apex at the origin, its axis coinciding with the $+z$ axis, and having a half-angle $\theta = constant$ (cone of constant θ)
- A half-plane containing the z-axis and making an angle $\varphi = constant$ with the xz-plane (plane of constant φ)

This is shown below.

We define a position vector M in the base vectors $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\varphi$ by its length $r = OM$ and the two angles θ, φ. This position vector is expressed as follow:

$$\overrightarrow{OM} = r\vec{e}_r \begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\overrightarrow{OM}^2 = r^2$$

its displacement vector is given by:

$$\overrightarrow{dM} = dr\vec{e}_r + r\sin\theta d\varphi\vec{e}_\varphi + rd\theta\vec{e}_\theta$$

$$\left(\overrightarrow{dM}\right)^2 = dr^2 + (r\sin\theta d\varphi)^2 + (rd\theta)^2$$

1.3. LINE INTEGRALS OF VECTOR FIELD ALONG A PARAMETERIZED CURVE

Line integrals are also called path or contour or curve integrals. A line integral is an integral in which the function to be integrated is determined along a curve in the coordinate system. The function which is to be integrated may be either a scalar field or a vector field. We can integrate a scalar-valued function or vector-valued function along a curve. The value of the line integral can be evaluated by adding all the values of points on the vector field. One interpretation of the line integral of a vector field is the amount of work that a force field does on a particle as it moves along a curve.

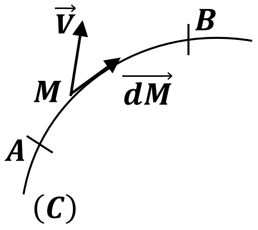
1.3.1. Elementary displacement

Let a vector field $\vec{V}(M)$ and a vector element of line length $\overrightarrow{MM'} = \overrightarrow{dM}$ noted also $\vec{dl}$. The elementary displacement dC of $\vec{V}(M)$ is expressed as:

$$dC = \vec{V}.\overrightarrow{dM}$$

1.3.2. Path displacement

We consider a path AB on a curve (C). It is advisable to fix the direction of travel on the curve (C). By definition, the circulation of a vector denoted $\vec{V}$ around an open curve (C) is given the following line integral:



$$C_{AB} = \int_{\widehat{AB}} dC = \int_{\widehat{AB}} \vec{V}.\overrightarrow{dM}$$

For a closed curve the same circulation is given by:

$$C = \oint \vec{V}.\overrightarrow{dM}$$

For example, if the vector field is a field of forces, the circulation is nothing but work.

1.3. FLUX OF THE VECTOR

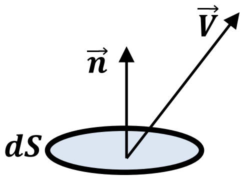
The flux ϕ of $\vec{V}$ through an open surface (S) that is not limited by a certain volume is given by:

$$\phi = \iint_S \vec{V}.\vec{n}dS$$

1.4.1. Elementary flux

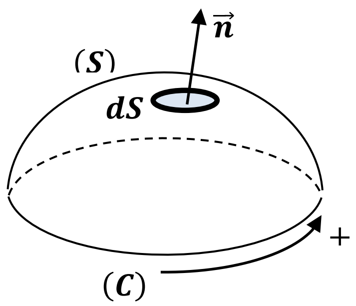
The flux of $\vec{V}$ through an elementary oriented surface $\overrightarrow{dS}$ is given by:

$$d\phi = \vec{V}.\overrightarrow{dS} = \vec{V}.\vec{n}dS$$



where $\vec{n}$ is the unit vector normal to the elementary surface dS, which should be properly oriented, taking into account the conventions that will be specified.

1.4.2. Flux through an open surface



Let (C) be the contour on which the surface (S) rests. Once (C) oriented, the direction of the unit vector $\vec{n}$ is defined by the corkscrew rule (direction in which the corkscrew advances when it is turned in the positive direction chosen on (C)). We then have

$$\phi = \iint_S d\phi = \iint_S \vec{V}.\vec{n}dS$$

1.4.2. Flux through a closed surface

The flux of $\vec{V}$ through a closed surface (S), delimited by a certain volume, is given by:

$$\phi = \oiint_S \vec{V}.\vec{n}\,dS = \oiint_S f(r)\,dS = 4\pi r^2 f(r)$$

$\vec{n}$ is the normal external to an element of the surface denoted dS.

