Algebra tutorial series 1

Reminder
Recall that $(E,+,\cdot)$ is a $\mathbb{R}-$vector space if

1. $(E,+)$ is an abelian group,
2. **(a)** $\forall x,y\in E,\ \forall\alpha\in\mathbb{R},\ \alpha\cdot(x+y)=\alpha\cdot x+\alpha\cdot y,$
   (b) $\forall x\in E,\ \forall\alpha,\beta\in\mathbb{R},\ (\alpha+\beta)\cdot x=\alpha\cdot x+\beta\cdot x,$
   (c) $\forall x\in E,\ \forall\alpha,\beta\in\mathbb{R},\ \alpha\cdot(\beta\cdot x)=(\alpha\beta)\cdot x,$
   (d) $1\cdot x=x.$

Exercise 1 *Show that the sets below are vector spaces (on $\mathbb{R}$) :*

— $E_1=\mathbb{R}^2$ *with addition " + " and multiplication " · " by a real number, defined by :*

$$\forall(x,y),(x',y')\in\mathbb{R}^2,\quad\forall\alpha\in\mathbb{R},\quad(x,y)+(x',y')=(x+x',y+y'),\quad\alpha\cdot(x,y)=(\alpha x,\alpha y).$$

— $E_2=\{P\in\mathbb{R}_2[X]/\ \ degP\le 2\}$ *the set of polynomials of degree less than or equal to 2, with coefficients in $\mathbb{R}$, with addition $P+Q$ of polynomials and multiplication by a real number $\lambda\cdot P$.*
$\forall P,Q\in E,\quad P=aX^2+bX+c,\quad Q=a'X^2+b'X+c',\quad P+Q=(a+a')X^2+(b+b')X+(c+c').$
and $\forall\alpha\in\mathbb{R}\quad\alpha\cdot P=(\alpha\cdot a)X^2+(\alpha\cdot b)X+(\alpha\cdot c).$

— $E_3=\{(u_n):\mathbb{N}\longrightarrow\mathbb{R}\}$ *: the set of real sequences with the addition of the sequences defined by $(u_n)+(v_n)=(u_n+v_n)$ and the multiplication by a réel number $\lambda\cdot(u_n)=(\lambda\times u_n)$*

— $E_4=\{f:[0,1]\longrightarrow\mathbb{R}\}$ *: the set of real-valued functions defined on the interval $[0,1]$, with addition of the functions $f+g$ and multiplication by a real number $\lambda\cdot f$.*

Exercise 2 *Study in which cases $\mathbb{R}^2$ is a vector space on $\mathbb{R}$ for the laws noted respectively $\oplus$ and $\otimes$:*

1. $\forall(x,y),(s,t)\in\mathbb{R}^2,\forall\alpha\in\mathbb{R}$
$$(x,y)\oplus(s,t)=(x+y,s+t);\quad\alpha\otimes(x,y)=(\alpha x,\alpha y).$$
2. $\forall(x,y),(s,t)\in\mathbb{R}^2,\forall\alpha\in\mathbb{R}$
$$(x,y)\oplus(s,t)=(x+s,y+t);\quad\alpha\otimes(x,y)=(\alpha x,0).$$
3. $\forall(x,y),(s,t)\in\mathbb{R}^2,\forall\alpha\in\mathbb{R}$
$$(x,y)\oplus(s,t)=(x.s,y.t);\quad\alpha\otimes(x,y)=(\alpha x,\alpha y).$$

Exercise 3 *In each of the following cases, say whether E_i is a subspace of E.*

1. $E=\mathbb{R}^2,$
$$\begin{aligned}E_1&=\left\{(x,y)\in\mathbb{R}^2\ /\ x+y=1\right\},\quad E_2=\left\{(x,y)\in\mathbb{R}^2\ /\ 2x+3y=0\right\},\\E_3&=\left\{(x,y)\in\mathbb{R}^2\ /\ xy\le 0\right\},\quad E_4=\left\{(x,y)\in\mathbb{R}^2\ /\ x\le y\right\}\\E_5&=\{(2x,3x)/\ \ x\in\mathbb{R}\}\end{aligned}$$
2. $E=\mathcal{C}(\mathbb{R},\mathbb{R}),$
$$E_1=\{f\in E\ /\ f(1)=f(0)\},\quad E_2=\{f\in E\ /\ f(1)-2f(0)=0\},$$
3. $E=\mathbb{R}_2[X]=\{P=aX^2+bX+c\ \ /\ \ a,b,c\in\mathbb{R}\}$
$$E_1=\{P\in E\ /\ P'(0)=2\},\qquad E_2=\{P\in E\ \ /\ \ P'(x)\ge 0,\ \ \forall x\in\mathbb{R}\}.$$

2.Vector families

Exercise 4 *Specify whether the following vectors* $\{e_1, \cdots\}$*form a free or generating family.*
Express, if possible, the vector a *as a linear combination of the vectors* e_1, e_2, e_3 *of* E*, in each of the following cases*

1. $E = \mathbb{R}, e_1 = 3.$
2. $E = \mathbb{R}^2, e_1 = (1,1), e_2 = (-1,2), e_3 = (1,0), \quad a = (2,4).$
3. $E = \mathbb{R}^2, e_1 = (1,1), e_2 = (-1,-1), e_3 = (2,2), \quad a = (1,0).$
4. $E = \mathbb{R}^3, e_1 = (1,1,0), e_2 = (1,0,1), e_3 = (0,1,1), \quad a = (1,1,1).$
5. $E = \mathbb{R}_2[X], e_1 = 1+3X, e_2 = X^2 - X, e_3 = X^2 + 1, \quad a = X^2 + X + 1, \quad a = X^3.$
6. $E = \mathcal{C}(\mathbb{R}, \mathbb{R}), e_1 : x \mapsto x, \quad e_2 : x \mapsto \cos x, \quad e_3 : x \mapsto \sin x. \quad a : x \mapsto \cos^2 x.$

.

Exercise 5 *In the vector space* $\mathbb{R}^3$*, we consider the two families of vectors :*
$A = \{v_1(2,0,-1), v_2(3,2,-4)\}$ *, and* $B = \{w_1(1,2,-3), w_2(0,4,-5)\}$.
— *Show that span*$(A) =$ *span*(B)

Exercise 6 *In the vector space* $\mathbb{R}^4$*, we consider the vectors*

$$v_1 = (1,1,1,1), v_2 = (0,1,2,1), v_3 = (1,0,-2,3) et v_4 = (1,1,2,-2).$$

1. *Determine the rank of the family* $A = \{v_1, v_2, v_3, v_4\}$ *. Is the set* A *free ?*
2. *Determine real numbers* α *and* β *so that the vector* $u = (1,1,\alpha,\beta) \in span(A)$.