Chapter contents

1- Position vector in the diffrent coordinate systems (cartesian, polar, cylindical and spherical).
2- Velocity and acceleration in diffrent coordinate systems.
3- Curvilinear coordinate
4- Relative motion

I. GENERALITIES

Kinematics is a part of mechanics dealing with the movement of bodies, without entering into the relationship between the movement of the examined body (in particular of the point) and the forces acting on it. In the case of kinematics, we will consider what happens to the body in space over time. We will describe this type of relationship as the geometry of motion.

I-1 Time axis

Generally, time is always on the x-axis, because it allows for other quantities to be represented as functions of time, which is useful in physics. Time is also, in most cases, independent of other variables, which conventionally means that it should be on the x-axis.

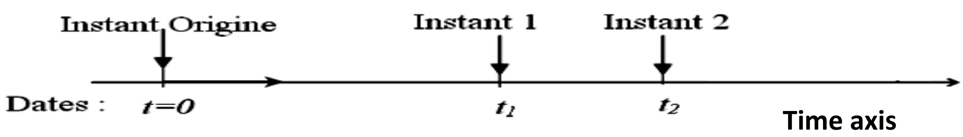


Figure1 Time axis

I-2 Reference frame (frame of reference)

A frame of reference is needed to describe an object's motion. It comprises a coordinate system and a clock used to establish factors like location and velocity of moving objects. In such frames, the three dimensions are commonly defined by means of axes and time with a clock.

In the physical world, the frame of reference of an object is defined by the **coordinate system** attached to it. The choice of the frame depends upon the problem at hand.
What is Frame of Reference?

A frame of reference is a set of coordinate axes that define the position of a particle in two- or three-dimensional space. The Cartesian coordinate system is the most basic frame of reference, in which a particle's position is specified by three coordinates x,y, and z.

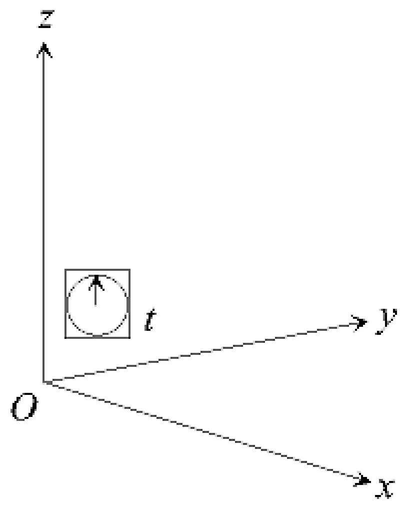


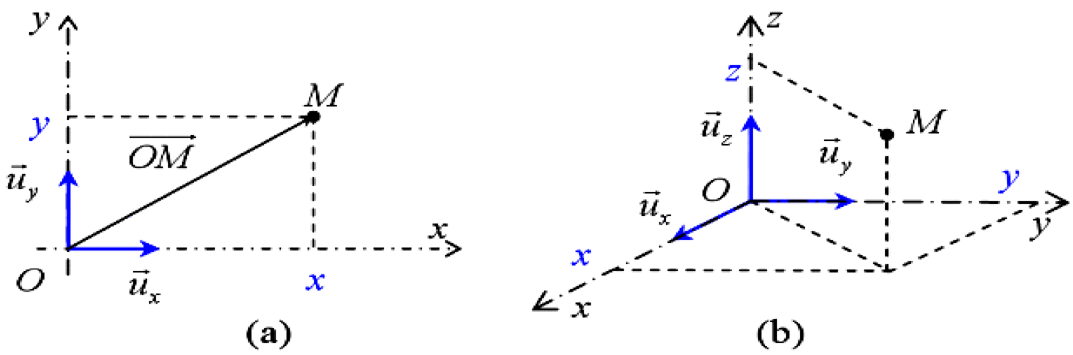
Figure2 Reference frame

I-3 Movement of the body

Changing the position of this body in relation to another one taken from a stationary body (**reference body**). In the case of mechanics, the Earth is usually taken as the reference frame.

I-4 Reference system

A system that is fixed and bound to a reference body. The most common reference system is a rectangular coordinate system (Cartesian system) .



x, y, z-coordinates of the moving point M with respect to the fixed coordinate system (reference system).

I-5 Equations of trajectory

A trajectory is a **path** taken up by a moving particle that is following through space as a function of time. Mathematically, a trajectory is described as a position of an object over a particular time. In order to describe the movement of this particle, it is necessary to determine how particular coordinates change with time.

$$x = f(t) \quad , \quad y = g(t) \quad , \quad z = h(t)$$

We will call the above equations the kinematic equations of motion. In these equations the time is a parameter. After removing time, we get the relations between the x, y, z coordinates (i.e. the path of motion) called the equation of trajectory.

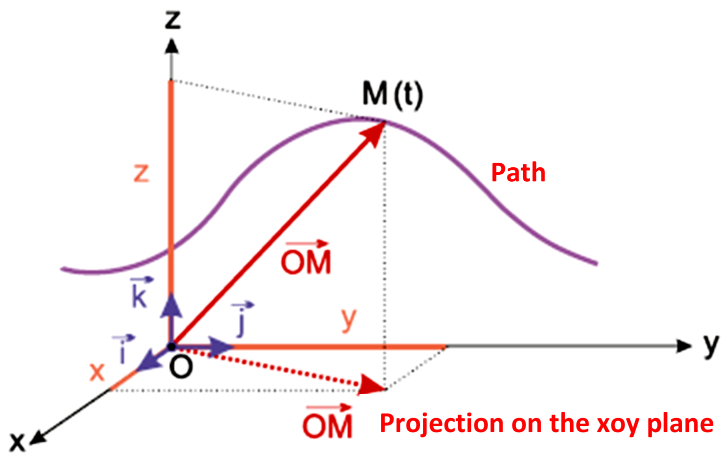


Figure 3 Trajectory of M

I-6 Position vector

The position vector of a particle is a <u>vector</u> drawn from the origin of the reference frame to the particle. It expresses both the distance of the point M from the origin O and its direction from the origin. In three dimensions, the position vector $\overrightarrow{OM}$ can be expressed as :

$$\overrightarrow{OM} = x\vec{u}_x + y\vec{u}_y + z\vec{u}_z$$

Where x , y , and z are the coordinates and $\left(\vec{u}_x, \vec{u}_y, \vec{u}_z\right)$ is the orthonormal base of the reference frame (O, x, y, z). The magnitude of the position vector gives the distance between the point M and the origin O.

In general, an object's position vector will depend on the frame of reference; ***<u>different frames will lead to different values for the position vector.</u>***

I-7 Displacement vector

- **Distance:** Is the length of the path travelled by an object between two points in space. From its definition, the distance is a scalar and it is always a positive quantity.
- **Displacement:** Is the change in the position of a particle . If at time $t = t_1$ the object is at position $\overrightarrow{OM}$, and at a later time $t = t_2 > t_1$ the object is at position$\overrightarrow{OM'}$. The **displacement vector** is defined as $\overrightarrow{MM'}$

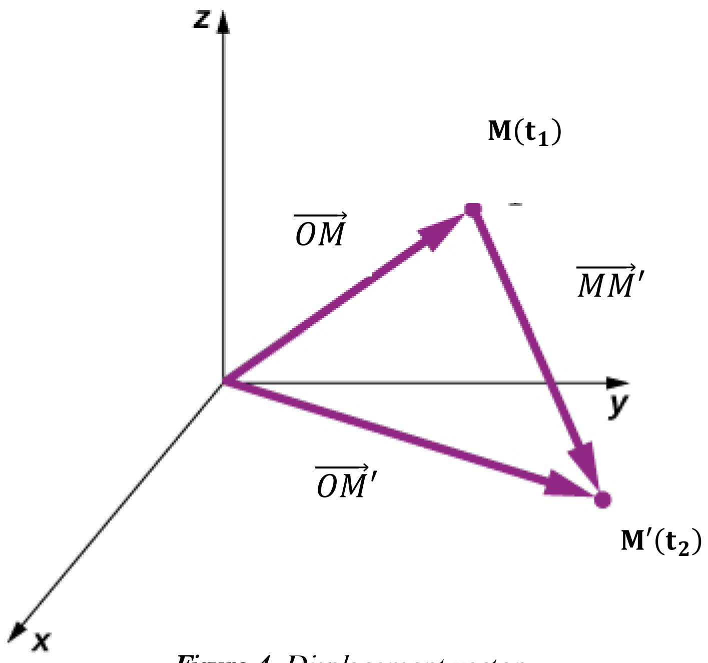


Figure 4 Displacement vector

I-8 Velocity :

Velocity is a crucial topic in physics. Many qualities of a body, such as kinetic energy is influenced by its velocity. The term velocity describes how quickly or slowly an object is moving. It can be defined as the rate of change of the object's position with respect to time and frame of reference. It is critical to have a thorough knowledge of the notions of instantaneous velocity and average velocity.