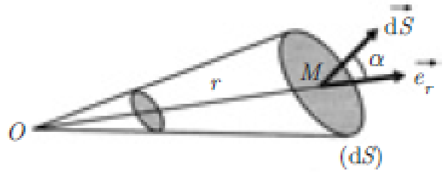
Example: Spherically symmetric field

Calculate the flux of the vector $V(M) = f(r)\vec{e}_r$ through a sphere with center O and radius r. let ***f(r)*** is constant when moving on the sphere.
We simply have:

$$\phi = \oiint_S \vec{V}.\vec{n}\,dS = \oiint_S f(r)\,dS = 4\pi r^2 f(r)$$

1.5. SOLID ANGLE

1.5.1. Elementary solid angle



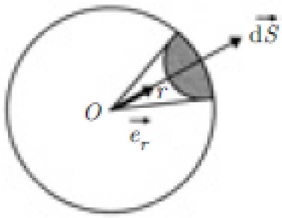
By definition, the solid angle $d\Omega$ under which we see an elementary oriented surface $\overrightarrow{dS}$ from a given point $\boldsymbol{O}$ is:

$$d\Omega = \frac{\overrightarrow{dS}.\vec{e}_r}{r^2} = \frac{dScos\alpha}{r^2}$$

In the case where the element dS is taken on the sphere with center $\boldsymbol{O}$ and radius $\boldsymbol{r}$, we simply have:

$$d\Omega = \frac{dS}{r^2}\vec{n}.\vec{e}_r = \frac{dS}{r^2}$$

1.5.2. Solid angle unit

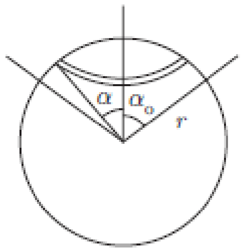


The solid angle through which all space can be seen is:

$$\Omega = \frac{1}{r^2}\oiint_S dS = \frac{4\pi r^2}{r^2} = 4\pi\ steradiant$$

For a half space: $\boldsymbol{\Omega = 2\pi\ stéradiants}$

for a cone with half angle at sommet $\boldsymbol{\alpha_0}$:



$$dS = 2\pi r sin\alpha r d\alpha = 2\pi r^2 sin\alpha d\alpha$$

$$\Omega = \oiint_S \frac{dS}{r^2} = \int_0^{\alpha_0} 2\pi sin\alpha d\alpha$$

$$\Omega = 2\pi(1 - cos\alpha_0)$$

1.6. VECTOR OPERATORS

1.6.1. Gradient

The operator $\overrightarrow{grad}$ (or also denoted by $\vec{\nabla}$, polar vector operator nabla) associated with a scalar function ***f (x, y, z)*** is a vector with components $(df/dx, df/dy, df/dz)$.

as:

$$df = \frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz$$

One can deduced:

$$df = \overrightarrow{grad}f.\overrightarrow{dM}$$

the relationship we use to define the gradient in any coordinate system.

1.5.1.1. Gradient properties

A level surface is the locus of points where the scalar function $\boldsymbol{f}$ is a constant. Level surfaces are defined by: ***f (x,y,z) = cte***.

**** Gradient direction:**

Let a level surface $\boldsymbol{f}\ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{\lambda f}$.

For a point $\boldsymbol{M}$ moving on this surface, we have:

$$\boldsymbol{df} = \overrightarrow{\boldsymbol{gradf}}.\overrightarrow{\boldsymbol{dM}} = \mathbf{0}$$

The vactor $\overrightarrow{\boldsymbol{gradf}}$ is then normal to the level surface.
Consider now two points $\boldsymbol{M_1}$, $\boldsymbol{M_2}$ on two neighboring level surfaces $\boldsymbol{f} = \boldsymbol{\lambda_1}$ and $\boldsymbol{f} = \boldsymbol{\lambda_2} > \boldsymbol{\lambda_1}$. We have :

$$\boldsymbol{df} = \boldsymbol{\lambda_2} - \boldsymbol{\lambda_1} = \overrightarrow{\boldsymbol{gradf}}.\overrightarrow{\boldsymbol{M_1M_2}} > 0$$

The vector $\overrightarrow{\boldsymbol{gradf}}$ is oriented in the direction of increasing values of $\boldsymbol{f}$.

**** Gradient circulation:**

$$\boldsymbol{C}_{\widehat{\boldsymbol{AB}}} = \int_{\widehat{\boldsymbol{AB}}} \overrightarrow{\boldsymbol{grad}}\,\boldsymbol{f}.\overrightarrow{\boldsymbol{dM}} = \int_{\boldsymbol{f}(\boldsymbol{A})}^{\boldsymbol{f}(\boldsymbol{B})} \boldsymbol{df}$$

$$\int_{\widehat{\boldsymbol{AB}}} \overrightarrow{\boldsymbol{grad}}\,\boldsymbol{f}.\overrightarrow{\boldsymbol{dM}} = \boldsymbol{f}(\boldsymbol{B}) - \boldsymbol{f}(\boldsymbol{A})$$

It is equal to the variation of the function $\boldsymbol{f}$ and does not depend on the path taken. We say that the circulation of the vector $\overrightarrow{\boldsymbol{gradf}}$ is conservative. This relationship sometimes makes it easier to calculate the circulation of a vector along the path.

