University of Batna 2

Institute of Industrial Hygiene and Safety

**Module : Math 1 (L1)**

Dr.Bayarassou

# Chapter 1 : Logic and Mathematical Proof

## Introduction

The objective of this course is to provide students with a fundamental understanding of mathematical logic and proof techniques. In this chapter, we will first focus on the procedure of forming propositions through logical connectives. Following that, we will delve into the exploration of proof techniques.

## Contents

# 1 Proposition

**Definition 1.1**

*A proposition is a mathematical statement that can be either true or false, but it cannot be both.*

**Examples :** We usually use letters to denote proposition $P, Q, \dots$

| Propositions | Not Propositions |
|---|---|
| $P_1$ : Four is even | $Q_1$ : can you pass the sheet of paper |
| $P_2$ : $1 = 0$ | $Q_2$ : $3+6$ |
| $P_3$ : Triangle has three sides | $Q_3$ : for all real $x$, $(2x+1)e^{-x}$ |

**Remark 1.** *The trurh or falsity of a proposition is called its* ***truth value****.*

The two **truth values** of a proposition are «true» which denoted by " **T** " and «false» which denoted by " **F** ". These can be visualized in a table called a **truth table**, which helps determine the validity of certain propositions based on the truth values of other propositions. Therefore, to describe a proposition, we create its truth table as follows:

| P |
|---|
| T |
| F |

**Remark 2.** *Two propositions are equivalent if they have the same truth table, and we denote this by $P \Longleftrightarrow Q$. (equivalence is to logic as equality is to algebra.)*

| $P$ | $Q$ |
|---|---|
| $T$ | $T$ |
| $F$ | $F$ |

## 1.1 Logical Operations

Let $P$ be a proposition and $Q$ be another proposition, we usually combine simple statements $P$ and $Q$ into compound statements to create a new proposition based on **logical connectives** $(\wedge, \vee, \Longleftrightarrow, \Longrightarrow)$ which we will see below :

**Definition 1.2 (*Negation*)**

*If $P$ is a proposition, its negation not $(P)$ denoted $\overline{P}$, is true when $P$ is false and false when $P$ is true.*

**The truth table**

| $P$ | $\bar{P}$ |
|---|---|
| T | F |
| F | T |

**Examples**

$Q$ : "Today is Friday", its negation $\overline{Q}$ : "Today is not Friday"

$P : 10 > 2^3$ is true, while its negation $\overline{P} : 10 \leqslant 2^3$ is false.

## Rules regarding negation

Let $x$ and $y$ be two variables :

1. $\overline{x \leqslant y} \Longleftrightarrow x > y$
2. $\overline{x < y} \Longleftrightarrow x \geqslant y$
3. $\overline{x = y} \Longleftrightarrow x \neq y$

**Definition 1.3 (*Disjunction*)**

*If $P$ and $Q$ are propositions, their disjunction, $P$ or $Q$ denoted $P \vee Q$, is true if at least one of the propositions $P$ or $Q$ is true and false if $P$ and $Q$ are false.*

**The truth table:**

| P | Q | P ∨ Q |
|---|---|---|
| T | T | T |
| F | T | T |
| T | F | T |
| F | F | F |

**Examples**

**1**/ $P\,:\,1+3=6, \qquad Q\,:\,10 > 2^3$

$P \vee Q\,:\,(1+3=6) \vee \left(10 > 2^3\right)$ it is true because $Q$ is true.