Average Velocity

The average velocity is calculated by dividing the change in total displacement by the total time taken. The average velocity of an item is always less than or equal to its average speed :

$$\vec{V}_{avg} = \frac{\Delta\overrightarrow{OM}}{\Delta t}$$

Instantaneous Velocity

The rate of change of position over a relatively short time span is called instantaneous velocity or the velocity of an object at that instant of time. The instantaneous velocity may be calculated by multiplying the object's instantaneous speed by the direction in which it is traveling at the time.

It's also calculated as the average velocity divided by the minimum period. The ratio of total displacement to total time can be used to compute **average velocity**. The displacement is proportional to the time interval. The limit of this ratio between time and displacement is known as instantaneous velocity.

$$\vec{V}_{inst} = \lim_{\Delta t \to 0} \frac{\Delta \overrightarrow{OM}}{\Delta t}$$

Then :

$$\vec{V}_{inst} = \frac{d\overrightarrow{OM}}{dt}$$

Where :

$\vec{V}_{inst}$: is Instantaneous velocity at time t

$\overrightarrow{OM}$: denotes position vector

t: denotes time

I-9 Acceleration

Acceleration is the change in velocity divided by a period of time during which the change occurs. The SI units of velocity are m/s and the SI units for time are s, so the SI units for acceleration are m/s^2. Average acceleration is given by

$$\vec{a}_{avg} = \frac{\Delta \vec{V}}{\Delta t}$$

Average acceleration is distinguished from instantaneous acceleration, which is acceleration at a specific instant in time.

$$\vec{a}_{inst} = \lim_{\Delta t \to 0} \frac{\Delta \vec{V}}{\Delta t}$$

Then :

$$\vec{a}_{inst} = \frac{d\vec{V}}{dt}$$

Attention !

Speed and velocity both measure an object's rate of motion. However, speed is a scalar quantity, which means that it can be described with a numerical value. Velocity is a vector quantity, which depends on direction as well as magnitude.

- speed : the rate of distance traveled by a moving object over time
- velocity : the rate of displacement of a moving object over time

Basically, an object's speed tells you how fast it's going. Its velocity tells you how fast it's going in a certain direction. You use speed measurements in your daily life, but physicists depend on velocity measurements more frequently in their work.

II- COORDINATE SYSTEMS

II-1 Cartesian coordinate system

In the reference frame (O, x, y, z) a particle M is identified by its space coordinates corresponding to the algebraic measure of (x, y, z) the projection of M successively on the 3 axes of the reference frame. These 3 coordinates are of the same nature and homogeneous at a length.

The orthonormal base associated with this axis system is denoted $\left(\vec{u}_x, \vec{u}_y, \vec{u}_z\right)$. It is a basis that does not change over time: these vectors keep the same direction and the same norm over time. We still say that the base is **fixed** in the frame. These vectors can be represented anywhere in space but in general they are represented at the origin point O.

Knowing the position vector $\overrightarrow{OM}$ also makes it possible to locate the particle M. The components of this vector, in the Cartesian base $\left(\vec{u}_x, \vec{u}_y, \vec{u}_z\right)$ correspond to the coordinates of the point M:

$$\overrightarrow{OM} = x\vec{u}_x + y\vec{u}_y + z\vec{u}_z$$

(x, y, z) are the Cartesian coordinates of the point M .

(x, y, z) are the components of the position vector $\overrightarrow{OM}$ in the Cartesian basis $\left(\vec{u}_x, \vec{u}_y, \vec{u}_z\right)$

<u>Velocity</u>

The position vector is a function of time $\overrightarrow{OM} = \overrightarrow{OM}(t)$ and the instantaneous velocity then corresponds to the time derivative of the position vector:

$$\vec{V}(t) = \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t}$$

<u>Important</u> :

The velocity vector is a vector tangent to the trajectory at the point considered.

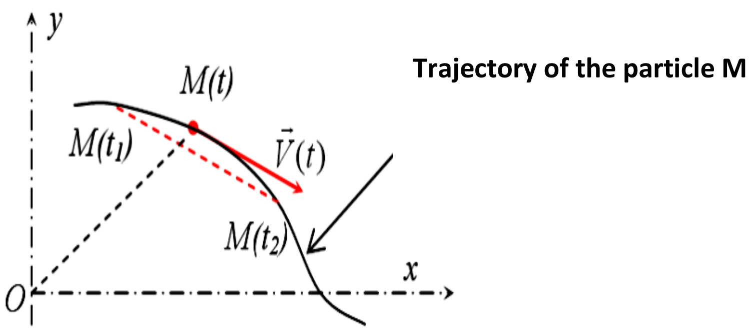


Figure 5 Velocity

The base $(\vec{u}_x, \vec{u}_y, \vec{u}_z)$ is made up of "fixed" vectors in the reference frame: their direction, their direction, their norm do not change over time. Using the expression of the position vector in Cartesian coordinates and the rules for deriving a sum of functions, we have:

$$\vec{V}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{d(x\vec{u}_x + y\vec{u}_y + z\vec{u}_z)}{dt}$$

$$\vec{V}(t) = \frac{dx}{dt}\vec{u}_x + \frac{dy}{dt}\vec{u}_y + \frac{dz}{dt}\vec{u}_z$$

$$\vec{V}(t) = \dot{x}\vec{u}_x + \dot{y}\vec{u}_y + \dot{z}\vec{u}_z$$

<u>Note</u>: By convention and to simplify the expression, the "derivation of a variable with respect to time" is noted by the variable surmounted by a point for the first derivative, 2 points for the second derivative, etc :

$$\frac{dX}{dt} = \dot{X} \quad ; \quad \frac{d^2X}{dt^2} = \ddot{X}$$

$$\vec{V}(V_x = \dot{x}, V_y = \dot{y}, V_z = \dot{z})$$

The value V corresponds to the magnitude of the velocity :

$$\| \vec{V}(t) \| = V = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

<u>Acceleration :</u>

The acceleration vector corresponds to the time derivative of the velocity, that is to say also to the second derivative of the position vector:

$$\vec{a} = \frac{d\vec{V}(t)}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2}$$

$$\vec{a} = \frac{d\vec{V}(t)}{dt} = \frac{d[\dot{x}\vec{u}_x + \dot{y}\vec{u}_y + \dot{z}\vec{u}_z}{dt} = \frac{d\dot{x}}{dt}\vec{u}_x + \frac{d\dot{y}}{dt}\vec{u}_y + \frac{d\dot{z}}{dt}\vec{u}_z$$

$$\vec{a} = \frac{d\vec{V}(t)}{dt} = \ddot{x}\vec{u}_x + \ddot{y}\vec{u}_y + \ddot{z}\vec{u}_z$$

$$\vec{a} = (a_x = \dot{V}_x = \ddot{x}\ ; a_y = \dot{V}_y = \ddot{y}\ ; a_z = \dot{V}_z = \ddot{z})$$

II-2 Polar coordinate system (in a plan)

The particle M is perfectly located if we know the distance $OM = \rho$ (Greek letter rho) and the angle θ (Greek letter theta) that the segment makes with the axis Ox (see figure)

