Exercise Sheet N° 05 Correction Numerical Analysis I

Exercise 1 :

1) The trapezoid formula is given by :
$I_T(f) = h\left[\frac{y_0}{2} + y_1 + y_2 + y_3 + ... + \frac{y_n}{2}\right]$, with $h = \frac{b-a}{n} = x_{i+1} - x_i$, for every $i = 0, 1, ..., n$, $y_i = f(x_i)$ and $x_i = a + i \times h$.

From the table we obtain that $n = 10$, $[a, b] = [0, 1]$ and $h = 0.1 = \frac{1}{10}$.

By replacing y_i with their respective values in I_t , we obtain that

$$I_T(f) = \frac{1}{10} \times \left[\frac{0.1}{2} + 0.17 + 0.13 + 0.15 + 0.23 + 0.25 + 0.21 + 0.22 + 0.25 + 0.23 + \frac{0.26}{2}\right].$$

So $I_T(f) = 0.202$.

Simpson's formula is given by: $I_s(f) = \frac{h}{3} \times [y_0 + 4\sigma_1 + 2\sigma_2 + y_n]$ with $n = 2p$ (n even).

with $\sigma_1 = y_1 + y_3 + y_5 + ... + y_{n-1}$ and $\sigma_2 = y_2 + y_4 + y_6 + ... + y_{n-2}$.

By summing the y_i having even indices then the $y_i$having the odd indices except the first and last (y_0 and y_{10}) we obtain

$\sigma_1 = y_1 + y_3 + y_5 + y_7 + y_9 = 1.02$ and $\sigma_2 = y_2 + y_4 + y_6 + y_8 = 0.82$, SO

$$I_s(f) = \frac{1}{30} \times [0.1 + 4 \times 1.02 + 2 \times 0.82 + 0.26] = 0.2026666667.$$

2) In this part we calculate approximate values of $\int_0^1 xf(x)dx$, the new y_i are $\tilde{y}_i = x_i \times y_i$, so we add another line in the table which defines the function f to have

x_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x_i)$	0.1	0.17	0.13	0.15	0.23	0.25	0.21	0.22	0.25	0.23	0.26
$\tilde{y}_i$	0	0.017	0.026	0.045	0.092	0.125	0.126	0.154	0.2	0.207	0.26

SO $I_T(xf) = \frac{1}{10} \times \left[\frac{0}{2} + 0.017 + 0.026 + 0.045 + 0.092 + 0.125 + 0.126 + 0.154 + 0.2 + 0.207 + \frac{0.26}{2}\right] = 0.1122$.

For Simpson's formula $\sigma_1 = \tilde{y}_1 + \tilde{y}_3 + \tilde{y}_5 + \tilde{y}_7 + \tilde{y}_9 = 0.548$ and $\sigma_2 = \tilde{y}_2 + \tilde{y}_4 + \tilde{y}_6 + \tilde{y}_8 = 0.444$, So

$$I_s(xf) = \frac{1}{30} \times [0 + 4 \times 0.548 + 2 \times 0.444 + 0.26] = 0.1113333333.$$

Exercise 2 :

1) We want to give approximate values for the integral $I = \int_0^1 \frac{dx}{x^2+1}$ The trapezoid formula is given by $I_T(f) = h\left[\frac{y_0}{2} + y_1 + y_2 + y_3 + ... + \frac{y_n}{2}\right]$, with $h = \frac{b-a}{n} = x_{i+1} - x_i$, for every $i = 0, 1, ..., n$, $y_i = f(x_i)$ and $x_i = a + i \times h$.

In this case $[a,b]=[0,1]$, $n=10$, $f(x)=\frac{1}{x^2+1}$ and $h=\frac{b-a}{n}=0.1=\frac{1}{10}$.

Calculate the y_i which are the images of $x_i=\frac{i}{10}, i=0,1,..,10$, by the function f, we use the highest precision possible, because the quadrature formulas are not exact formulas, they are approximate formulas for calculating integrals. So the y_i will be given with at least 8 number after the comma and they will be written in a table (by line or by column) of the form

i	x_i	$f(x_i)=y_i$
0	0	1
1	0.1	0.9900990099
2	0.2	0.9615384615
3	0.3	0.9174311927
4	0.4	0.8620689655
5	0.5	0.8
6	0.6	0.7352941176
7	0.7	0.6711409396
8	0.8	0.6097560976
9	0.9	0.5524861878
10	1	0.5

By replacing the y_i by their respective values in $I_T(f)$ we obtain

$I_T(f)=\frac{1}{10}\times\left[\frac{y_0}{2}+y_1+y_2+y_3+.....+\frac{y_{10}}{2}\right]=0.7849814972.$

Simpson's formula is given by

$I_s(f)=\frac{h}{3}\times[y_0+4\sigma_1+2\sigma_2+y_n]$ with,

$\sigma_1=y_1+y_3+y_5+....+y_{n-1}$ and $\sigma_2=y_2+y_4+y_6+....+y_{n-2}$ and $n=2p$ (n even).

