UNIVERSITE BATNA 2
FACULTE DE TECHNOLOGIE
DEPARTEMENT DE MECANIQUE

MODULE : Optimisation en Biomécanique
DATE 19/1/22

SOLUTION DE L'EXAMEN FINAL

EXERCICE N°1 (8points)

a)

$$P(\lambda) = det(A - \lambda I_3) = det\begin{bmatrix} -1-\lambda & 0.5 & 0 \\ 0.5 & -1-\lambda & 0.5 \\ 0 & 0.5 & -1-\lambda \end{bmatrix} \quad \text{(1pt)}$$

$$P(\lambda) = (-1-\lambda)det\begin{bmatrix} -1-\lambda & 0.5 \\ 0.5 & -1-\lambda \end{bmatrix} - (0.5)det\begin{bmatrix} 0.5 & 0.5 \\ 0 & -1-\lambda \end{bmatrix} \quad \text{(0.5pt)}$$

$$P(\lambda) = (-1-\lambda)((-1-\lambda)^2 - 0.25) - 0.25(-1-\lambda) \quad \text{(0.5pt)}$$

$$P(\lambda) = (-1-\lambda)((-1-\lambda)^2 - 0.25 - 0.25) = (-1-\lambda)((-1-\lambda)^2 - 0.5) \quad \text{(0.5pt)}$$

$$P(\lambda) = (-1-\lambda)\left(-1-\lambda+\sqrt{0.5}\right)\left(-1-\lambda-\sqrt{0.5}\right) \quad \text{(0.5pt)}$$

Les racines de l'équation caractéristiques :
$P(\lambda) = (-1-\lambda)\left(-1-\lambda+\sqrt{0.5}\right)\left(-1-\lambda-\sqrt{0.5}\right) = 0$ (0.5pt)
Sont $\lambda_1 = -1$ (0.5pt) $\lambda_2 = -1+\sqrt{0.5}$ (0.5pt) et $\lambda_3 = -1-\sqrt{0.5}$ (0.5pt)

Ces racines représentent les valeurs propres de A.

b). Elles sont toutes strictement négatives. Donc la matrice A n'est pas définie positive. (3pts)

EXERCICE N°1 (12 points)

$$f(x) = x_1^2 + x_2^2 + (x_1 + x_2 - 3x_3)^2 + x_3^4 + 2x_3^3 - 5x_3^2$$

a) Déterminons le gradient de f. (formule du gradient (1pt))

$f(x) = x_1^2 + x_2^2 + (x_1 + x_2 - 3x_3)^2 + x_3^4 + 2x_3^3 - 5x_3^2.$

On a :

$\frac{\partial f(x)}{\partial x_1} = 2x_1 + 2(x_1 + x_2 - 3x_3) = 4x_1 + 2x_2 - 6x_3$ (0.5 pt)

$\frac{\partial f(x)}{\partial x_2} = 2x_2 + 2(x_1 + x_2 - 3x_3) = 4x_2 + 2x_1 - 6x_3$ (0.5 pt)

$\frac{\partial f(x)}{\partial x_3} = -6(x_1 + x_2 - 3x_3) + 4x_3^3 + 6x_3^2 - 10x_3 = -6x_1 - 6x_2 + 8x_3 + 4x_3^3 + 6x_3^2$ (0.5 pt)

$\nabla f(x) = 0$ est équivalente au système d équations suivants

$$\begin{cases} 4x_1 + 2x_2 - 6x_3 & = 0 \\ 4x_2 + 2x_1 - 6x_3 & = 0 \\ -6x_1 - 6x_2 + 8x_3 + 4x_3^3 + 6x_3^2 & = 0 \end{cases} \quad (1)$$

b) Vérifions que $\boldsymbol{x^*} = (\boldsymbol{-2, -2, -2})$ est un point critique de *f*.