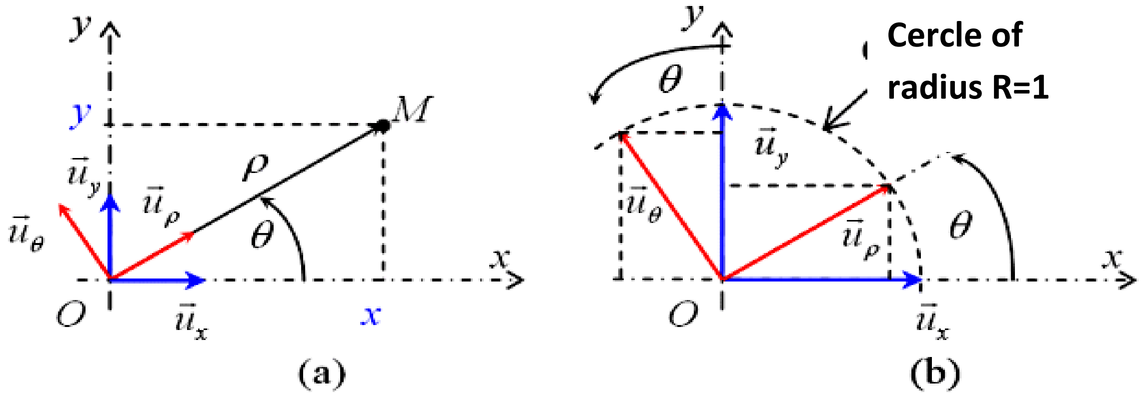


Figure 6 : the polar coordinates (ρ, θ) *and the associated basis* $(\vec{u}_\rho, \vec{u}_\theta)$

The length of the segment $OM = \rho$ corresponds to the ***radial coordinate*** (denoted ρ or r). The angle θ is the ***angular coordinate***. This angle is measured relative to the abscissa axis Ox

The position vector $\overrightarrow{OM}$ can be written : $\overrightarrow{OM} = \| \overrightarrow{OM} \| \vec{u}_\rho = \rho \vec{u}_\rho$

The unit vector $\vec{u}_\rho$ is according to the direction O to M : it is the radial vector (according to the radius).

A new direct orthonormal basis $(\vec{u}_\rho, \vec{u}_\theta)$ is obtained by associating $\vec{u}_\rho$ with the unit vector $\vec{u}_\theta$ directly perpendicular (in the trigonometric sense): it is the orthoradial vector (perpendicular to the radius)

The base of the polar coordinate system $(\vec{u}_\rho, \vec{u}_\theta)$is a base defined from the position of the particle . If the point M is moving the vector $\vec{u}_\rho$ (and therefore $\vec{u}_\theta$) changes direction: the base is "***mobile***" in the frame. These vectors can be represented anywhere in space but they are often represented either at the origin O point or at the point M itself.

Relation between polar coordinates an cartesian ones

With the notations in Figure 3, the relationships between the Cartesian and polar coordinate systems are:

Cartesian $\rightarrow$ polar crossing

$$OM = \rho = \sqrt{x^2 + y^2}$$

$$\cos\theta = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}\ ;\ \sin\theta = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}\ ;\ \tan\theta = \frac{\sin\theta}{cos\theta} = \frac{y}{x} \qquad (3b)$$

$$\vec{u}_\rho = (\cos\theta)\vec{u}_x + (\sin\theta)\vec{u}_y$$

$$\vec{u}_\theta = (-\sin\theta)\vec{u}_x + (\cos\theta)\vec{u}_y$$

<u>Velocity</u>

The base $(\vec{u}_\rho, \vec{u}_\theta)$ is made up of "moving" vectors in the frame: these vectors change direction over time. Using the expression of the position vector in polar coordinates and the rules for deriving a product of functions, we have:

$$\vec{V}(t) = \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} = \frac{\mathrm{d}(\rho\vec{u}_\rho)}{\mathrm{d}t} = \frac{\mathrm{d}\rho}{\mathrm{d}t}\vec{u}_\rho + \rho\frac{\mathrm{d}\vec{u}_\rho}{\mathrm{d}t} = \dot{\rho}\vec{u}_\rho + \rho\frac{\mathrm{d}\vec{u}_\rho}{\mathrm{d}t}$$

From the expression of the vector $\vec{u}_\rho$, it appears as a function of the angular coordinate θ itself a function of time during the movement of the particle M.

The derivation of a compound function allows us to write:

$$\frac{\mathrm{d}\vec{u}_\rho}{\mathrm{d}t} = \frac{\mathrm{d}\vec{u}_\rho}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}\frac{\mathrm{d}\vec{u}_\rho}{\mathrm{d}\theta} = \dot{\theta}\vec{u}_\theta = \omega\vec{u}_\theta$$

$$\frac{\mathrm{d}\vec{u}_\theta}{\mathrm{d}t} = \frac{\mathrm{d}\vec{u}_\theta}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}\frac{\mathrm{d}\vec{u}_\theta}{\mathrm{d}\theta} = -\dot{\theta}\vec{u}_\rho = -\omega\vec{u}_\rho$$

The quantity $\dot{\theta}$ characterizes the variation of the polar angle over time and corresponds to the definition of <u>angular speed</u> (rad/s). It is often noted ω(Greek letter omega) and is expressed in radian/second. Using the expression again, the speed is then written:

$$\vec{V}(t) = \frac{d(\rho\vec{u}_\rho)}{dt} = \dot{\rho}\vec{u}_\rho + \rho\frac{d\vec{u}_\rho}{dt} = \dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta$$

$$\vec{V}(V_\rho = \dot{\rho}\ ;\ V_\theta = \rho\dot{\theta})$$

$$\|\vec{V}(t)\| = V = \sqrt{\dot{\rho}^2 + (\rho\dot{\theta})^2}$$

The quantities V_ρ and V_θ are respectively the ***radial*** and ***orthoradial*** components of the velocity in the polar base.

<u>Acceleration</u>

From the definition of the acceleration vector and the expression of the velocity we have:

$$\vec{a} = \frac{\mathrm{d}\vec{V}(t)}{\mathrm{d}t} = \frac{\mathrm{d}(\dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta)}{\mathrm{d}t} = \frac{\mathrm{d}(\dot{\rho}\vec{u}_\rho)}{\mathrm{d}t} + \frac{\mathrm{d}(\rho\dot{\theta}\vec{u}_\theta)}{\mathrm{d}t}$$

Using the usual rules for deriving a product and the expressions above we have :

$$\frac{d(\dot{\rho}\vec{u}_\rho)}{dt} = \frac{d(\dot{\rho})}{dt}\vec{u}_\rho + \dot{\rho}\frac{d(\vec{u}_\rho)}{dt} = \ddot{\rho}\vec{u}_\rho + \dot{\rho}[\dot{\theta}\vec{u}_\theta] = \ddot{\rho}\vec{u}\rho + \dot{\rho}\dot{\theta}\vec{u}_\theta$$

$$\frac{d(\rho\dot{\theta}\vec{u}_\theta)}{dt} = \frac{d(\rho)}{dt}\dot{\theta}\vec{u}_\theta + \frac{d(\dot{\theta})}{dt}\rho\vec{u}_\theta + \frac{d(\vec{u}_\theta)}{dt}\rho\dot{\theta} = \dot{\rho}\dot{\theta}\vec{u}_\theta + \rho\ddot{\theta}u_\theta + \rho\dot{\theta}[-\dot{\theta}\vec{u}_\rho]$$

$$\frac{d(\rho\dot{\theta}\vec{u}_\theta)}{dt} = -\rho\dot{\theta}^2\vec{u}_\rho + (\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\vec{u}_\theta$$

By grouping and ordering the different results we obtain the expression:

$$\vec{a} = \frac{d\vec{V}(t)}{dt} = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{u}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\vec{u}_\theta$$

The first term : $a_\rho = (\ddot{\rho} - \rho\dot{\theta}^2)$ corresponds to the **radial** component of the acceleration called the ***Normal acceleration*** denoted **a_N**

Le second terme $a_\theta = (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})$ corresponds to the ***orthoradiale*** component called the ***Tangential acceleration*** denoted **a_T**

II-3 Cylindrical coordinates

To obtain the cylindrical coordinate system, simply complete the polar coordinate system in the plane$(xOy$ $)$ with a third axis Oz .

