University of Batna 2

Faculty of Mathematics and Computer Science

Departement of Mathematics

Chapter 01: Real Numbers $\mathbb{R}$

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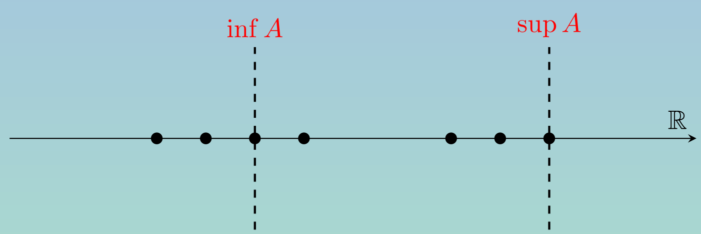

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Chapitre 1 Usual sets of numbers

Usual sets of numbers

Among the different types of numbers:

- $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers
- $\mathbb{Z} = (\mathbb{N}) \cup (-\mathbb{N}) = \{\dots, -2, -1, 0, 1, 2, \dots\}$ set of relative integers
- $\mathbb{Q} = \{\frac{p}{q}$ where $p \in \mathbb{Z}, q \in \mathbb{N}^*$ and p, q are coprime $p \wedge q = 1\}$ set of rational numbers.
- The set of real numbers, denoted $\mathbb{R}$ was introduced to complete the set $\mathbb{Q}$ of rational numbers we say that x is a real number if and only if either $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}, x$ is said to be irrational number $x \in (\mathbb{R} - \mathbb{Q})$.
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- These private sets of 0 are respectively noted by $\mathbb{N}^*, \mathbb{Z}^*, \mathbb{Q}^*, \mathbb{R}^*$.

Axiomatic definition of real numbers

The set $\mathbb{R}$ equipped with two internal laws, addition $(+)$, multiplication $(\times)$ or $(.)$ and a relation of comparison of the elements of $\mathbb{R}$ noted $(<)$ (less than or equal) satisfies the following axioms:

$(\mathbb{R}, +)$ is a commutative group, that's to say

- For all $x, y, z \in \mathbb{R}, (x + y) + z = x + (y + z)$ (associativity of addition)
- For all $x, y \in \mathbb{R}, x + y = y + x$ (commutativity of addition)
- For all $x \in \mathbb{R}, x + 0 = x$ (0 neutral element)
- For all $x \in \mathbb{R}$, there exists an element $-x \in \mathbb{R}$ such that $x + (-x) = 0$.

$(\mathbb{R}^*, \ .)$ is a commutative group, that's to say

- For all $x, y, z \in \mathbb{R}, (x.y).z = x.(y.z)$ (associativity of multiplication)
- For all $x, y \in \mathbb{R}, x.y = y.x$ (commutativity of multiplication)
- For all $x \in \mathbb{R}, x.1 = x$ (1 neutral element)
- For all $x \in \mathbb{R}^*$, there exists an element $x^{-1} = \frac{1}{x} \in \mathbb{R}$ such that $x.x^{-1} = 1$.

Multiplication is distributive with respect to addition, i.e:

- For all x, y and $z \in \mathbb{R}; x.(y + z) = x.y + x.z$

$(\mathbb{R}, \leq)$ is totally ordered, i.e:

- For all $x \in \mathbb{R}$, we have: $x \leq x$ (Reflexivity)
- For all $x, y \in \mathbb{R}$ we have: if $x \leq y$ and $y \leq x$ then $x = y$ (Antisymmetry)
- For all $x, y, z \in \mathbb{R}$ we have: if $x \leq y$ and $y \leq z$ then $x \leq z$ (Transitivity)
- For all $x, y \in \mathbb{R}$ we have: $x \leq y$ or $y \leq x$ (Total order)

Relation $(\leq)$ is compatible with addition and multiplication, i.e:

- For all $x, y, x', y' \in \mathbb{R}$ checking ($x \leq y$ and $x' \leq y'$) then $x + x' \leq y + y'$ (compatibility of the addition)

- For all $x, y, x', y' \in \mathbb{R}_+$ checking ($x \leq y$ and $x' \leq y'$) then $x.x' \leq y.y'$ (compatibility of multiplication)
- The relation ($x \leq y$ and $x \neq y$) for all $x, y \in \mathbb{R}$ means that $x < y$
- A real number x is said to be strictly positive if $0 < x$, the set of strictly positive real numbers is denoted by $\mathbb{R}_+^* =]0, +\infty[$
- A real number x is said to be strictly negative if $x < 0$, the set of strictly negative real numbers is denoted by $\mathbb{R}_-^* =]-\infty, 0[$
- For all $x, y \in \mathbb{R}$, we write $x - y$ instead of $x + (-y)$ and xy instead $x.y$

Notion of interval in $\mathbb{R}$

- A non-empty part E of $\mathbb{R}$ is an interval if and only if, for all $x, y \in \mathbb{R}$ verifies $x < y$ there exists $z \in E$ such that $x < z < y$.
- If a, b and x_0 denote real numbers such that $a < x_0 < y$ the open intervals of $\mathbb{R}$ are $]a, b[,]a, +\infty[,]-\infty, a[$ and $\mathbb{R} =]-\infty, +\infty[$, the intervals which are neither closed nor open are $[a, b[$ and $]a, b]$.
- If $a = b$; $[a, a] = \{a\}$ and $]a, a[= \emptyset$.
- a and b are called the limits of the interval and $(b - a)$ its length.
- The total order relation $(\leq)$ allows to define the absolute value function in $\mathbb{R}$.