**2**/ ($\mathbb{Z} \subset \mathbb{N}$) or (2 is a prime numbre)

**Definition 1.4 (*Conjunction*)**

*If $P$ and $Q$ are propositions, their conjunction, $P$ and $Q$ denoted $P \wedge Q$, is true only if both $P$ and $Q$ are true, otherwise, it is false.*

**The truth table :**

| P | Q | $P \wedge Q$ |
|---|---|---|
| T | T | T |
| F | T | F |
| T | F | F |
| F | F | F |

**Examples**

1/ $\left(\sin(\frac{\pi}{3}) = \frac{1}{2}\right)$ and $\left(\sqrt{3} \in \mathbb{Q}\right)$ 2/ $(1+5=6) \wedge (2\times 3 = 6)$

**Definition 1.5 (*Implication*)**

*If $P$ and $Q$ are propositions, the proposition $\overline{P} \vee Q$ is called implication, denoted by $P \Longrightarrow Q$ which is read as " $P$ implies $Q$ " or " if $P$, then $Q$ ". It is only false when $P$ is true and $Q$ is false; otherwise it is true.*

**The truth table :**

| P | Q | $P \Longrightarrow Q$ |
|---|---|---|
| T | T | T |
| F | T | T |
| T | F | F |
| F | F | T |

**Examples**

1. If I pass the final, then I'll graduate.
2. if $x \geqslant 0$, then the function $f(x) = \sqrt{x}$ is well defined.
3. $(2+3=7) \Longrightarrow (\sqrt{2}=2)$ .

**Definition 1.6 (*Equivalence*)**

*If $P$ and $Q$ are propositions, $P$ and $Q$ are equivalent if ($P \Longrightarrow Q$ and $Q \Longrightarrow P$ ), which is denoted by ($P \Longleftrightarrow Q$) and read as $P$ if and only if $Q$. The equivalence is true when $P$ and $Q$ are the same, i.e. both are false or both are true.*

**The truth table :**

| P | Q | $P \Longrightarrow Q$ | $Q \Longrightarrow P$ | $P \Longleftrightarrow Q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| F | T | T | F | F |
| T | F | F | T | F |
| F | F | T | T | T |

**Examples:**

1. $(z \in \mathbb{C}, z = \bar{z}) \Longleftrightarrow z \in \mathbb{R}$
2. $x^2 = 1 \Longleftrightarrow (x = 1)$ or $(x = -1)$

## 1.2 Basic Logical Laws

Let $P$, $Q$ and $R$ be logical propositions:

1. **C**ommutative Laws
   (a) $(P \wedge Q) \Longleftrightarrow (Q \wedge P)$
   (b) $(P \vee Q) \Longleftrightarrow (Q \vee P)$
2. **D**istributive Laws
   (a) $P \wedge (Q \vee R) \Longleftrightarrow (P \wedge Q) \vee (P \wedge R)$
   (b) $P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$
3. **A**ssociative Laws
   (a) $P \wedge (Q \wedge R) \Longleftrightarrow (P \wedge Q) \wedge R$
   (b) $P \vee (Q \vee R) \Longleftrightarrow (P \vee Q) \vee R$
4. **D**eMorgan's Laws
   (a) $\left[\overline{P \vee Q}\right] \Longleftrightarrow \left[\overline{P} \wedge \overline{Q}\right]$
   (b) $\left[\overline{P \wedge Q}\right] \Longleftrightarrow \left[\overline{P} \vee \overline{Q}\right]$
5. **C**ontrapositive : $[P \Longrightarrow Q] \Longleftrightarrow \left[\overline{Q} \Longrightarrow \overline{P}\right]$
6. $\left[\overline{P \Longleftrightarrow Q}\right] \Longleftrightarrow \left[P \wedge \overline{Q}\right] \vee \left[Q \wedge \overline{P}\right]$
7. $\left[\overline{P \Longrightarrow Q}\right] \Longleftrightarrow \left[P \wedge \overline{Q}\right]$
8. $\overline{(\overline{P})} \Longleftrightarrow P$

# 2 Quantifiers

The statement $(x > 1)$ is not complete, because it contains a variable $x$, and we cannot answer the question: the statement $(x > 1)$ is it true? because the answer depends on $x$. So, the proposition depending on a variable $x$ is complete if the variable is quantified by a quantifiers $\forall$ or $\exists$.

Let $P(x)$ be a proposition depending on a variable $x$, where $x$ is an element of a set $E$.

## 2.1 Universal Quantifier $\forall$

The universal Quantifier $\forall$ is used to express that the formula following holds for all values of the particular variable quantified.
The notation is : $\forall x, P(x)$, meaning " for all $x, P(x)$ is true ".