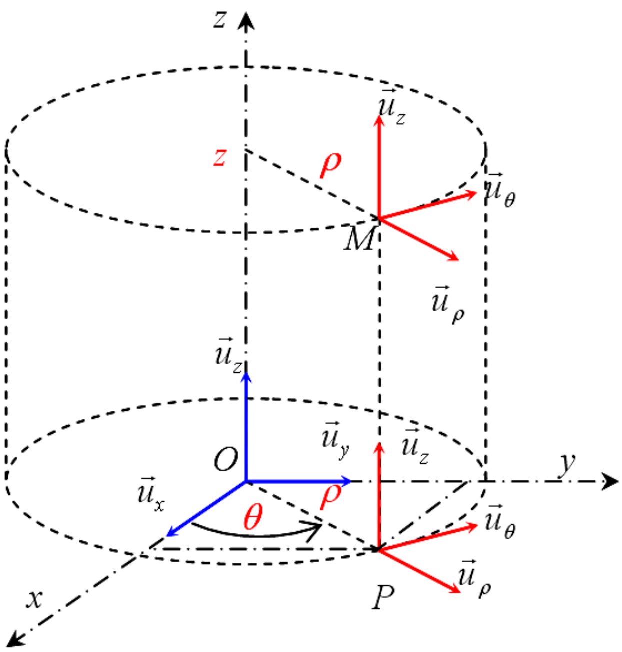


Figure 7: *cylindrical system* (ρ, θ, z) *and its associated basie* $(\vec{u}_\rho, \vec{u}_\theta, \vec{u}_z)$

The projection of M on the axis Oz gives the dimension z . The projection P of the point M in the plane (O,x,y) is located in polar coordinates (ρ,θ) .

$$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM} = \rho\vec{u}_\rho + z\vec{u}_z$$

$$\| \overrightarrow{OM} \| = OM = \sqrt{\rho^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$$

The cylindrical coordinates of M are :(ρ,θ,z)

The compenents of the position vector $\overrightarrow{OM}$ sont : $(\rho, 0, z)$

Velocity

Simply add the component along the axis Oz to the polar coordinate system to obtain the expression of the velocity. The basis vector $\vec{u}_z$ does not depend on time we have:

$$\vec{V}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{d(\rho\vec{u}_\rho + z\vec{u}_z)}{dt} = \frac{d(\rho\vec{u}_\rho)}{dt} + \frac{d(z\vec{u}_z)}{dt}$$

$$\vec{V}(t) = \dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta + \dot{z}\vec{u}_z$$

$$\vec{V}(V_\rho = \dot{\rho}\ ;\ V_\theta = \rho\dot{\theta}\ ;\ V_z = \dot{z})$$

$$\| \vec{V}(t) \| = V = \sqrt{\dot{\rho}^2 + (\rho\dot{\theta})^2 + \dot{z}^2}$$

Acceleration

Simply add the term corresponding to the derivation of the rating z. and we have:

$$\vec{a} = \frac{d\vec{V}(t)}{dt} = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{u}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\vec{u}_\theta + \ddot{z}\vec{u}_z$$

$$\vec{a} = \left(a_\rho = \ddot{\rho} - \rho\dot{\theta}^2\ ; a_\theta = 2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\ ; a_z = \ddot{z}\right)$$

II-4 Spherical coordinates

Spherical coordinates (see figure 8) allow you to locate a point on a sphere of radius $OM = r$ It is typically the location of a point on the Earth for which it is then sufficient to specify two angles: latitude and longitude.

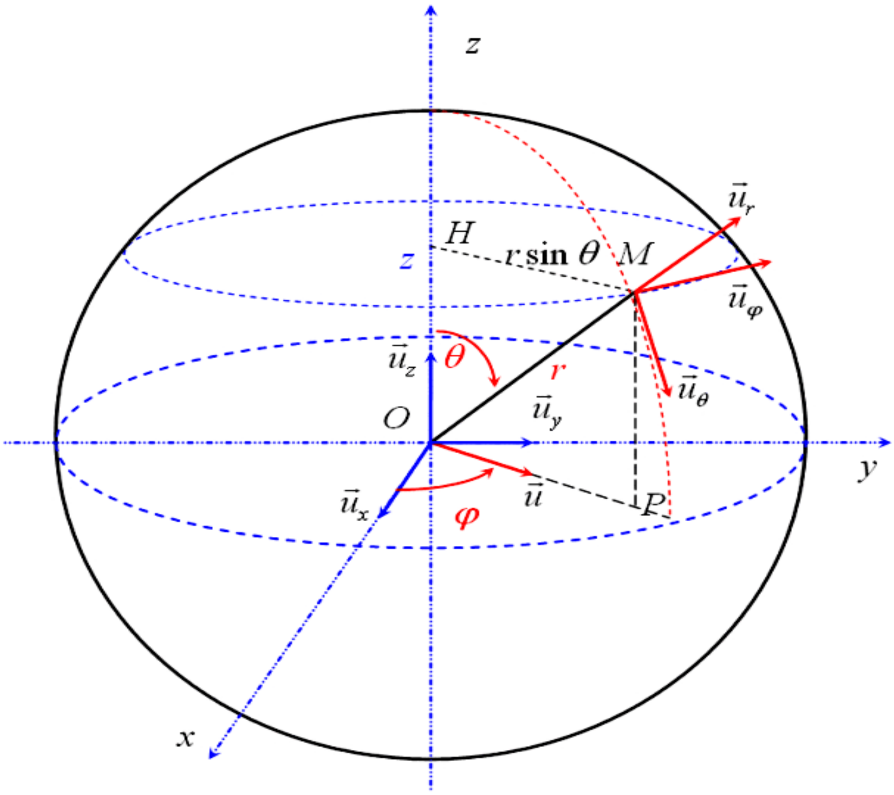


Figure 8 : the spherical system (r, θ, ϕ) *and its associated base* $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\phi)$

<u>Spherical coordinates</u> (r, θ, φ)

- The radial coordinate r corresponds to the distance from the origin O of the mark to the point M.
- The angular coordinate θ corresponds to the angle which made OM with the axis Oz . This angle, between 0 and π , is called colatitude (complementary angle of latitude) or zenith.
- The angular coordinate ϕ corresponds to the angle made by the plane defined by the axis Oz and OM with the axis Ox. This angle, between 0 and 2π , is called longitude or azimuth.

<u>Spherical basis</u>

The position vector is used to define the first vector of the base: $\overrightarrow{OM} = r\vec{u}_r$

- The unit vector $\vec{u}_r$ is according to the direction from O to M : it is the ***radial vector*** (according to the radius).
- When only the angle θ varies , the point M describes a semi-circle of radius r The unit vector $\vec{u}_\theta$ is tangent to this semicircle.
- When only the angle ϕ varies the point describes a circle of radius $r \sin \theta$. The unit vector $\vec{u}_\phi$ is tangent to this circle (following a parallel) oriented like ϕ.

The vectors $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\phi)$ form a direct orthonormal basis. This base is "mobile" in the reference frame.

Velocity

The expression of the velocity can be obtained from the expression of the elementary displacement . Using Figure 9, an elementary movement can be devided into:

- Radial elementary displacement $\mathrm{d}r$ according to $\vec{u}_r$ (the point moves away from the origin) The radial coordinate changes from r to $r + \mathrm{d}r$
- Elementary displacement according to θ (the point moves on the meridian) The colatitude changes from θ to $\theta + \mathrm{d}\theta$.
- Elementary displacement $r \sin \theta \,\mathrm{d}\phi$ according to $\vec{u}_\phi$ (the point moves along the parallel) The longitude changes from ϕ to $\phi + \mathrm{d}\phi$.

