Algebra tutorial series 2

Exercise 1 *Let A and B be two sets defined by :* $A =]0,6[, \quad B = \{x \in \mathbb{N}, \qquad x < 3\}$
Determine :

$$A \cap B, \quad A \cup B, \quad \complement_A^B, \quad A \cap \mathbb{N}, \quad \complement_{\mathbb{R}}^A.$$

Exercise 2 *Let A, B and C be three subsets of a set E. Show that*

1. $\complement_E^{A\cup B} = \complement_E^A \cap \complement_E^B$.
2. $\complement_E^{A\cap B} = \complement_E^A \cup \complement_E^B$.
3. $A \subset B \Leftrightarrow \complement_E^B \subset \complement_E^A$.

Where $\complement_E^A$ denotes the complement of a set A in E.
Simplify the following sets :

1. $\complement_E^{A\cup B} \cap \complement_E^{\left(C\cup \complement_E^A\right)}$;
2. $\complement_E^{A\cap B} \cup \complement_E^{\left(C\cap \complement_E^A\right)}$;

Exercise 3 *Determine the sets A and B that simultaneously satisfy the following conditions :*

1. $A \cup B = \{1, 2, 3, 4, 5\}$,
2. $A \cap B = \{3, 4, 5\}$,
3. $1 \notin A \setminus B$,
4. $2 \notin B \setminus A$.

Exercise 4 ***Symmetric difference***
Let A and B be two parts of a set E. We call the symmetric difference of A and B, and we denote $A\Delta B$, the set defined by :

$$A\Delta B = (A \cup B) \setminus (A \cap B).$$

1. *Make a drawing, then calculate $A\Delta B$ for $A = \{0, 1, 2, 3\}$ and $\quad B = \{2, 3, 4\}$.*
2. *Determine the sets $A\Delta E, A\Delta A$ and $A\Delta\emptyset$.*
3. *Suppose $A\Delta B = A \cap B$. Prove by contradiction that : $A = \emptyset, (B = \emptyset)$.*
4. *Let $C \in P(E)$. Show that $A\Delta B = A\Delta C$ if and only if $B = C$.*

Exercise 5 *Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$.*

1. *Determine $f(\{-2, 0, 2\}), f^{-1}(\{0\})$.*
2. *Is the map f injective (one-to-one) ? Justify.*
3. *Is the map f surjective (onto) ? Justify.*
   Consider the sets $A = [-3, 2], B = [0, 4]$.
4. *Compare the sets $f(A \cap B)$ and $f(A) \cap f(B)$*
5. *What condition must satisfy f so that $f(A \cap B) = f(A) \cap f(B)$.*

Exercise 6

A. Let $f : \mathbb{R} \to \mathbb{R}$ defined by :

$$f(x) = \begin{cases} 1, & x < 0 \\ 1 + x, & x \geq 0 \end{cases}$$

1. *Determine the following sets : $f(\mathbb{R}), f^{-1}(\{0\}), f^{-1}(\{1\}), f^{-1}(\{-1\}), f^{-1}([1, 2])$.*
2. *Is f injective ? is f surjective ?*

B. Let the map $g : \mathbb{R} \setminus \left\{\frac{1}{2}\right\} \to \mathbb{R}^{\star}$ *such that :* $g(x) = \dfrac{9}{2x-1}$.

1. *Show that* g *is a bijection. Determine its inverse function* g^{-1}.
2. *Determine* $g^{-1}([-5, 2])$.

Exercise 7 *Let* $\mathcal{R}$ *be the binary relation defined in* $\mathbb{R}$ *by :*

$$\forall x \in \mathbb{R}, \quad x\mathcal{R}y \Leftrightarrow x^2 - y^2 = x - y.$$

1. *Show that* $\mathcal{R}$ *is an equivalence relation.*
2. *Calculate the equivalence class of an element* x *of* $\mathbb{R}$.
3. *Determine the equivalence class of* 0, *deduce that of* 1.

Exercise 8 *On* $\mathbb{R}^2$, *let* $\prec$ *be the relation given by*

$$\forall (x, y), (x', y') \in \mathbb{R}^2, \quad (x, y) \prec (x', y') \Leftrightarrow [(x < x') \, or \, (x = x' \quad and \quad y \leq y')].$$

1. *Show that* $\prec$ *is a total order relation on* $\mathbb{R}^2$.
2. *Classify the following elements in ascending order for the relation* $\prec$ *:*

$$(-1, 1), \quad (-1, -1), \quad (1, -1) \quad and \quad (0, 0).$$

Additional exercises

Exercise 1 *Let* E *be a non-empty set. Show that :*

1. *What about* A *and* B *such that* $A \cup B = A \cap B$?
2. $\forall A, B \in P(E) : A \cup B = A \cap B \Rightarrow A = B$.
3. $\forall A, B, C \in P(E) : (A \cap B = A \cap C \quad and \quad A \cup B = A \cup C) \Rightarrow B = C$.
4. $\forall A, B, C \in P(E) : A - (B \cup C) = (A - B) \cap (A - C)$.

Exercise 2 *What is the image of the sets :* $\mathbb{R}, [0, 2\pi], [0, \pi/2]$, *and the inverse image of the sets :* $[0, 1], [3, 4], [1, 2]$ *by the application* $x \mapsto \sin x$.

Exercise 3 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$, *defined by* $f((x, y)) = 2x - y$.

1. *Determine* $f^{-1}(\{1\})$ *and represent it graphically.*
2. *Is* f *injective ?*
3. *Show that* f *is surjective.*