Correction n°1

Logic and Mathematical Proof

Solution 1

1/ Using quantifiers, we write the propositions as follows:

1. $\exists k \in \mathbb{N},\ 102 = 3k$
2. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R},\ x^3 = y^3 \Longrightarrow x = y$
3. $\forall n \in \mathbb{N}, \exists m \in \mathbb{N},\ n < m$

2/ The negation of propositions:

1. $(\mathbf{P_1})$ is false because $\exists x = -1 \in \mathbb{R}, (-1+1)^2 = 0$. Thus $\bar{\mathbf{P}}_\mathbf{1} : \exists x \in \mathbb{R}, (x+1)^2 \leqslant 0$
2. $(\mathbf{P_2})$ is false and $\bar{\mathbf{P}}_\mathbf{2} : \forall x \in \mathbb{R}, x^2 \neq -1$
3. $(\mathbf{P_3})$ is false because $2x^2 + 3x = 0$ has two solutions $x_1, x_2 \in \mathbb{R}$. Hence $\bar{\mathbf{P}}_\mathbf{3}$:$\exists x \in \mathbb{R},\ 2x^2 + 3x = 0$
4. $(\mathbf{P_4})$ is false and $\bar{\mathbf{P}}_\mathbf{4} : \exists (x, y, z) \in \mathbb{R}^3,\ (xz = yz) \wedge (x \neq y)$

3/ The contrapositive of the propositions:

1. $(n \neq 2)$ and (n is even) $\Longrightarrow$ n is not prime
2. $\forall n \geqslant 2$, n is odd, then $(n^2 - 1)$ is divisible by 8.

Solution 2

1• Simplification of propositions :

$$\bullet [P \Longrightarrow (Q \Longrightarrow R)] \Longleftrightarrow [P \Longrightarrow (\bar{Q} \vee R)] \Longleftrightarrow \bar{P} \vee (\bar{Q} \vee R)$$

$$\bullet [(P \wedge Q) \Longrightarrow R] \Longleftrightarrow \overline{(P \wedge Q)} \vee R \Longleftrightarrow (\bar{P} \vee \bar{Q}) \vee R$$

According to the associativity of the connective **or** $(\vee)$, we deduce that the two propositions are equivalent: $[P \Longrightarrow (Q \Longrightarrow R)] \Longleftrightarrow [(P \wedge Q) \Longrightarrow R]$

2• The truth table of $\overline{(P \Longrightarrow Q)} \Longleftrightarrow (P \wedge \bar{Q})$ is

P	Q	$\bar{Q}$	$P \Longrightarrow Q$	$\overline{P \Longrightarrow Q}$	$P \wedge \bar{Q}$	$\overline{(P \Longrightarrow Q)} \Longleftrightarrow (P \wedge \bar{Q})$
T	T	F	T	F	F	T
F	F	T	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T

The table show that $\overline{P \Longrightarrow Q}$ and $P \wedge \bar{Q}$ have the same truth table. Therefore, the two propositions are equivalent. ie $\overline{(P \Longrightarrow Q)} \Longleftrightarrow (P \wedge \bar{Q})$ is true.

Solution 3

Let us show by exhaustion that:

$$\text{If } x \text{ is a real number, then } |x+3| - x > 2$$

We consider two cases: $x \geqslant -3$ and $x < -3$.

case 1: x $\geqslant$ −3. Then $|x+3| = x+3$, so we have $|x+3| - x = x+3-x = 3 > 2$, so the proposition holds.

case 2: x < −3. Then $|x+3| = -(x+3)$, so we have $|x+3| - x = -x-3-x = -2x-3$. Since $x < -3$, we must have $-x > 3$, so $-2x-3 > 2(3)-3 = 3 > 2$. Therefore, the proposition holds.

Since the proposition holds in all cases, it must be true. Hence if $x \in \mathbb{R}$, then $|x+3| - x > 2$.

Solution 4

Let us show by contradiction that:

$$\text{If } n^2+5 \text{ is odd, then } n \text{ is even, for every integer } n.$$

Suppose by contradiction, that n^2+5 is odd and n is also odd. By definition, there exists integers k and ℓ so that, $n^2+5 = 2k+1$ and $n = 2\ell+1$. Hence, we have

$$\begin{aligned} 2k+1 &= n^2+5 \\ &= (2\ell+1)^2+5 \\ &= 4\ell^2+4\ell+1+5 \\ &= 2(2\ell^2+2\ell+3) \end{aligned}$$

Therefore, $2k+1$ is even. This is clearly impossible, and hence we cannot have that n^2+5 is odd and n is also odd. Thus, if n^2+5 is odd, we must have n as even

Solution 5

Let us show by induction that:

$$\forall n \in \mathbb{N}, \quad n^3+2n \text{ is divisible by } 3$$

Let $P(n)$ be the proposition : n^3+2n is divisible by 3

1. **base case:** When $n = 0$, we have $0^3 + 2(0) = 0+0 = 0$ and 0 is divisible by 3.
   Thus, $P(0)$ is correct.

2. **induction hypothesis :** Let $n \in \mathbb{N}$, Assume that $P(n)$ is correct. That means : $\exists k \in \mathbb{N}$, $n^3 + 2n = 3k$, We will now show that $P(n+1)$ is correct :

   Where,

$$\begin{aligned} (n+1)^3+2(n+1) &= n^3+3n^2+3n+1+2n+2 \\ &= n^3+2n+3(n^2+n+1) \\ &= 3k+3(n^2+n+1) \\ &= 3(k+n^2+n+1) \end{aligned}$$

   Let $m = k+n^2+n+1$, then $\exists m = k+n^2+n+1 \in \mathbb{N}$, $(n+1)^3+2(n+1) = 3m$.
   We deduce that $(n+1)^3+2(n+1)$ is divisible by 3. Thus, $P(n+1)$ is correct.

3. **Conclusion :** By mathematical induction, $\forall n \in \mathbb{N}, \quad n^3+2n$ is divisible by 3.