Exercise Sheet N° 02 Correction Numerical Analysis I

Exercise 1 :

Let the equation $f(x) = sinx - x + 1 = 0$.
We write $f(x) = f_1(x) - f_2(x)$ such that :$f_1(x) = sinx$ and $f_2(x) = x - 1$.

Graphically this equation admits a single root in the interval $[1, 2]$.

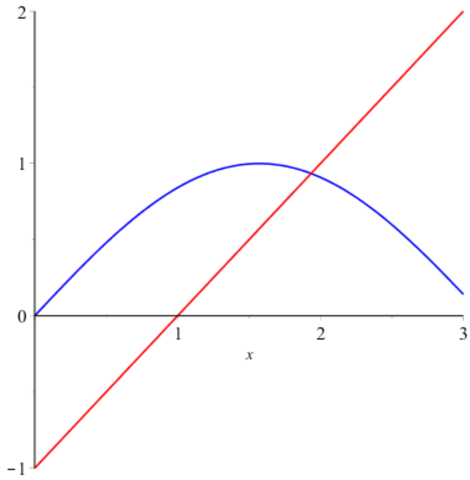

Figure 1: $f_1(x) = blue, f_2(x) = red.$

Numerically :
We use the theorem of intermediate values

1) According to the theorem of intermediate values

- ⋄ Existence:

$$\left.\begin{array}{r} f(1) = 0.841471 \succ 0 \\ f(2) = -0.090722 \prec 0 \end{array}\right\}, f(1) \times f(2) \prec 0 \Rightarrow \exists \alpha \in]1, 2[\text{ such that } f(\alpha) = 0.$$

- ⋄ Uniqueness:

$f^{'}(x) = cosx - 1 \prec 0$ on the $[1, 2]$
$\Rightarrow f^{'}(x) \neq 0$ on the $[1, 2] \Rightarrow \alpha$ is unique.

2) The Newton-Raphson algorithm is given by

$$\begin{cases} x_0 = \text{initial approximation} \\ x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)}. \end{cases}$$

x_0 must check: $f(x_0) \times f^{''}(x_0) \succ 0$.
$f^{''}(x) = -sinx < 0 \Rightarrow x_0 = 2.$
The algorithm becomes :

$$\begin{cases} x_0 = 2 \\ x_{n+1} = x_n - \frac{sin(x_n) - x_n + 1}{cos(x_n) - 1}. \end{cases}$$

- Algorithm convergence:

  If $f \in C^2[a,b]$, and $f^{'}$, $f^{''}$ have constant signs on the $[a,b]$, then the algorithm of Newton-Raphson converges to the exact value of α.

$$\left.\begin{array}{r} f^{'}(x) = cos(x) - 1 \prec 0, \text{ on the [1,2]} \\ f^{''}(x) = -sinx \prec 0, \text{on the [1,2]} \end{array}\right\} \Rightarrow \text{convergence of the algorithm.}$$

3) $x_0 = 2$

   $x_1 = 2 - \frac{sin(2)-2+1}{cos(1.934564)-1} = 1.935951152$

   $x_2 = 1.935951152 - \frac{sin(1.935951152)-1.935951152+1}{cos(1.935951152)-1} = 1.934563874$

   we have $|x_2 - x_1| = 0.0014 \prec 10^{-2} \Rightarrow \alpha \simeq x_2 = 1.934563874$.

Exercise 2 :

$f(x) = x^3 + 2x^2 + 10x - 20 = 0$.

1) We use the theorem of intermediate values

   - Existence:

$$\left.\begin{array}{r} f(1) = -7 \prec 0 \\ f(2) = 16 \succ 0 \end{array}\right\}, f(1) \times f(2) \prec 0 \Rightarrow \exists \alpha \in]1,2[\text{ such that } f(\alpha) = 0.$$

   - Uniqueness:

     $f^{'}(x) = 3x^2 + 4x + 10$, whose discriminant is negative, which implies that $f^{'} > 0$ so f is strictly increasing, hence the uniqueness of the root in $[1,2]$.

2) $f(x) = 0 \Leftrightarrow x(x^2 + 2x + 10) = 20 \Leftrightarrow x = \dfrac{20}{x^2 + 2x + 10}$.

   Let's show that $F([1,2]) \subset ([1,2])$.

   $F^{'}(x) = \dfrac{-40(x+1)}{(x^2+2x+10)^2} < 0 \Rightarrow F$ is decreasing, so

   $\forall x \in [1,2] \quad F(2) \le F(x) \le F(1)$, $\frac{10}{9} = 1.1111 \le F(x) \le \frac{20}{13} = 1.5384$.

3) $F^{''}(x) = \dfrac{120(x^2+2x+2)}{(x^2+2x+2)^3} > 0$, which implies that $F^{'}$ is increasing, so $F^{'}(1) \le F^{'}(x) \le F^{'}(2) \le 0$, so $\left|F^{'}(x)\right| \le \left|F^{'}(1)\right| = 0.473 \le \dfrac{1}{2}$.

4) F continues from $[1,2]$ into itself, moreover F is contracting, because $\left|F^{'}(x)\right| \le k \le 1$.

   Which means that the iterative method $x_{n+1} = F(x_n)$ converges to α the fixed point F, root of f.

5) $x_0 = 1, x_1 = 1.5384, x_2 = 1.295019, x_3 = 1.401825, ..., x_8 = 1.368241$.