In the particular case of a closed path, we have:

$$\boldsymbol{C}_{\widehat{\boldsymbol{AB}}} = \oint \overrightarrow{\boldsymbol{grad}}\,\boldsymbol{f}.\overrightarrow{\boldsymbol{dM}} = \mathbf{0}$$

**** When is a vector field $\vec{V}$ a gradient?**

- When the value of its circulation does not depend on the path followed.
- When $\vec{\boldsymbol{\nabla}}\boldsymbol{\Lambda}\vec{\boldsymbol{V}} = \mathbf{0}$
- We show that, for a vector $\vec{\boldsymbol{V}}$ to be a gradient field, it is necessary and sufficient that the cross partial derivatives of its components are equal two by two, i.e.:

$$\frac{\partial \boldsymbol{V_x}}{\partial \boldsymbol{y}} = \frac{\partial \boldsymbol{V_y}}{\partial \boldsymbol{x}}, \frac{\partial \boldsymbol{V_y}}{\partial \boldsymbol{z}} = \frac{\partial \boldsymbol{V_z}}{\partial \boldsymbol{y}}, \frac{\partial \boldsymbol{V_z}}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{V_x}}{\partial \boldsymbol{z}}$$

1.6.2. Divergence

The operator $\overrightarrow{d\iota v}$ (or $\vec{\boldsymbol{\nabla}}.$) associated with a vector $\vec{\boldsymbol{V}}$ is the scalar product of $\vec{\boldsymbol{\nabla}}$ by this vector $\vec{\boldsymbol{V}}$ such as:

$$\boldsymbol{div\,\vec{V} = \vec{\nabla}.\vec{V}}$$

1.6.2.1. Divergence properties

** Divergence et flux d'un vecteur

By definition, the flux differential $\boldsymbol{d\phi}$ of $\vec{\boldsymbol{V}}$ through an elementary volume $\boldsymbol{d\tau}$ is related to the divergence of $\vec{\boldsymbol{V}}$ by:

$$\boldsymbol{d\phi = div\,\vec{V} d\tau}$$

The divergence of a vector field represents the flow of this vector out of the unit volume. We can deduce that:

$$\boldsymbol{\phi = \oiint_{(S)} \vec{V}.\overrightarrow{dS} = \iiint_{(\tau)} div\vec{V} d\tau}$$

This formula, known as the Green-Ostrogradsky formula, sometimes facilitates the calculation of the flux of a vector through a closed surface.

** Physical meaning of divergence

The divergence of a vector field $\vec{\boldsymbol{V}}$ expresses the volume flux density of this vector through a given volume, as explained by Green's theorem above.

1.6.3. Rotational or curl

The operator $\overrightarrow{\boldsymbol{rot}}$ (or $\vec{\boldsymbol{\nabla}}\boldsymbol{\Lambda}$) associated with a vector $\vec{\boldsymbol{V}}$ is the vector product of $\vec{\boldsymbol{\nabla}}$ cross this vector:

$$\boldsymbol{\overrightarrow{rot}\,\vec{V} = \overrightarrow{curl}\,\vec{V} = \vec{\nabla}\Lambda\vec{V}}$$

1.6.3.1. Curl properties

** Physical meaning of the rotational

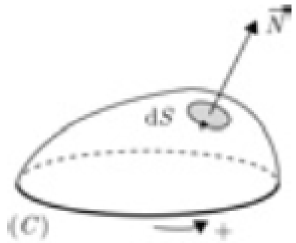
Consider the flow of water in a river whose particle velocity is considered as a vector field whose expression is:

$$\boldsymbol{\vec{v} = y^2\vec{e}_x + 0\vec{e}_y}$$

The velocity of the particles varies only as a function of $\boldsymbol{y}$. If we place a twig of wood in the fluid, we observe a rotation effect which is exerted on this twig since the movement takes place in the direction of $\boldsymbol{x}$ and the velocity only varies with $\boldsymbol{y}$. It is as if the twig of wood is subjected to a torque of rotation and therefore it will start to turn toward itself. The amount of rotation will be ± large. The rotation of the twig can be in one direction or in the other. This rotation effect must be represented by a vector which measures this amount of rotation and determines the direction of rotation. It can only be represented by a vector called the rotational or curl.

**** Rotational and circulation of vector**

By definition, the circulation differential of $\vec{V}$ on a closed contour (C) is related to the rotational of $\vec{V}$ by:



$$dC = \overrightarrow{rot}\,\vec{V}.\overrightarrow{dS}$$

where dS is an element of any surface (S) that rests on (C).
This relationship allows us to define the coordinate of the rotational in any direction of the unit vector $\vec{n}$. We can deduce:

$$C = \oint_{(C)} \vec{V}.\overrightarrow{dM} = \iint_{(S)} \overrightarrow{rot}\,\vec{V}.\overrightarrow{dS}$$

This formula, called Stokes formula, sometimes facilitates the calculation of the circulation of a vector along a closed contour.

1.7. LAPLACIAN

The Laplacian operator (denoted Δ) is defined by:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

It can be applied to a scalar function:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

or to a vector :

$$\Delta\vec{V} = \frac{\partial^2\vec{V}}{\partial x^2} + \frac{\partial^2\vec{V}}{\partial y^2} + \frac{\partial^2\vec{V}}{\partial z^2} = \Delta V_x\vec{e}_x + \Delta V_y\vec{e}_y + \Delta V_z\vec{e}_z$$

The interest of all these vector operators is on the one hand, to allow a concise writing of the so-called "local" equations (example: Maxwell's equations), and on the other hand, to facilitate the calculations, due to the vector relations which exist between them, and to the integral transformations that they allow to perform.