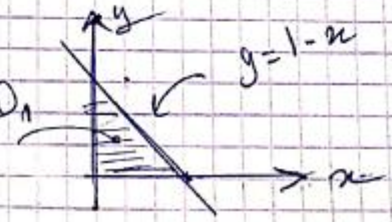
Ex 3 :

A. Calculons $\iint_{D_1} \left(\frac{x-y}{x+y}\right)^4 dy\,dx$ par le changement de variable : $u = x-y$ et $v = x+y$



• Exprimons x, y en fonction de u, v :

$$\begin{cases} u = x-y \\ v = x+y \end{cases} \Rightarrow \begin{cases} u+v = 2x \\ v-u = 2y \end{cases} \Rightarrow \begin{cases} x = \frac{u+v}{2} \\ y = \frac{v-u}{2} \end{cases}$$

• Calculons le Jacobien J :

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = +\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

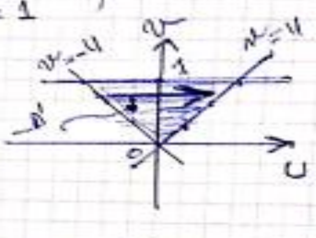
l'intégrale devient alors : $\iint_{D'} \frac{u^4}{v^4} \cdot \frac{1}{2} \cdot du\,dv$

• Les bornes d'intégration : D'

$$\left.\begin{array}{l} x = 0 \Rightarrow \frac{u+v}{2} = 0 \Rightarrow v = -u \\ y = 0 \Rightarrow \frac{v-u}{2} = 0 \Rightarrow v = u \\ y = 1-x \Rightarrow \frac{v-u}{2} = 1 - \frac{u+v}{2} \Rightarrow v = 1 \end{array}\right\} \Rightarrow \begin{array}{l} -v \leq u \leq v \\ 0 \leq v \leq 1 \end{array}$$

Domaine régulier en u (irrégulier en v) $\overset{Ou}{\Longrightarrow}$

donc intégrons selon Ou :



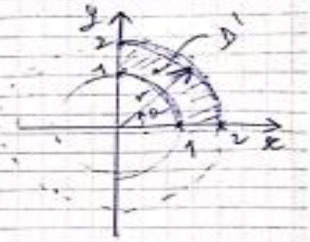
$$\int_0^1 \int_{-v}^{v} \frac{u^4}{v^4} \cdot \frac{1}{2} \cdot du\,dv =$$

$$\frac{1}{2}\int_0^1 \left[\int_{-v}^{v} \frac{u^4}{v^4} du \right] dv = \frac{1}{2}\int_0^1 \left(\left[\frac{v^5}{5v^4}\right] - \left[\frac{-v^5}{5v^4}\right] \right) dv =$$

$$\frac{1}{2}\int_0^1 \frac{2v^5}{5v^4} dv = \frac{1}{5}\int_0^1 v\,dv = \frac{1}{5}\left[\frac{v^2}{2}\right]_0^1 =$$

$$\frac{1}{5}\left[\frac{1}{2} - 0\right] = \frac{1}{10}$$

B. Calculons $\iint_{D_2} dx\,dy$ en utilisant les coordonnées polaires :



- les coordonnées polaires :

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

- le Jacobien $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} =$

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) = r$$

- les bornes d'intégration : D'

$$\begin{cases} 1 \leq x^2+y^2 \leq 4 \Rightarrow 1 \leq r^2 \leq 4 \Rightarrow 1 \leq r \leq 2 \\ x \geq 0 \Rightarrow r\cos\theta \geq 0 \Rightarrow \cos\theta \geq 0 \\ y \geq 0 \Rightarrow r\sin\theta \geq 0 \Rightarrow \sin\theta \geq 0 \end{cases} \Rightarrow \text{1}^{\text{er}} \text{ quart} \quad 0 \leq \theta \leq \pi/2$$

l'intégrale devient alors : $\iint_{D'} \frac{1}{r^2} \cdot r \cdot dr\,d\theta$

$$\int_0^{\pi/2} \int_1^2 \frac{r\,dr\,d\theta}{r^2} =$$

$$\left(\int_1^2 \frac{dr}{r}\right) \cdot \left(\int_0^{\pi/2} d\theta\right) = \left[\ln r\right]_1^2 \cdot \left[\theta\right]_0^{\pi/2} =$$

$$\left[\ln 2 - \ln 1\right] \cdot \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2} \ln 2.$$