We therefore obtain the expression:

$$\mathrm{d}\overrightarrow{OM} = \mathrm{d}\vec{l} = \mathrm{d}r\vec{u}_r + r\mathrm{d}\theta\vec{u}_\theta + r \sin \theta \mathrm{d}\phi\vec{u}_\phi$$

Relation between spherical coordinates an cartesian ones

The projection H of the point M on the axis Oz gives :

$z = OH = r \cos \theta$

if P is the projection of M on the plan xOy we have : $OP = r \sin \theta$

the coordinates x and y of the point M are those of the point P so :

$x = OP \cos \phi = r \sin\theta \cos\phi$

$y = OP \sin \phi = r \sin\theta \sin\phi$

The unit vector $\vec{u}$ according to OP has the expression : $\vec{u} = \cos\phi \vec{u}_x + \sin\phi \vec{u}_y$

the unit vector $\vec{u}_\phi$ est directement perpendiculaire à $\vec{u}$. It do the angle $(\phi+\pi/2)$ with the axis Ox and can be written as :

$$\vec{u}_\phi = -\sin\phi\vec{u}_x + \cos\phi\vec{u}_y$$

The unit vector $\vec{u}_r$ has the expression :

$$\vec{u}_r = \sin\theta\vec{u} + \cos\theta\vec{u}_z = \sin\theta[\cos\phi\vec{u}_x + \sin\phi\vec{u}_y] + \cos\theta\vec{u}_z$$

$$\vec{u}_r = \sin\theta\cos\phi\vec{u}_x + \sin\theta\sin\phi\vec{u}_y + \cos\theta\vec{u}_z$$

And finally , the unit vector $\vec{u}_\theta$ is directly orthogonal to $\vec{u}_r$ and written :

$$\vec{u}_\theta = \cos\theta\vec{u} - \sin\theta\vec{u}_z = \cos\theta[\cos\phi\vec{u}_x + \sin\phi\vec{u}_y] - \sin\theta\vec{u}_z$$

$$\vec{u}_\theta = \cos\theta\cos\phi\vec{u}_x + \cos\theta\sin\phi\vec{u}_y - \sin\theta\vec{u}_z$$

Then the expression of the velocity will be done by :

$$V(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{l}}{dt} = \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta + r\sin\theta\dot{\phi}\vec{u}_\phi$$

Acceleration

The acceleration vector corresponds to the time derivative of the velocity, that is to say also to the second derivative of the position vector :

$$\vec{a} = \frac{d\vec{V}(t)}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2}$$

The acceleration in spherical coordinates is written in the form :

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\,\dot{\varphi}^2\sin^2\theta\right)\overrightarrow{u_r} + \left(r\ddot{\theta} + 2\dot{r}\,\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta\right)\overrightarrow{u_\theta}$$
$$+ \left(r\dot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta\right)\overrightarrow{u_\varphi}$$

II-5 Curvilinear coordinate

When we vary the position of the point M in an elementary way by describing the trajectory, the curvilinear abscissa of the point M goes from **s** to **s + ds** between the instant **t** and the instant **t + dt** (see figure 11). The elementary displacement of the point M is tangent to the trajectory and is then written:

$$d\overrightarrow{OM} = \overrightarrow{MM'} = ds\,\vec{u}_t$$

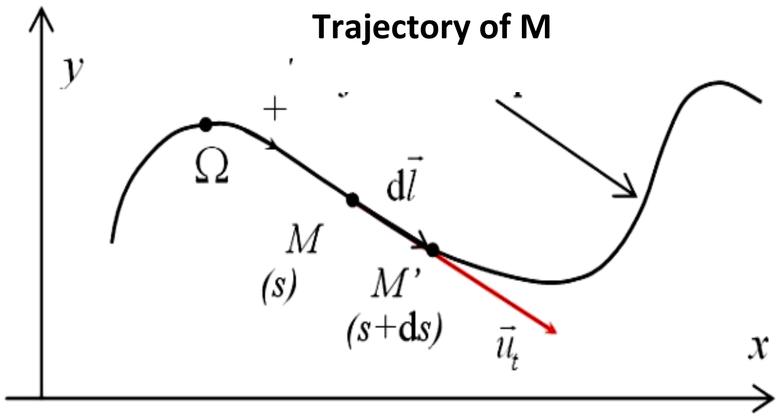


Figure 9 : Déplacement élémentaire dans le repère de Frenet

The velocity has the expression :

$$\vec{V} = \frac{ds}{dt}\vec{u}_t = \dot{s}\vec{u}_t = v\vec{u}_t$$

The quantity $v = \dot{s}$ corresponds to the algebraic value of the velocity (positive if the point moves in the chosen positive direction)

$$\|\vec{V}\| = \|\dot{s}\vec{u}_t\| = |\dot{s}|\,\|\vec{u}_t\| = |\dot{s}| = |v| = V$$

In the Frenet base the speed is written:

$$\vec{V} = \dot{s}\vec{u}_t = v\vec{u}_t$$

And the acceleration :

$$\vec{a} = \frac{d\dot{s}}{dt}\vec{u}_t + \dot{s}\frac{d\vec{u}_t}{dt}$$

The expression of the acceleration in the Frenet base is :

$$\vec{a} = \ddot{s}\vec{u}_t + \frac{v^2}{\rho}\vec{u}_n = \frac{dv}{dt}\vec{u}_t + \frac{v^2}{\rho}\vec{u}_n$$

Noticed

We can verify that this result is always true whatever the concavity of the trajectory.

- The normal component of the acceleration $a_n = \dfrac{v^2}{\rho}$ is always positive: it is always turned towards the center of curvature of the trajectory at the point considered.

It indicates that the direction of the speed vector changes and is all the more important as the radius of curvature R_c is small (sharp turn). If the movement is rectilinear (infinite radius of curvature) this term is zero.

- The tangential component of the acceleration $a_t = \frac{dv}{dt}$ indicates whether the value of the speed changes. If the movement is uniform this term is zero.

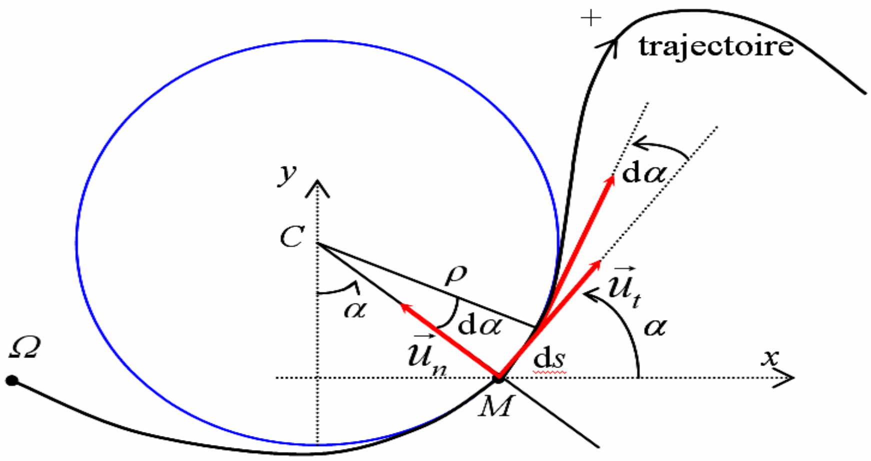


Figure 10 : Frenet base and elementary displacement ds

III MOVEMENTS

III-1 Rectilinear Movement

III-1-1 Uniform Rectilinear Movements URM :

A body has **constant velocity motion** or uniform rectilinear motion when its trajectory is a straight line and its velocity is constant. This implies that it covers equal distances in equal times.

It is characterized by:

- a rectilinear trajectory (a straight line)
- a constant speed (v = cte)
- zero acceleration (a = 0)

Velocity vector characteristics for uniform rectilinear motion

Its norm is constant and equal to the initial speed at the origin of the times: $v = v_0$.

