Solution TD n° 2 : Suite intégrales simples et multiples

Exercice n° 1

a) Calcul d'aire de D : $\Rightarrow$ D dans le plan oxy

$$I = \int_{-1}^{1}\left[\int_{x^2}^{4-x^3} dy\right]dx = \int_{-1}^{1} y\Big|_{x^2}^{4-x^3} dx = \int_{-1}^{1}(4-x^3-x^2)\,dx$$

$$= 4x - \frac{x^4}{4} - \frac{x^3}{3}\Big|_{-1}^{1} = 4(1) - \frac{(1)^4}{4} - \frac{(1)^3}{3} - \frac{(-1)^3}{3} - \frac{(-1)^4}{4} - 4(-1)$$

$$\boxed{I = \frac{22}{3}}$$

b) Calcul d'aire Σ de la sphère $x^2+y^2+z^2=R^2$, $z \geq 0$

malgré l'existance de la variable z, mais nous allons calculer l'aire Σ avec une intégrale double

avec la formule $\Rightarrow$ $\Sigma = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dx\,dy.$

$$z^2 = R^2 - x^2 - y^2$$

$$\Rightarrow z = \sqrt{R^2 - x^2 - y^2}$$

$$\frac{\partial z}{\partial x} = \frac{-2x}{2\sqrt{R^2-x^2-y^2}} = \frac{-x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{2\sqrt{R^2-x^2-y^2}} = \frac{-y}{\sqrt{R^2-x^2-y^2}}$$

$$\Sigma = \iint_D \sqrt{\left(\frac{-x}{\sqrt{R^2-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{R^2-x^2-y^2}}\right)^2 + 1}\; dx\,dy$$

$$= \iint_D \sqrt{\frac{\cancel{x^2} + \cancel{y^2} + R^2 - \cancel{x^2} - \cancel{y^2}}{R^2-x^2-y^2}}\; dx\,dy$$

$$\Sigma = \iint_D \frac{R}{\sqrt{R^2 - x^2 - y^2}}\, dx\, dy = \iint_D \frac{R}{\sqrt{R^2 - (x^2+y^2)}}\, dx\, dy.$$

utilisons les coordonnées polaire pour résoudre l'intégrale :

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad |J| = r \qquad \begin{array}{l} 0 \leq r \leq R \\ 0 \leq \theta \leq 2\pi \end{array}$$

$$\Sigma = R\int_0^{2\pi} \left[\int_0^R \frac{1}{\sqrt{R^2 - r^2}}\, r\, dr\right] d\theta$$

nous avons $\int (f(x))' dx = f(x)$.

alors $\Sigma = R\int_0^{2\pi}\left[\int_0^R -\left(\sqrt{R^2 - r^2}\right)' dr\right] d\theta$

$$= R\int_0^{2\pi} -\sqrt{R^2 - r^2}\,\Big|_0^R\, d\theta = -R\int_0^{2\pi}\left(\sqrt{R^2 - R^2} - \sqrt{R^2 - 0^2}\right) d\theta$$

$$= R^2\int_0^{2\pi} d\theta = R^2\theta\,\Big|_0^{2\pi} = 2\pi R^2$$

donc l'aire de la sphère $= 2 \times 2\pi R^2$

$$\boxed{\Sigma = 4\pi R^2}$$

Exercice n°2

a) Les coordonnées cylindriques :-

on a :