   After 8 iterations we can clearly see that we are very close to the value given by Leonardo of Pisa.

Exercise 3 :

Let the equation $f(x) = x - 2 - lnx = 0$.

1) We use the theorem of intermediate values

◇ Existence:

$$\left.\begin{array}{r} f(3) = -0.098612 \prec 0 \\ f(4) = 0.6137057 \succ 0 \end{array}\right\}, f(3) \times f(4) \prec 0 \Rightarrow \exists \alpha \in]3,4[\text{ such that } f(\alpha) = 0.$$

◇ Uniqueness:

$f'(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x} \succ 0$ in the $[3,4]$, so α is unique.

2) The Newton-Raphson algorithm is given by

$$\left\{\begin{array}{c} x_0\text{=initial approximation} \\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \end{array}\right.$$

$x_0$must check the condition $f(x_0) \times f''(x_0) \succ 0$.
$f'(x) = 1 - \dfrac{1}{x} \Rightarrow f''(x) = \dfrac{1}{x^2} \succ 0$ in the $\mathbb{R}^* \Rightarrow x_0 = 4$.

◇ Algorithm convergence:

If $f \in C^2[a,b]$, and f', f'' keep constant signs on $[a,b]$, then the algorithm of Newton-Raphson converges to the exact value of α.

$$\left.\begin{array}{r} f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x} \succ 0 \text{ in the [3,4]}, \\ f''(x) = \frac{1}{x^2} \succ 0 \text{ sur } \mathbb{R}^*, (\text{ or in } [3,4]) \end{array}\right\} \Rightarrow \text{convergence of the algorithm.}$$

3) convergence criterion $|x_{n+1} - x_n| \le \varepsilon = 10^{-4}$.

$$\left\{\begin{array}{l} x_0 = 4, x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.1817815 \\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.1462848 \text{ et } \; x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.14619332. \end{array}\right.$$

$|x_3 - x_2| \le 10^{-4}. \Rightarrow \alpha \simeq 3.14619332$.

4) The error due to this algorithm verifies the following estimate :

$$|\alpha - x_n| \le \frac{|f(x_n)|}{m_1} \qquad \textit{where } m_1 = \min_{x\in[3,4]} \left|f'(x)\right|.$$

$f''(x) = \frac{1}{x^2} \succ 0$ in the $\mathbb{R}^* \Rightarrow \left|f'\right| = f'$ is an increasing function $\Rightarrow$

$$\min_{x\in[3,4]} \left|f'(x)\right| = \min_{x\in[3,4]} f'(x) = f'(3) = \frac{2}{3} = m_1.$$

So

$$|\alpha - x_3| \le \frac{3}{2} \times |f(x_3)| \simeq 0.1017 \times 10^{-6}.$$

Therefore x_3 is an approximate value of α.

5) The method of successive approximations or the fixed point which is based on equivalence

$$f(x) = 0 \Leftrightarrow x = F(x).$$

Now we define F. For this we have

$$f(x) = x - 2 - \ln x = 0 \Leftrightarrow x - 2 = \ln x \Leftrightarrow \left\{\begin{array}{c} x = 2 + \ln x \\ \text{or} \\ x = \exp(x-2) \end{array}\right.$$

The choice $x = exp(x-2)$ eis unacceptable, because the images of the real numbers of $[3,4]$ do not all belong à to this interval
($F(4) > 4$), so we take $x = 2 + lnx = F(x)$.
Consequently the fixed point algorithm associated with this equation is given by :

$$\begin{cases} x_0\text{=initial approximatio} \\ x_{n+1} = F(x_n) = 2 + lnx_n. \end{cases}$$

- Algorithm convergence:
  If $\left|F^{'}(x)\right| \leq k < 1$ in the $[3,4]$, then this algorithm converges to the exact value of α
  $F(x) = 2 + \ln x \Rightarrow F^{'}(x) = \frac{1}{x} > 0$ in the $[3,4] \Rightarrow F^{''}(x) = -\frac{1}{x^2} < 0$ in the $\mathbb{R}^*$ (or in the [3,4]).
  Then $F^{'}$ is a decreasing function on $[3,4]$. Hence $0 < F^{'}(4) \leq \left|F^{'}(x)\right| = F^{'}(x) \leq F^{'}(3) = \frac{1}{3} = k < 1$.
  Consequently, we have the desired result.

6) The error of this algorithm is given by ($k < 1 \Leftrightarrow lnk < 0$)

$$e_n = |x_n - \alpha| \leq \frac{k^n}{1-k}|x_1 - x_0| \leq 10^{-4} \Longleftrightarrow k^n \leq \frac{10^{-4}(1-k)}{|x_1 - x_0|} \Longleftrightarrow n \geq \frac{\ln\left[\frac{10^{-4}(1-k)}{|x_1-x_0|}\right]}{\ln k}$$

$x_0 = 4 \Rightarrow x_1 = 3.386294361$, then $n > 8.31$. So we take $n = 9$.

7) Calculation of the approximate value.

$$\begin{aligned} x_0 &= 4 \quad &: x_5 = 3.148516297 \\ x_1 &= 3.386294361 &: x_6 = 3.146931325 \\ x_2 &= 3.219736215 &: x_7 = 3.146427796 \\ x_3 &= 3.169299436 &: x_8 = 3.146267776 \\ x_4 &= 3.153510565 &: x_9 = 3.146216917. \end{aligned}$$

So $\alpha \simeq 3.146216917$.