Its direction is constant and corresponds to the direction of the movement

Position of a point in uniform rectilinear motion

Since the movement takes place along a straight line we can choose a reference in which the latter coincides with the abscissa axis. We then have OM = x

Since in this case the speed of the point M corresponds to the derivative of its abscissa x as a function of time then reciprocally the abscissa x corresponds to an antiderivative of the speed The time equation of the position is written :

$$v = \frac{dx}{dt} = cste \Rightarrow dx = vdt \quad donc \int_{x_0}^{x} dx = \int_{t_0}^{t} vdt = v\int_{t_0}^{t} dt$$

So generally :

$$x - x_0 = v \cdot (t - t_0)$$

We distinguish two cases:

$$t_0 = 0 \;;\; x_0 = 0 \quad \Rightarrow \quad x = v \cdot t \;;$$

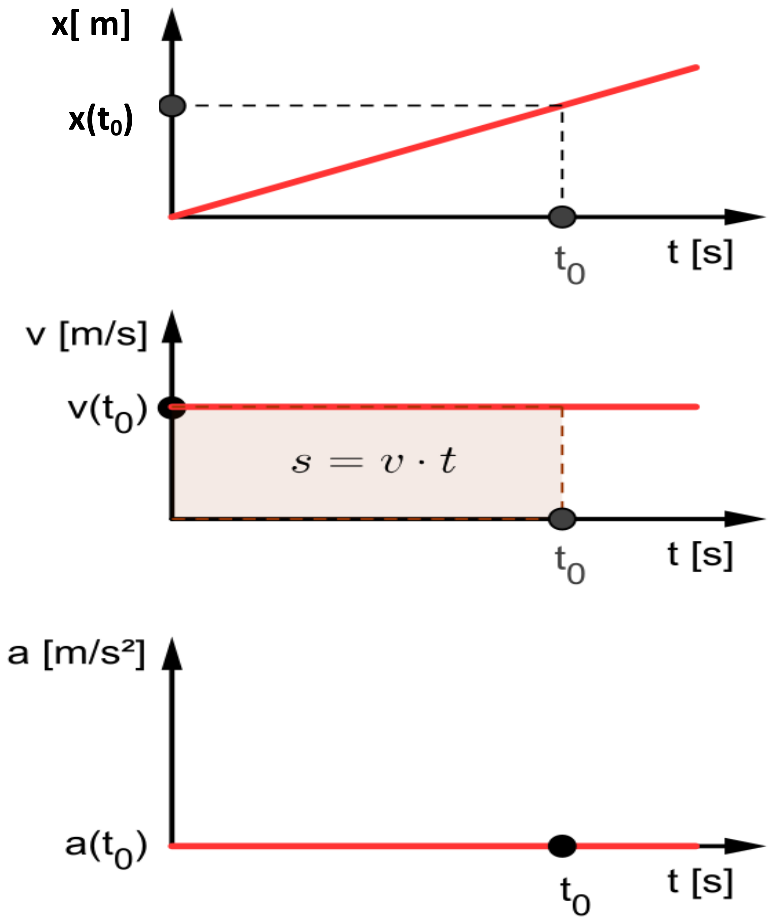
$$t_0 = 0 \;;\; x_0 = x_0 \quad \Rightarrow \quad x = v \cdot t + x_0 \;;$$

- if the speed is oriented in the same direction as the abscissa axis then vo > o and the abscissa is an increasing function.
- if the speed vo is oriented in the direction opposite to the abscissa axis then vo < o and the abscissa is a decreasing function.

Acceleration of a point in uniform rectilinear motion

Acceleration is the time derivative of the velocity. The latter beingconstant then its derivative is zero. The acceleration is therefore zero: a = 0

Graphs to describe URM :



III-1-2 Uniformly Varied Rectilinear Motion (UVRM)

The accelerated movement is a movement in which velocity changes with time. If there is an increase in velocity, the acceleration is positive and If the velocity diminishes, the acceleration is negative (deceleration).

Characteristics of the acceleration vector for uniformly varied rectilinear motion

Its norm is constant and equal to the initial acceleration: $a = a_0$. Its direction corresponds to that of the movement.

Speed of a uniformly varied moving point

Since acceleration corresponds to the time derivative of speed , then speed is a primitive of acceleration:

$$a_x = \frac{dv}{dt} = \frac{d^2x}{dt^2} = cste$$

The speed of a point in uniformly accelerated motion is therefore a function of time:

$v = a_x t + v_o$ where $a_x = a_0$ is the acceleration at $t = 0$ s

v_o is the speed at $t = 0$ s

Remarks

If the speed is zero at t = 0 then the speed is a linear function of time

Position of a point in uniformly varied motion

Since velocity is a time derivative of position , then position is a primitive of velocity.

$$v = a_x t + v_o$$

Therefore the primitive corresponding to the abscissa has the form:

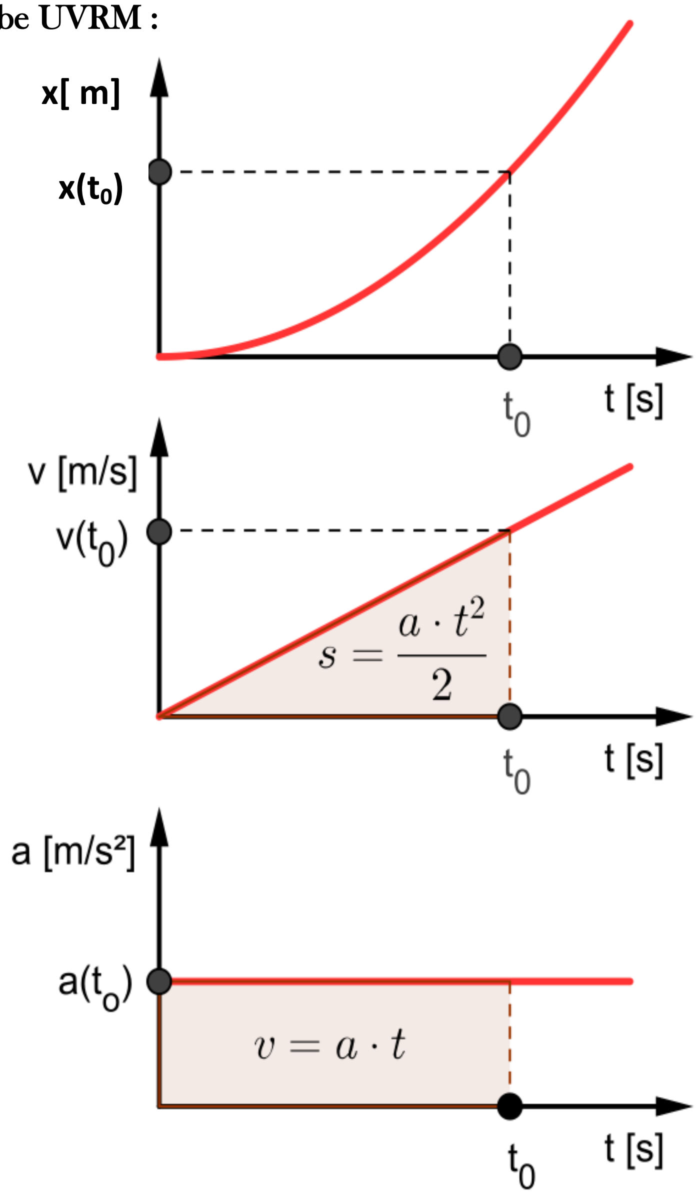
$$x = \frac{1}{2} a_x t^2 + v_o t + x_0$$

With $a_x = a_0$ acceleration of the mobile

v_o initial speed at t = 0

x_0 initial abscissa at t = 0

Graphs to describe UVRM :



III-2 Circular movement

The trajectory of the point is a circle characterized by its center O and its radius R . It is logical to choose the origin of the reference frame in the center of the circle and the axis Oz perpendicular to the plane containing the trajectory. The polar coordinate system is well suited for this type of movement. The time equations of motion can be written $\rho = R$ = const and $\theta = \theta(t)$ The shape of the function will qualify the type of circular motion.

Depending on the form of the function $\theta(t)$ the movement will be said to be circular and:

- Uniform if $\theta(t) = \omega_o t + \theta_o$ and $\dot{\theta} = \omega_o$ = constante
- Uniformly varied (accelerated or decelerated) if $\ddot{\theta} = \ddot{\theta}_o$ = constant either and
  $\dot{\theta} = \omega = \ddot{\theta}_o t + \dot{\theta}_o$ $\theta(t) = \frac{1}{2}\ddot{\theta}_o t^2 + \dot{\theta}_o t + \theta_o$
- Sinusoïdal if $\theta(t) = \theta_m \cos(\omega t + \phi)$

III-2-1 Uniform Circular Movement UCM :

The time equation $\theta(t)$ is obtained by integration of the equation :

$$\frac{d\theta}{dt} = \dot{\theta}(t) = \omega_o = cste$$

With, at the initial moment $\theta(t = 0) = \theta_o$ (initial elongation),

We have: $\theta = \omega_o t + \theta_o$

The velocity is written

$$\overrightarrow{OM}(t) = \rho \vec{u}_\rho(t) = R\vec{u}_\rho$$

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(R\vec{u}_\rho)}{dt} = R\frac{d\vec{u}\rho}{dt} = R\dot{\theta}\vec{u}_\theta = R\omega_o\vec{u}_\theta$$

