Le schéma (S3) est au moins d'ordre 2 ssi

$$\phi(t,y,0) = f(t,y)$$

$$\frac{\partial \phi}{\partial h}(t,y,0) = \frac{1}{2} f^{(1)}(t,y).$$

On a : $\phi(t,y,0) = (a+b) f(t,y)$.

$$\frac{\partial \phi}{\partial h}(t,y,h) = \frac{\partial}{\partial h}\left[a f(t,y) + b f\left(t+\frac{2}{3}h, y+\frac{2}{3}h f\left(t+\frac{h}{3}, y+\frac{h}{3} f(t,y)\right)\right)\right]$$

$$= b \frac{\partial}{\partial h} f\left(t+\frac{2}{3}h, y+\frac{2}{3}h f\left(t+\frac{h}{3}, y+\frac{h}{3} f(t,y)\right)\right)$$

$$= b\left[\frac{\partial f}{\partial t}\left(t+\frac{2h}{3}, y+\frac{2h}{3} f\left(t+\frac{h}{3}, y+\frac{h}{3} f(t,y)\right)\right)\cdot\frac{2}{3} + \frac{\partial f}{\partial y}\left(t+\frac{2h}{3}, y+\frac{2h}{3} f\left(t+\frac{h}{3}, y+\frac{h}{3}\right.\right.\right.$$
$$\left.\times \frac{2}{3} f\left(t+\frac{h}{3}, y+\frac{h}{3} f(t,y)\right)\right).$$

$$= \frac{2b}{3}\left[\frac{\partial f}{\partial t}\left(t+\frac{2h}{3}, y+\frac{2h}{3} f\left(t+\frac{h}{3}, y+\frac{h}{3} f(t,y)\right)\right) + \frac{\partial f}{\partial y}\left(t+\frac{2h}{3}, y+\frac{2h}{3} f\left(t+\frac{h}{3}, y+\frac{h}{3} f(t,y)\right)\right)\right.$$
$$\left.\times f\left(t+\frac{h}{3}, y+\frac{h}{3} f(t,y)\right)\right]$$

Donc $\frac{\partial \phi}{\partial h}(t,y,0) = \frac{2b}{3}\left[\frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y) \times f(t,y)\right]$

$$= \frac{2b}{3} f^{(1)}(t,y)$$

donc les conditions sont $\begin{cases} a+b = 1 \Rightarrow a = 1-b \\ \frac{2b}{3} = \frac{1}{2} \Rightarrow b = \frac{3}{4} \end{cases} \Rightarrow \begin{cases} a = \frac{1}{4} \\ b = \frac{3}{4} \end{cases}.$

4). Montrons que (S3) ~~est exacte~~ est d'ordre 3.

On a : $\phi(t,y,h) = \frac{1}{4} f(t,y) + \frac{3}{4} f\left(t+\frac{2}{3}h, y+\frac{2h}{3} f\left(t+\frac{h}{3}, y+\frac{h}{3} f(t,y)\right)\right]$