**Examples :**

1. $\forall x \in \mathbb{R}, x^2+1>0$ meaning : for all real numbers $x$, $x^2+1$ is strictly positive.
2. $\forall x \in [1,+\infty[, (x^2>1)$ .
3. $\forall x \in \mathbb{R}, (x^2>1)$ .

## 2.2 Existential Quantifier $\exists$

The existential quantifier $\exists$ expresses that the formula following holdes for some (at least one) value of that quantified variable.
The notation is : $\exists x, P(x)$, meaning " there is at least one $x, P(x)$ is true ".

**Examples :**

1. $\exists x \in ]0,+\infty[, x^2-6x+1=0$ meaning : there exists at least one real $x$ strictly positive such that $x^2-6x+1$ equals 0.
2. $\exists x \in \mathbb{R}, (x^2=-1)$.
3. $\exists x \in \mathbb{R}, (x(x-1)<0)$.

**Remark 3.** *The notation $\exists!$ is called **uniqueness quantifier** states that "there exists a unique".*

- $\exists! n \in \mathbb{N}^*, \dfrac{n(n+1)}{2}=3$ *states : there exists a unique natural number $n \neq 0$ shuch that $\dfrac{n(n+1)}{2}$ equals* $3$ .

### 2.3 Quantifiers Order

When quantifiers in the same proposition are of the same quantity ( all universal or all existential), the order in which they occur does not matter. But when they are mixed, the order becomes crucial. we show two cases :

- **case** 1 : $\forall x, \exists y$ states that $y$ depends on $x$.
- **case** 2 : $\exists x, \forall y$ states that $x$ is not related to $y$.

**Example**

1. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 0$ (True)
   We can find $y$ that makes $x + y = 0$, for all integers $x$ there exists $y$ such that: $y = 5 - x$.
2. $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 0$ (False)
   There is no $y$ that will make $x + y = 0$, for all integers $x$.
3. $\exists M \in \mathbb{R}, \forall n \geqslant 1, \dfrac{1}{n+1} \leqslant M$
   there exists at least one real $M$ which is not related to $n$ such that for all $n$ greater than or equal to 1, $\dfrac{1}{n+1} \leqslant M$. Indeed, $\exists M = \dfrac{1}{2}, \forall n \geqslant 1, \dfrac{1}{n+1} \leqslant \dfrac{1}{2}$

**Negation of Quantified Propositions**

When we negate a quantified proposition, the existential and universal quantifiers complement one another.

1. $\overline{\forall x \in E, P(x)} \Longleftrightarrow \exists x \in E, \overline{P(x)}$.
2. $\overline{\exists x \in E, P(x)} \Longleftrightarrow \forall x \in E, \overline{P(x)}$.
3. $\overline{\forall x \in E, \exists y \in E, p(x,y)} \Longleftrightarrow \exists x \in E, \forall y \in E, \overline{P(x,y)}$.
4. $\overline{\exists x \in E, \forall y \in E, p(x,y)} \Longleftrightarrow \forall x \in E, \exists y \in E, \overline{P(x,y)}$.

## 3 Proof Techniques

Mathematical statements can typically be phrased as an implication $P \Longrightarrow Q$, and we have many ways to prove it. One of the following modes is used :

### 3.1 Direct Proof

Direct proof is the simplest and easiest method of proof available to us, There are only two steps :

1. Assume that $P$ is true.
2. Use $P$ to show that $Q$ must be true.

**Example** prove that $\forall a, b \in \mathbb{Q} \Longrightarrow a+b \in \mathbb{Q}$.

**Proof:**

let $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$. The set of all rational numbers $\mathbb{Q}$ are defined as $\frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$.
Write $a = \frac{p}{q}$ with $p \in \mathbb{Z}$, $q \in \mathbb{N}^*$. and $b = \frac{p'}{q'}$ with $p' \in \mathbb{Z}$, $q' \in \mathbb{N}^*$.
Now

$$a+b = \frac{p}{q} + \frac{p'}{q'} = \frac{pq' + qp'}{qq'}$$

Where, the numerator $pq' + qp'$ is an element of $\mathbb{Z}$; and the denominator $qq'$ is an element of $\mathbb{N}^*$. So $a+b$ is written as $a+b = \frac{p''}{q''}$ with $p'' = pq' + qp' \in \mathbb{Z}$, and $q'' = qq' \in \mathbb{N}^*$. Thus $a+b \in \mathbb{Q}$. ■