$0 \leq x \leq 2$

$0 \leq y \leq \sqrt{4-x^2}$

$0 \leq z \leq 4$

pour déssiner le domaine :-

$x = 0$

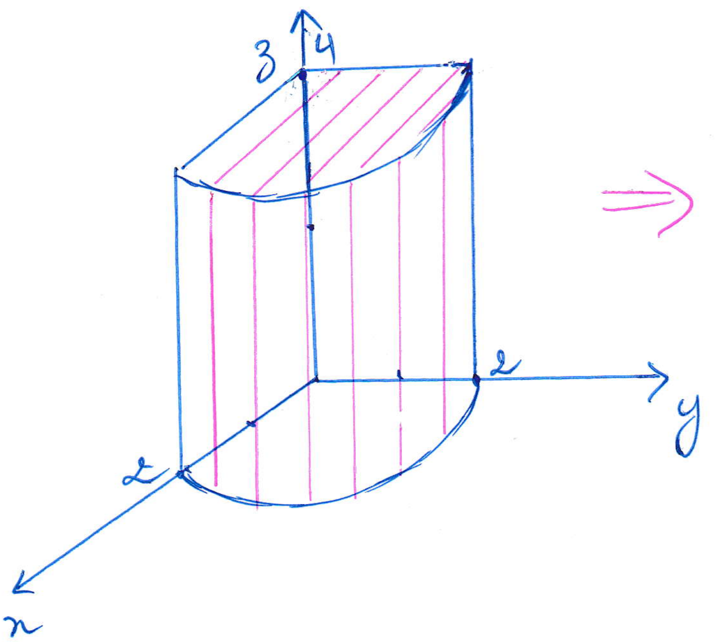
$x = 2$

$y = 0$

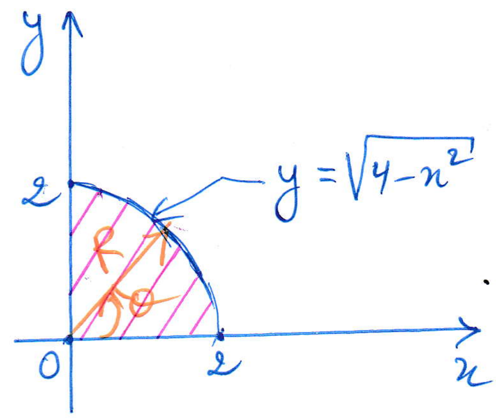
$y = \sqrt{4-x^2} \Rightarrow y^2 = 4 - x^2 \Rightarrow y^2 + x^2 = 4 \Rightarrow$ équation d'un cercle $C(0,0)$ $R = 2$.

$z = 0$

$z = 4$



$\Rightarrow$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$|J| = r$

$0 \leq r \leq 2$

$0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq z \leq 4$

$$I = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^4 z\sqrt{x^2+y^2}\, dz\, dy\, dx \Rightarrow \int_0^2 \left[\int_0^{\frac{\pi}{2}} \left[\int_0^4 r \cdot r \cdot z \cdot dz \right] d\theta \right] dr$$

$$= \int_0^2 \left[\int_0^{\frac{\pi}{2}} \left[\int_0^4 r^2 z\, dz \right] d\theta \right] dr = \int_0^2 \left[\int_0^{\frac{\pi}{2}} r^2 \cdot \frac{z^2}{2} \Big|_0^4 d\theta \right] dr$$

$$= 8 \int_0^2 r^2 \theta \Big|_0^{\frac{\pi}{2}} dr = 8 \int_0^2 r^2 \left(\frac{\pi}{2} - 0\right) dr =$$

$$I = 4\pi \int_0^2 r^2 \, dr = 4\pi \cdot \frac{r^3}{3}\Big|_0^2 = \boxed{\frac{32\pi}{3}}$$

b) <u>coordonnées sphériques :-</u>

$$I = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} (x^2+y^2) \, dz \, dy \, dx$$

$$-4 \leq x \leq 4$$
$$-\sqrt{16-x^2} \leq y \leq \sqrt{16-x^2}$$
$$0 \leq z \leq \sqrt{16-x^2-y^2}$$