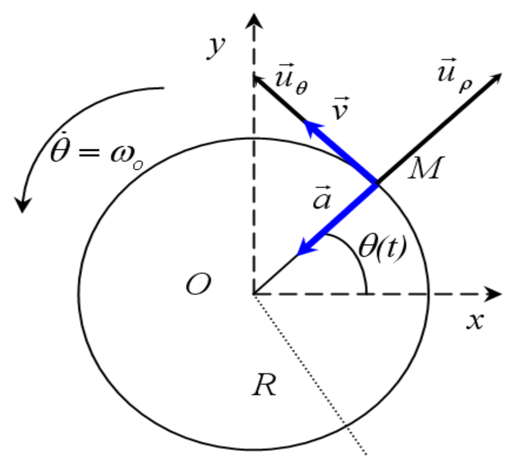


Figure 11: *Velocity and acceleration in the case of uniform circular motion*

The acceleration is written

The expression for the acceleration vector is simplified. The angular speed being constant, the tangential component of the acceleration vector is zero. All that remains is the normal component:

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}(R\omega_o\vec{u}_\theta)}{\mathrm{d}t} = R\omega_o\frac{\mathrm{d}\vec{u}_\theta}{\mathrm{d}t}$$

$$\vec{a} = R\omega_o(-\dot{\theta}\vec{u}_\rho) = -R\omega_o^2\vec{u}_\rho$$

$$\vec{a} = \vec{a}_n = \frac{v^2}{R}\vec{u}_n = R\omega_o^2\vec{u}_n = -R\omega_o^2\vec{u}_\rho$$

III-2-2 Uniformly Varied Circular Motion: UVCM

The kinematic characteristics of uniformly varied circular motion are: $\ddot{\theta} = \ddot{\theta}_o$ (the angular acceleration is constant and is equal to the initial angular acceleration).

The time equation for elongation is then obtained by integrating the time equation for angular velocity: $\dot{\theta} = \omega = \ddot{\theta}_o t + \dot{\theta}_o$

Which gives :

$$\theta(t) = \frac{1}{2}\ddot{\theta}_o t^2 + \dot{\theta}_o t + \theta_o$$

The linear velocity is written using the expression in polar coordinates which gives:

$$\vec{v} = \frac{\mathrm{d}(\overrightarrow{OM})}{\mathrm{d}t} = \frac{\mathrm{d}(R\vec{u}_\rho)}{\mathrm{d}t} = R\frac{\mathrm{d}\vec{u}_\rho}{\mathrm{d}t} \Rightarrow$$

$$\vec{v} = R\dot{\theta}\vec{u}_\theta = R\omega(t)\vec{u}_\theta$$

This result can be found using the expression for the speed in polar coordinates by setting $\rho = R = constante$.

The derivative of the velocity vector reveals two terms.

We have the linear acceleration:

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = R\frac{\mathrm{d}(\omega\vec{u}_\theta)}{\mathrm{d}t} = R\frac{\mathrm{d}\omega}{\mathrm{d}t}\vec{u}_\theta + R\omega\frac{\mathrm{d}\vec{u}_\theta}{\mathrm{d}t} \Rightarrow$$

$$\vec{a} = -R\omega^2\vec{u}_\rho + R\dot{\omega}\vec{u}_\theta$$

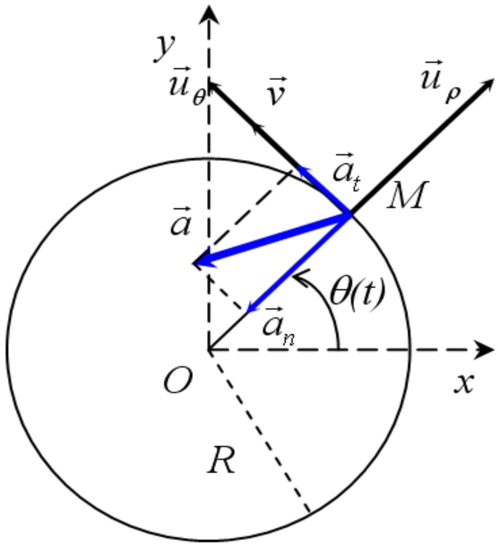


Figure 12: Velocity and acceleration vectors in the case of an UVCM.

IV- RELATIVE MOTION

Consider :

- R a fixed reference frame
- R_1, a movining reference frame
- A particle M, in motion, defined by its coordinates (x,y,z) in the frame R and by (x_1, y_1, z_1) in the frame R_1

By changing coordinates we can go from the movement of M relative to R_1 to the movement of M relative to R.

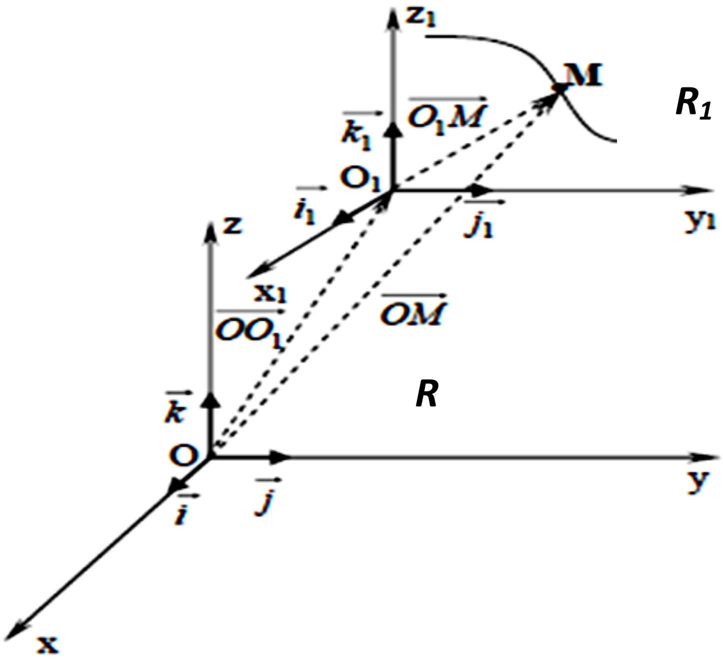
Simply apply the vector relationship: $\overrightarrow{OM} = \overrightarrow{OO_1} + \overrightarrow{O1M}$

To study this movement, we use the following definitions:

- R is the absolute reference or absolute frame of reference and ; the movement of point M relative to “R” is called absolute movement.
- R_1 is the relative reference or relative frame of reference and ; the movement of point M relative to “R_1” is called relative movement.
- The movement of “R_1” relative to “R” is called drive movement

Reference frame definitions

- **Inertial Reference Frame:** An inertial reference frame is non-accelerating and non-rotating. For now, we will also assume that it is non-moving.
- **Non-Inertial Reference Frame:** A non-inertial reference frame has an origin that accelerates and/or axes that rotate.
- **Body-fixed Reference Frame:** A body-fixed reference frame is fixed to a body that is usually moving.



Composition of velocities :

We have :

$$\overrightarrow{OM} = \overrightarrow{OO_1} + \overrightarrow{O_1M}$$

$$\vec{v}_a = \frac{d\overrightarrow{OM}}{dt} = \frac{d\overrightarrow{OO_1}}{dt} + \frac{d\overrightarrow{O_1M}}{dt}$$

Va is the absolute velocity of **M**

We know that in the reference frame R_1:

$$\overrightarrow{O_1M} = x_1\vec{i_1} + y_1\vec{j_1} + z_1\vec{k_1}$$

Then :

$$\vec{v_a} = \underbrace{\frac{d\overrightarrow{OO_1}}{dt} + x_1\frac{d\vec{i_1}}{dt} + y_1\frac{d\vec{j_1}}{dt} + z_1\frac{d\vec{k_1}}{dt}}_{\vec{v_e}} + \underbrace{\frac{dx_1}{dt}\vec{i_1} + \frac{dy_1}{dt}\vec{j_1} + \frac{dz_1}{dt}\vec{k_1}}_{\vec{v_r}}$$