## 3.2 Proof by Exhaustion

Proof by exhaustion ( also known as **Proof by Cases**) is the proof that something is true by showing that it is true for each and every case that could possibly be considered. We can generalise the method of proof by exhaustion using two steps:

1. Establish the cases that apply to the statement.
2. Prove that the statement is true for each case. Each case of the statement needs to be proved separately, one by one. We need to exhaust all of the cases to verify the statement.

**Example**

Show that $\forall n \in \mathbb{N} \Longrightarrow \frac{n(n+1)}{2} \in \mathbb{N}$.

**Proof:**

**Step 1:** Split the statement into a finite number of cases. Thus we can write $n = 2k$ or $n = 2k+1$, with $k$ being any natural number.

**Step 2:** Now prove that the statement is true for each case.

**case 1:** $n$ is even, $\exists k \in \mathbb{N}$ such as $n = 2k \Longrightarrow \frac{2k(2k+1)}{2} = k(2k+1) \in \mathbb{N}$.

**case 2:** $n$ is odd, $\exists k \in \mathbb{N}$ such as $n = 2k+1 \Longrightarrow \frac{(2k+1)(2k+1)+1}{2} = \frac{(2k+1)(2k+2)}{2} = (2k+1)(2k+1) \in \mathbb{N}$

As both cases satisfy the statement, we have proved that the given statement is correct. ■

**Exercice :** Show that, if $n$ is a positive integer, then $n(n^4-1)$ is divisible by 2.

## 3.3 Proof by Contrapositive

Proof by Contrapositive is focused on the following formula :

$$[P \Longrightarrow Q] \Longleftrightarrow [\bar{Q} \Longrightarrow \bar{P}]$$

So, in order to prove the proposition $P \Longrightarrow Q$, we prove that $\bar{Q}$ is true, then $\bar{P}$ is true.
**Example** for $n \in \mathbb{N}$. prove that, if $n^2$ is even, then $n$ is even.

**Proof:**

We prove this proposition by proving its contrapositive statement : if $n$ is odd, then $n^2$ is odd.
As $n$ is odd, So $\exists k \in \mathbb{N}$ such that $n = 2k+1$. Then $n^2 = (2k+1)^2 = 4k^2+4k+1 = 2\ell+1$ with $\ell = 2k^2+2k \in \mathbb{N}$. Hence $n^2$ is odd.
Conclusion : By contraposition : if $n^2$ is even, then $n$ is even. ■

**Exercice :** Prove that, for all integers $a$ and $b$, if $a+b$ is odd, then $a$ is odd or $b$ is odd.

**Important results**

1. $n$ is odd $\Longleftrightarrow n^2$ is odd.
2. $n$ is even $\Longleftrightarrow n^2$ is even.

## 3.4 Proof by Contradiction

A proof by contradiction is a common proof technique that is based on a very simple principle; something that leads to a contradiction can not be true, and if so, the opposite must be true. It process as :

1. Assume that $P$ is true and $Q$ is false.
2. Look for a contradiction.
3. State that because of the contradiction, it can't be the case that the statement is false, so it must be true.

**Example** Prove that if $ab$ is irrational, then at least one of $a$ and $b$ are also irrational.

**Proof:** Let us assume that $ab$ is irrational, but $a$ and $b$ are rational.

Write $a = \frac{c}{d}$, and $b = \frac{e}{f}$, with $c, d, e, f \in \mathbb{Z}$ and $d, f \neq 0$.

