Faculty of Mathematics and Computer Science

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Chapter 03: Binary Relations on a Set

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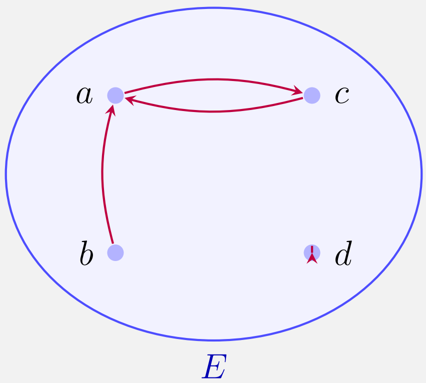

$R \subseteq E \times E$: a binary relation on E

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1 Basic Definitions

Binary relations are a fundamental concept in mathematics and computer science. They allow us to define relationships between the elements of two sets (or even a single set).

Definition

Let E be a set. A binary relation on E is a subset of the Cartesian product $E \times E$. In other words, a binary relation R on E is a set of ordered pairs (a, b), where $a, b \in E$. We then write:

$$R \subseteq E \times E$$

Remark

Each element of the relation R is a pair (a, b) where a is related to b. We can also describe this relation more intuitively: we say that a is related to b if and only if $(a, b) \in R$.
This can be written as:

$$a\,R\,b \quad \text{or simply} \quad a \sim b,$$

depending on the context.

Example

1. Let $E = \mathbb{R}$ and define the relation R by:

$$aRb \iff a^2 = b^2.$$

   In this case, two real numbers a and b are related if they have the same absolute value.

2. Let $E = \mathbb{Z}^*$ (the set of nonzero integers) and define the relation R by:

$$aRb \iff \exists k \in \mathbb{Z}^* \text{ such that } b = ka.$$

   This means that b is a multiple of a.

2 Properties of Binary Relations

A binary relation can have several important properties that characterize its behavior. The main ones are the following:

Reflexivity: The relation R is reflexive if, for any element $a \in E$, we have aRa. In other words, each element is related to itself:

$$\forall a \in E,\ aRa.$$

Symmetry: The relation R is symmetric if:

$$\forall a, b \in E,\ aRb \Rightarrow bRa.$$

Antisymmetry: The relation R is antisymmetric if:

$$\forall a, b \in E,\ (aRb) \wedge (bRa) \Rightarrow a = b.$$

Transitivity: The relation R is transitive if:

$$\forall a, b, c \in E,\ (aRb) \wedge (bRc) \Rightarrow aRc.$$

2.1 Order Relation

Definition

A binary relation R on a set E is called an order relation if and only if it satisfies the following properties:

- R is reflexive,
- R is antisymmetric,
- R is transitive.

2.1.1 Total and Partial Order Relations

Definition

Let R be an order relation on a set E.

- The relation R is called a total order on E if:

$$\forall a, b \in E,\ (aRb) \text{ or } (bRa).$$

- The relation R is called a partial order on E if:

$$\exists a, b \in E \text{ such that } (a \not R\, b) \text{ and } (b \not R\, a).$$

Example

- Let $E = \mathbb{R}$ and define $aRb \iff a \leq b$. Then R is a total order relation on $\mathbb{R}$.
- Let $E = \mathbb{Z}^*$ and define $aRb \iff \exists k \in \mathbb{Z}^*$ such that $b = ka$. Then R is a partial order relation on $\mathbb{Z}^*$.

2.2 Equivalence Relation

Definition

A binary relation R on a set E is called an equivalence relation if and only if it satisfies the following three properties:

- R is reflexive,
- R is symmetric,
- R is transitive.

2.2.1 Equivalence Class and Quotient Set

Definition

Let R be an equivalence relation on a set E and let $a \in E$. The equivalence class of a (denoted $[a]$ or $\dot{a}$) is the set of all elements of E that are equivalent to a according to the relation R:

$$[a] = \dot{a} = \{ b \in E \mid bRa \}.$$

Definition

Let E be a set and R an equivalence relation on E. The quotient set of E by R, denoted E/R, is the set of all equivalence classes:

$$E/R = \{ [a] \mid a \in E \}.$$

Example

Let $E = \mathbb{R}$ and define the relation R by

$$aRb \iff a^2 - b^2 = a - b.$$

1. Show that R is an equivalence relation on $\mathbb{R}$.
2. Find the equivalence classes of 0, 1, and 2.
3. Determine the quotient set E/R.