Using the table we obtain

$\sigma_1=y_1+y_3+y_5+y_7+y_9=3.93115733$

$\sigma_2=y_2+y_4+y_6+y_8=3.16865764$, So

$$I_s(f)=\frac{1}{30}\times[1+4\times3.93115733+2\times3.16865764+0.5]=0.7853981535.$$

2) **Exact value**: Now calculating the exact value of this integral

$$\int_0^1\frac{dx}{1+x^2}=[\arctan x]_{x=0}^{x=1}=\arctan(1)-\arctan(0)=0.7853981635.$$

$|I-I_T(f)|=0.00041667$

$|I-I_s(f)|=0.00000001.$

consequently the Simpson formula is more precise than the trapezoid formula.

Exercise 3 :

1) We want to give approximate values for the integral

$I=\int_0^{\frac{\pi}{4}}\frac{\sin(x)}{\cos^2(x)}dx$ The trapezoid formula is given by

$I_T(f)=h\left[\frac{y_0}{2}+y_1+y_2+y_3+.....+\frac{y_n}{2}\right]$, with $h=\frac{b-a}{n}=x_{i+1}-x_i$, for every $i=0,1,...,n$, $y_i=f(x_i)$ and $x_i=a+i\times h$.

In this case $[a,b]=\left[0,\frac{\pi}{4}\right]$, $n=10$, $f(x)=\frac{\sin(x)}{\cos^2(x)}$ and $h=\frac{b-a}{n}=\frac{\pi}{40}$.

The unit of measurement of x_i is the **Radian**.

Calculate the y_i which are the images of $x_i=a+i\times h=\frac{i\times\pi}{40}$, $i=0,1,....,10$. by the function f, we use the most high precision possible, because the quadrature formulas are not exact formulas, they are approximate formulas for the calculation of integral. So the

y_i will be given with at least 8 number after the comma and they will be written in a table (by line or by column) of the form

i	x_i	$f(x_i) = y_i$
0	0	0
1	$\frac{\pi}{40}$	0.07894506812
2	$\frac{2\pi}{40} = \frac{\pi}{20}$	0.1603587223
3	$\frac{3\pi}{40}$	0.2469006435
4	$\frac{4\pi}{40} = \frac{\pi}{10}$	0.3416407865
5	$\frac{5\pi}{40} = \frac{\pi}{8}$	0.4483415292
6	$\frac{6\pi}{40} = \frac{3\pi}{20}$	0.5718537807
7	$\frac{7\pi}{40}$	0.7187097368
8	$\frac{8\pi}{40} = \frac{\pi}{5}$	0.8980559532
9	$\frac{9\pi}{40}$	1.123190406
10	$\frac{10\pi}{40} = \frac{\pi}{4}$	1.414213562

By replacing the y_i by their respective values in $I_T(f)$ we obtain :

$$I_T(f) = \frac{\pi}{40}\left[\frac{y_0}{2} + y_1 + y_2 + y_3 + ... + \frac{y_{10}}{2}\right] = 0.4158764491.$$

Simpson's formula is given by $I_s(f) = \frac{\pi}{120} \times [y_0 + 4\sigma_1 + 2\sigma_2 + y_{10}]$ where,
$\sigma_1 = y_1 + y_3 + y_5 + ... + y_{n-1}$ and $\sigma_2 = y_2 + y_4 + y_6 + ... + y_{n-2}$ et $n = 2p$ (n even).
Using the table we obtain
$\sigma_1 = y_1 + y_3 + y_5 + y_7 + y_9 = 2.616087384$
$\sigma_2 = y_2 + y_4 + y_6 + y_8 = 1.971909243$, so

$$I_s(f) = \frac{\pi}{120} \times [0 + 4 \times 2.616087384 + 2 \times 1.971909243 + 1.414213562] = 0.4142289812.$$

2) **<u>Exact value</u>**: Now calculating the exact value of this integral, we use the next change of variable : $u = \cos(x) \Rightarrow du = -\sin(x)dx$, $x = 0 \Rightarrow u = \cos(0) = 1$, $x = \frac{\pi}{4} \Rightarrow u = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, so :

$$\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^2(x)}dx = -\int_1^{\frac{\sqrt{2}}{2}} \frac{du}{u^2} = \int_{\frac{\sqrt{2}}{2}}^1 u^{-2}du = \left[-\frac{1}{u}\right]_{\frac{\sqrt{2}}{2}}^1 = -1 + \frac{2}{\sqrt{2}} = -1 + \sqrt{2} = 0.4142135624.$$

$|I - I_T(f)| = 0.0016628867$

$|I - I_s(f)| = 0.000015419.$

Consequently the Simpson formula is more precise than the trapezoid formula.