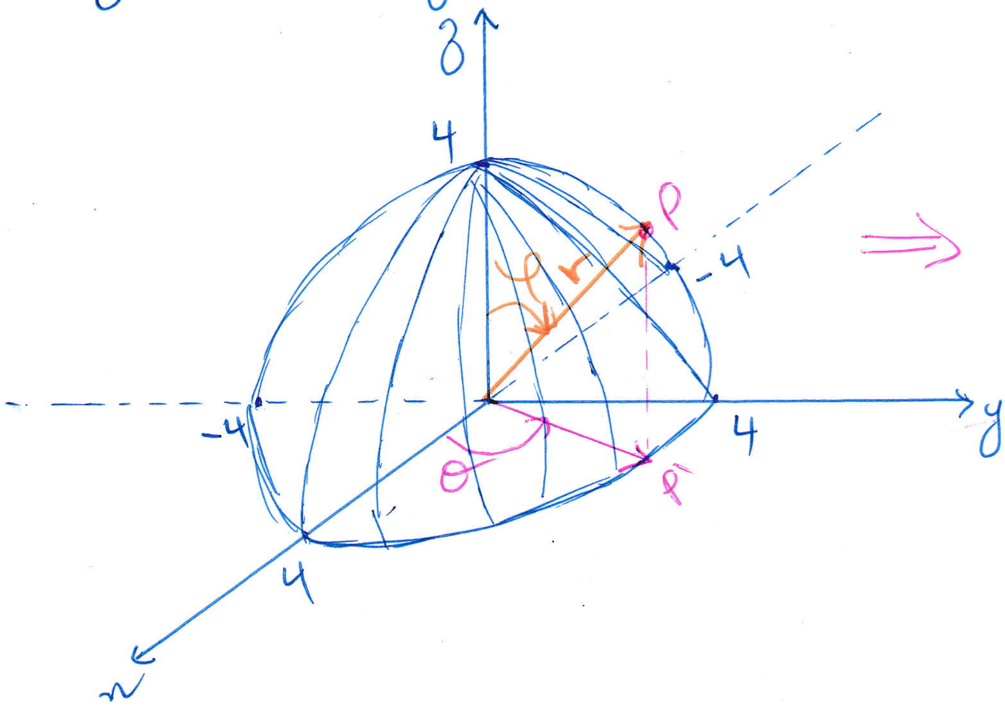
$x = 4 \quad , x = -4$

$y = \sqrt{16-x^2} \Rightarrow y^2 + x^2 = 16$ cercle $C(0,0)$ $R = 4$

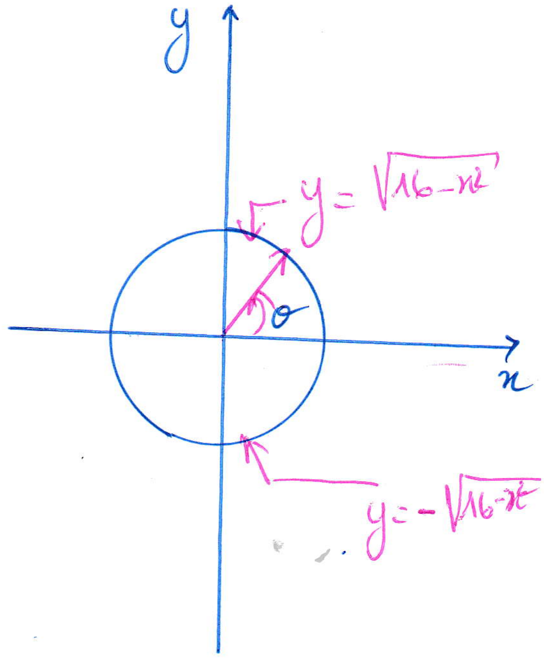
$y = -\sqrt{16-x^2} \Rightarrow y^2 + x^2 = 16$ " " "

pour la sphère

$z^2 = 16 - x^2 - y^2 \Rightarrow z^2 + x^2 + y^2 = (4)^2$. et $z \geq 0$



$\Rightarrow$



Les coordonnées sphériques

$x = r\cos\theta\sin\varphi$

$y = r\sin\theta\sin\varphi$ $\qquad |J| = r^2\sin\varphi$

$z = r\cos\varphi$

on a : $0 \le r \le 4$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \frac{\pi}{2}$

$x^2+y^2 = r^2\sin^2\varphi$.

$$I = \int_0^4\int_0^{\frac{\pi}{2}}\int_0^{2\pi} r^2\sin^2\varphi \cdot r^2\sin\varphi \, d\theta\, d\varphi\, dr$$

$$= \int_0^4\int_0^{\frac{\pi}{2}}\int_0^{2\pi} r^4\sin\varphi(1-\cos^2\varphi)\, d\theta\, d\varphi\, dr$$

$$= \int_0^4\int_0^{\frac{\pi}{2}} r^4\sin\varphi(1-\cos^2\varphi)\, 2\pi\, d\varphi\, dr$$

$$= 2\pi\int_0^4 \underbrace{\left[\int_0^{\frac{\pi}{2}} r^4\sin\varphi(1-\cos^2\varphi)\, d\varphi\right]}_{I_1} dr$$

$I_1 \Rightarrow$ changement de variables : $u = \cos\varphi$

$du = -\sin\varphi\, d\varphi$

$\Rightarrow d\varphi = -\frac{du}{\sin\varphi}$

$\varphi = 0 \Rightarrow u = 1$

$\varphi = \frac{\pi}{2} \Rightarrow u = 0$

alors $I_1 = \int_1^0 r^4 \cancel{\sin\varphi}(1-u^2) \times -\frac{du}{\cancel{\sin\varphi}}$

$$= -\int_1^0 r^4(1-u^2)\, du = -r^4\left(u-\frac{u^3}{3}\right)\Big|_1^0$$

$I_1 = -r^4\left(0 - \frac{3-1}{3}\right) = \boxed{\frac{2}{3}r^4}$

$$I = 2\pi\int_0^4 \frac{2}{3}r^4\, dr = 4\pi\,\frac{r^5}{15}\Big|_0^4 = \frac{4\pi}{15}\times(4)^5 = \boxed{\frac{\pi}{15}(4)^6}$$