Absolute value in $\mathbb{R}$

Definition

The absolute value in $\mathbb{R}$ is a function denoted $|.|$ defined from $\mathbb{R}$ to $\mathbb{R}$ by: for all $x \in \mathbb{R}, |x| = \max(x, -x)$, i.e: $|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$

Proposition

For all $x, y \in \mathbb{R}$, we have:

- $|x| = 0 \Leftrightarrow x = 0$
- $|xy| = |x||y|$ in particular $|x|^2 = |x^2| = x^2$
- $|x| \leq \alpha \Leftrightarrow -\alpha \leq x \leq \alpha$
- $|x + y| \leq |x| + |y|$
- $||x| - |y|| \leq |x - y|$
- $|x| \geq \alpha \Leftrightarrow x \in]-\infty, \alpha] \cup [\alpha, +\infty[$

Bounded part in $\mathbb{R}$

Definition

Let E be a non-empty subset of $\mathbb{R}$, we say that:

- $M \in \mathbb{R}$ is an upper bound of E if, **for all $x \in E, x \leq M$.**

The smallest of the majorants when it exists is called the **upper bound** of E. It is a maximum if it belongs to E. it is noted $\sup E$ or $\max E$.

- $m \in \mathbb{R}$ is a lower bound of E if, **for all $x \in E, m \leq x$.**

The largest of the lower bounds when it exists is called the **lower bound** of E, it is noted $\inf E$. It is a minimum if it belongs to E and it is noted $\min E$.

Remark

$\sup E$ and $\inf E$ when they exist are unique and $[\inf E, \sup E]$ is the smallest closed interval containing E.

Upper Bound Axiom

Every non-empty part of $\mathbb{R}$ and bounded above admits an upper bound.

Remark

Every non-empty part of $\mathbb{R}$ and bounded below has a lower bound.

Proposition

Let E be a bounded subset of $\mathbb{R}$, M and $m \in \mathbb{R}$, then:

- $M = \sup \mathrm{E} \Leftrightarrow \begin{cases} \text{For all } x \in E, x \leq M \\ \text{For all } \varepsilon > 0, \text{there exists } x_0 \text{ such as } M - \varepsilon < x_0 \end{cases}$

- $m = \inf \mathrm{E} \Leftrightarrow \begin{cases} \text{For all } x \in E, m \leq x \\ \text{For all } \varepsilon > 0, \text{there exists } x_0 \text{ such as } x_0 < m + \varepsilon \end{cases}$

Examples:

- $E = \{-1, 0, 1\}, \min \mathrm{E} = -1, \max \mathrm{E} = 1$
- $E = [0, 1], \min \mathrm{E} = 0, \max \mathrm{E} = 1$
- $E = [0, 1[, \min \mathrm{E} = 0, \sup \mathrm{E} = 1$
- $E =]0, 1], \inf \mathrm{E} = 0, \max \mathrm{E} = 1$
- $E =]0, 1[, \inf \mathrm{E} = 0, \sup \mathrm{E} = 1$

Archimedes' Axiom

Proposition

For all $x, y \in \mathbb{R}^*$, there exists $n \in \mathbb{N}^*$ such that $nx > y$ ($\mathbb{R}$ is Archimedean).

Remark

$\mathbb{N}$ is unbounded above, therefore $\mathbb{Z}$ is unbounded (it suffices to take $x = 1$ in the Archimedes axiom).

Completed number line $\overline{\mathbb{R}}$

Definition

We call the completed number line $\mathbb{R}$ the set obtained by adding to $\mathbb{R}$ the two distinct elements $-\infty$ and $+\infty$ verifying, for all $x \in \overline{\mathbb{R}}, -\infty \leq x \leq +\infty$ ($\overline{\mathbb{R}} = \mathbb{R} \cup]-\infty, +\infty[$)

Operations on $\mathbb{R}$ extend partly to $\overline{\mathbb{R}}$ by setting

- For all $x \in \overline{\mathbb{R}}, \begin{cases} x + (+\infty) = (+\infty) + x \\ x + (-\infty) = (-\infty) + x \end{cases}$
- For all $x \in \overline{\mathbb{R}},\ x(+\infty) = (+\infty)x = \begin{cases} +\infty, & \text{if } x > 0 \\ -\infty, & \text{if } x < 0 \end{cases}$
- For all $x \in \overline{\mathbb{R}},\ x(-\infty) = (-\infty)x = \begin{cases} -\infty, & \text{if } x > 0 \\ +\infty, & \text{if } x < 0 \end{cases}$
- $(+\infty) + (+\infty) = +\infty,\ (-\infty) + (-\infty) = -\infty,\ (+\infty)(+\infty) = +\infty,$

  $(-\infty)(-\infty) = +\infty, (-\infty)(+\infty) = -\infty.$

Remark

The sum $(+\infty) + (-\infty)$ and the product $0(+\infty)$ are not defined.

Reasoning by recurrence

The recurrence principle allows us to show that an assertion $P(n)$, depending on naturel number n, is true for all $n \in \mathbb{N}$, the demonstration by recurrence takes place in two steps:

- We prove $P(n)$ is true for $n = n_0$ (initial condition)

- We assume $P(n)$ is true and we show that $P(n + 1)$ is true (final condition).

Once this is established, we conclude that $P(n)$ is true for all $n \geq n_0$.

Example:

Show that $\forall n \geq 1; 1 + 2 + \ldots n = \frac{n(n+1)}{2}$.