Thus $ab = \frac{ce}{df}$, and as $c, d, e, f$ are integers, then this implies that $ab$ is rational, which is a contradiction.

Consequently, if $ab$ is irrational, then at least one of $a$ and $b$ are also irrational. ■

**Example :** Prove that $\sqrt{2}$ is irrational.

**Proof:** Assume by contradiction, that $\sqrt{2}$ is a rational number. Then there exists $p, q \in \mathbb{Z}, (q \neq 0)$ where $\frac{p}{q} = \sqrt{2}$ and where $\frac{p}{q}$ is in lowest terms, that is, $p$ and $q$ have no common factor other than 1.

$$
\begin{aligned}
\frac{p}{q} = \sqrt{2} &\Longrightarrow \frac{p^2}{q^2} = 2 \\
&\Longrightarrow p^2 = 2q^2 \\
&\Longrightarrow p^2 \text{ is an even integer} \\
&\Longrightarrow p \text{ is an even integer} \\
&\Longrightarrow 4 \text{ is a factor of } p^2 \\
&\Longrightarrow q^2 \text{ is an even} \\
&\Longrightarrow q \text{ is an even}
\end{aligned}
$$

Hence both $p$ and $q$ have a common factor, namely 2, which is a contradiction. ■

## 3.5 Proof By Counterexample

This proof structure allows us to prove that a property is not true by providing an example where it does not hold.

1. State your proposition and what you need to prove clearly.
2. Find a counter example, we can do this by testing examples and eventually we will be able to find a good example.

**Remark 4.** *When proving by counterexample, it is not enough to state the counterexample. One must also explain why it is a counterexample.*

**Example :** Prove that : If $n$ is prime, then $2^n - 1$ is prime.

**Proof:** The case $n = 11$ is a counterexample, indeed,

$$2^{11} - 1 = 2047 = 23 \times 89 \text{ is not prime even though } n = 11 \text{ is prime.}$$ ■

**Exercice :** Give a counterexample to disprove the statement :

- All even numbers are not prime.

## 3.6 Proof by Induction

Proof by induction is a very powerful method in which we use recursion to demonstrate an infinite number of facts in a finite amount of space. Let the proposition $P(n), n \geqslant n_0$ be true for $n \in \mathbb{N}$. The method of proof by Induction follow three steps :

1. **base case** Show that $P(n_0)$ is true for the initial value of $n$.
2. **induction hypothesis** Assume that $P(n)$ is true for some $n$, and show that this implies that $P(n+1)$ is true.
3. **induction step** Then, by the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$.

**Example** proves that, for all $n \in \mathbb{N}$, $2^n > n$.

**Proof:** for $n \geqslant 0$, let $P(n)$ be the statement that

$$2^n > n.$$

We will show that $P(n)$ is true for all $n \geqslant 0$.

**base case**: When $n = 0$ we have $2^0 = 1 > 0$. So $P(0)$ is true.

**Induction hypothesis**: Assume that $P(n)$ is true for some positive integer $n$. We will prove that $P(n+1)$ is true.

$$\begin{aligned} 2^{n+1} = 2^n + 2^n &> n + 2^n \quad \text{because from } P(n) \text{ we have } 2^n > n \\ &> n + 1 \quad \text{because } 2^n \geqslant 1 \end{aligned}$$

Then $P(n+1)$ is true.

**Induction step**: by mathematical induction $P(n)$ is true for all $n \geqslant 0$, ie $2^n > n$ for all $n \geqslant 0$ ■

**Exercices** Prove by induction that :

- $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$ for every positive integer n.
- $9^n - 1$ is divisible by 8 for all non-negative integers.

The students wishing to delve deeper and practice more exercices is invited to consult these references :

# References

[1] J. Rivaud, Algèbre : Classes préparatoires et Université Tome 1, Exercices avec solutions, Vuibert.

[2] http://exo7.emath.fr/ficpdf/fic00002.pdf

[3] https://www.bibmath.net/ressources/index.php?action=affiche&quoi=bde/logique.html