$\vec{v_a} = \vec{v_e} + \vec{v_r}$ Is the absolute velocity

$\vec{v_r} = \frac{dx_1}{dt}\vec{i_1} + \frac{dy_1}{dt}\vec{j_1} + \frac{dz_1}{dt}\vec{k_1}$ Is the relative velocity

$$\vec{v_e} = \frac{d\overrightarrow{OO_1}}{dt} + x_1\frac{d\vec{i_1}}{dt} + y_1\frac{d\vec{j_1}}{dt} + z_1\frac{d\vec{k_1}}{dt}$$

Composition of accelerations

$$\vec{a_a} = \frac{d\vec{v_a}}{dt} = \frac{d\vec{v_e}}{dt} + \frac{d\vec{v_r}}{dt} = \frac{d}{dt}\left[\frac{d\overrightarrow{OO_1}}{dt} + x_1\frac{d\vec{i_1}}{dt} + y_1\frac{d\vec{j_1}}{dt} + z_1\frac{d\vec{k_1}}{dt}\right] + \frac{d}{dt}\left[\frac{dx_1}{dt}\vec{i_1} + \frac{dy_1}{dt}\vec{j_1} + \frac{dz_1}{dt}\vec{k_1}\right]$$

$$\Rightarrow \vec{a_a} = \frac{d^2\overrightarrow{OO_1}}{dt^2} + \frac{dx_1}{dt}\frac{d\vec{i_1}}{dt} + x_1\frac{d^2\vec{i_1}}{dt^2} + \frac{dy_1}{dt}\frac{d\vec{j_1}}{dt} + y_1\frac{d^2\vec{j_1}}{dt^2} + \frac{dz_1}{dt}\frac{d\vec{k_1}}{dt} + z_1\frac{d^2\vec{k_1}}{dt^2} + \frac{d^2x_1}{dt^2}\vec{i_1} + \frac{dx_1}{dt}\frac{d\vec{i_1}}{dt} + \frac{d^2y_1}{dt^2}\vec{j_1} + \frac{dy_1}{dt}\frac{d\vec{j_1}}{dt} + \frac{d^2z_1}{dt^2}\vec{k_1} + \frac{dz_1}{dt}\frac{d\vec{k_1}}{dt}$$

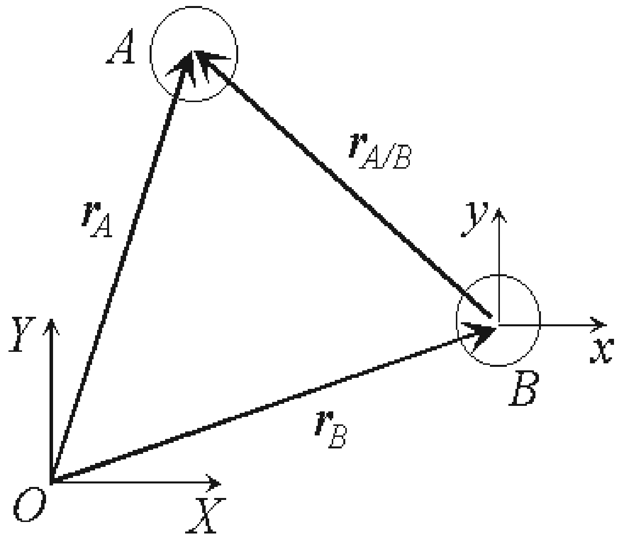
$$\Rightarrow \vec{a_a} = \underbrace{\frac{d^2\overrightarrow{OO_1}}{dt^2} + x_1\frac{d^2\vec{i_1}}{dt^2} + y_1\frac{d^2\vec{j_1}}{dt^2} + z_1\frac{d^2\vec{k_1}}{dt^2}}_{\vec{a_e}} + \underbrace{\frac{d^2x_1}{dt^2}\vec{i_1} + + \frac{d^2y_1}{dt^2}\vec{j_1} + \frac{d^2z_1}{dt^2}\vec{k_1}}_{\vec{a_r}}$$

$$\underbrace{+2\left[\frac{dx_1}{dt}\frac{d\vec{i_1}}{dt} + \frac{dy_1}{dt}\frac{d\vec{j_1}}{dt} + \frac{dz_1}{dt}\frac{d\vec{k_1}}{dt}\right]}_{\vec{a_c}}$$

Motion Relative to Translating Axes

In the course so far particle motion has been described using position vectors that were referred to fixed reference frames. The positions, velocities and accelerations determined in this way are referred to as absolute. Often it isn't possible or convenient to use a fixed set of axes for the observation of motion. Many problems are simplified considerably by the use of a moving reference frame.

In the following we will restrict our attention to moving reference frames that translate but do not rotate.



Consider two particles A and B moving along independent trajectories in the plane, and a fixed reference O. Let r_A and r_B be the positions of particles A and B in the fixed reference. Instead of observing the motion of particle A relative to the fixed reference as we have done in the past, we will attach a non-rotating reference to particle B and observe the motion of A relative to the moving reference at B.

We define its speed with respect to B to be :

$$\vec{r}_{AB} = \overrightarrow{BA} = \vec{r}_A - \vec{r}_B .$$

Where $$\vec{V}_{AB} = \frac{d\vec{r}_{AB}}{dt} = \frac{d\vec{r}_A}{dt} - \frac{d\vec{r}_B}{dt} \Rightarrow \boxed{\vec{V}_{AB} = \vec{V}_A - \vec{V}_B}$$

The speed of B relative to the observer $O:\overrightarrow{V_B} = \frac{d\vec{r}_B}{dt}$.

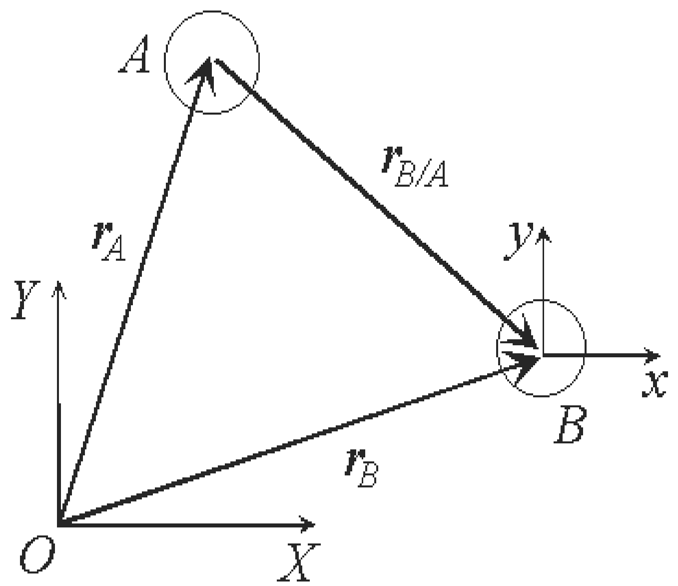
$$\vec{V}_{BA} = \frac{d\vec{r}_{BA}}{dt}$$

And we define its speed with respect to A to be :

Then :
$$\vec{r}_{BA} = \overrightarrow{AB} = \vec{r}_B - \vec{r}_A$$

$$\vec{V}_{BA} = \frac{d\vec{r}_{BA}}{dt} = \frac{d\vec{r}_B}{dt} - \frac{d\vec{r}_A}{dt} \Rightarrow \boxed{\vec{V}_{BA} = \vec{V}_B - \vec{V}_A}$$

Note that $\vec{V}_{AB} = -\vec{V}_{BA}$, that is to say that the speed of A relative to B is equal to the speed of B relative to A, but the two speeds are in opposite directions.



We obtain the two relative accelerations of the two moving material points by time deriving , each of the two expressions of the relative speeds posed previously:

$$\vec{a}_{AB} = \frac{d\vec{V}_{AB}}{dt} = \frac{d\vec{V}_A}{dt} - \frac{d\vec{V}_B}{dt} \qquad \boxed{\vec{a}_{AB} = \vec{a}_A - \vec{a}_B}$$

$$\vec{a}_{BA} = \frac{d\overrightarrow{V_{BA}}}{dt} = \frac{d\overrightarrow{V_B}}{dt} - \frac{d\overrightarrow{V_A}}{dt} \qquad \boxed{\vec{a}_{BA} = \vec{a}_B - \vec{a}_A}$$

Here too, it should be noted that $\vec{a}_{BA} = -\vec{a}_{AB}$ that is to say that the two accelerations are equal but in opposite directions.