$4(-2) + 2(-2) - 6(-2) \qquad = -12 + 12 = 0$ (0.5 pt)

$4(-2) + 2(-2) - 6(-2) = -12 + 12 = 0$ (0.5 pt)

$-6(-2) - 6(-2) + 8(-2) + 4(-2)^3 + 6(-2)^2 \quad = 0$ (0.5 pt)

Donc $\boldsymbol{x^*} = (\boldsymbol{-2, -2, -2})$ est un point critique de *f*.

c) Déterminons la matrice Hessienne de *f* au point $\boldsymbol{x^*} = (\boldsymbol{-2, -2, -2})$

formule (1pt)

$\frac{\partial^2 f_2}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial f_2}{\partial x_1}\right) = \frac{\partial}{\partial x_1}(4x_1 + 2x_2 - 6x_3) = 4$ (0.5 pt)

$\frac{\partial^2 f_2}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{\partial f_2}{\partial x_2}\right) = \frac{\partial}{\partial x_1}(4x_2 + 2x_1 - 6x_3)$=2 (0.5 pt)

$\frac{\partial^2 f_2}{\partial x_1 \partial x_3} = \frac{\partial}{\partial x_1}\left(\frac{\partial f_2}{\partial x_3}\right) = \frac{\partial}{\partial x_1}(-6x_1 - 6x_2 + 8x_3 + 4x_3^3 + 6x_3^2)$= -6 (0.5 pt)

$\frac{\partial^2 f_2}{\partial x_2 \partial x_1} = \frac{\partial(4x_1 + 2x_2 - 6x_3)}{\partial x_2} = 2$ (0.5 pt)

(0.5 pt)

$$\frac{\partial^2 f_2}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(\frac{\partial f_2}{\partial x_2}\right) = \frac{\partial}{\partial x_2}(4x_2 + 2x_1 - 6x_3) = 4$$

(0.5 pt)

$$\frac{\partial^2 f_2}{\partial x_2 \partial x_3} = \frac{\partial(-6x_1 - 6x_2 + 8x_3 + 4x_3^3 + 6x_3^2)}{\partial x_2} = -6$$

(0.5 pt)

$$\frac{\partial^2 f_2}{\partial x_3 \partial x_1} = \frac{\partial(4x_1 + 2x_2 - 6x_3)}{\partial x_3} = -6$$

(0.5 pt)

$$\frac{\partial^2 f_2}{\partial x_3 \partial x_2} = \frac{\partial(4x_2 + 2x_1 - 6x_3)}{\partial x_3} = -6$$

$$\frac{\partial^2 f_2}{\partial x_3^2} = \frac{\partial}{\partial x_3}\left(\frac{\partial f_2}{\partial x_3}\right) = \frac{\partial}{\partial x_3}(-6x_1 - 6x_2 + 8x_3 + 4x_3^3 + 6x_3^2) = 8 + 12x_3^2 + 12x_3$$ (0.5 pt)

La matrice hessienne est donc

$$\nabla^2 f_2 = \begin{bmatrix} 4 & 2 & -6 \\ 2 & 4 & -6 \\ -6 & -6 & 8 + 12x_3^2 + 12x_3 \end{bmatrix}$$ pour tout x (0.5 pt)

d) Déterminer la nature du point critique $\boldsymbol{x^* = (-2, -2, -2)}$.

Au point critique $x^* = (-2, -2, -2)$, on a : $\nabla^2 f_2(-2, -2, -20) = \begin{bmatrix} 4 & 2 & -6 \\ 2 & 4 & -6 \\ -6 & -6 & 32 \end{bmatrix}$,

$d_1 = 4$ $\quad$ $d_2 = det\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = 12$ (0.5 pt)

Et $d_3 = det\begin{bmatrix} 4 & 2 & -6 \\ 2 & 4 & -6 \\ -6 & -6 & 32 \end{bmatrix} = 4det\begin{bmatrix} 4 & -6 \\ -6 & 32 \end{bmatrix} - 2det\begin{bmatrix} 2 & -6 \\ -6 & 32 \end{bmatrix} - 6det\begin{bmatrix} 2 & 4 \\ -6 & -6 \end{bmatrix}$ (0.5 pt)

$d_3 = 4(128 - 36) - 2(64 - 36) - 6(-12 + 24) = 240$ (0.5 pt)

$d_1 = 4 > 0$, $d_2 = 12 > 0$ et $d_3 = 240 > 0$. Donc $\nabla^2 f(-2, -2, -2)$ est définie positive. Le point critique $x^* = (-2, -2, -2)$ est un point de minimum de f_2. (0.5 pt)