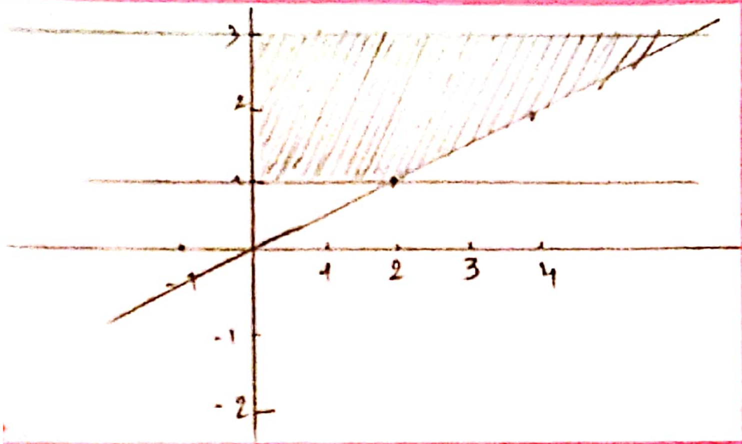
Intégrales doubles

AKLI
AHAL

Exercice 01 :

$D_1 = \{(x,y) \in \mathbb{R}^2 \ / \ 1 \leq y \leq 3, \ 0 \leq x \leq 2y\}$

1 Représentation graphique :



D_1 est régulier selon y

FUB 2 :

$$I = \int_1^3 \int_0^{2y} f(x,y)\, dx\, dy.$$

2. Intervertir l'ordre d'intégration

$D_1 = \{(x,y) \in \mathbb{R}^2 \ / \ 0 \leq x \leq 6, \ \varphi_1(x) \leq y \leq \varphi_2(x)\}$

$$\varphi_1(x) = \begin{cases} 1 & \text{si} \quad 0 \leq x \leq 2 \\ \frac{x}{2} & \text{si} \quad 2 \leq x \leq 6 \end{cases}$$

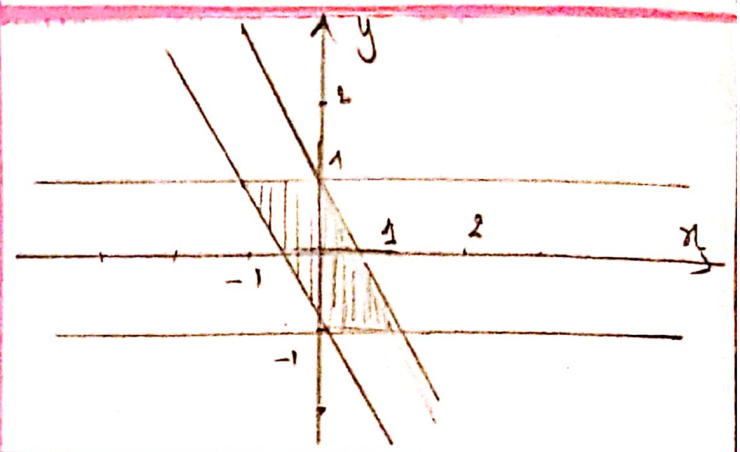
$\varphi_2(x) = 3$

Appliquons FUB 1 :

$$I = \int_0^6 \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy\, dx$$

$$= \int_0^2 \int_1^3 f(x,y)\, dy\, dx + \int_2^6 \int_{\frac{x}{2}}^3 f(x,y)\, dy\, dx$$

$D_2 = \{ y = 1-2x, \ y = -1, \ y = -1-2x, \ y = 1$



D_2 est régulier selon x et y

$D_2 = \{(x,y) \in \mathbb{R}^2 / -1 \leq x \leq 1, \ \varphi_1(x) \leq y \leq \varphi_2(x)\}$

$$\varphi_1(x) = \begin{cases} -1-2x & \text{si} \ -1 \leq x \leq 0 \\ -1 & \text{si} \quad 0 \leq x \leq 1 \end{cases}$$

$$\varphi_2(x) = \begin{cases} 1-2x & \text{si} \quad 0 \leq x \leq 1 \\ +1 & \text{si} \ -1 \leq x \leq 0 \end{cases}$$

$D_2 = \{(x,y) \in \mathbb{R}^2 \ / \ -1 \leq x \leq 0 \ ; -1-2x \leq y \leq 1\}$

$\cup \{(x,y) \in \mathbb{R}^2 / \ 0 \leq x \leq 1, \ -1 \leq y \leq 1-2x\}$

FUB 1 :

$$I = \int_{-1}^{0} \int_{-1-2x}^{1} f(x,y)\, dy\, dx + \int_0^1 \int_{-1}^{1-2x} f(x,y)\, dy\, dx$$

Intervertir l'ordre d'intégration

$D_2 = \{(x,y) \in \mathbb{R}^2 / -1 \leq y \leq 1, \ \phi_1(y) \leq x \leq \phi_2(y)\}$

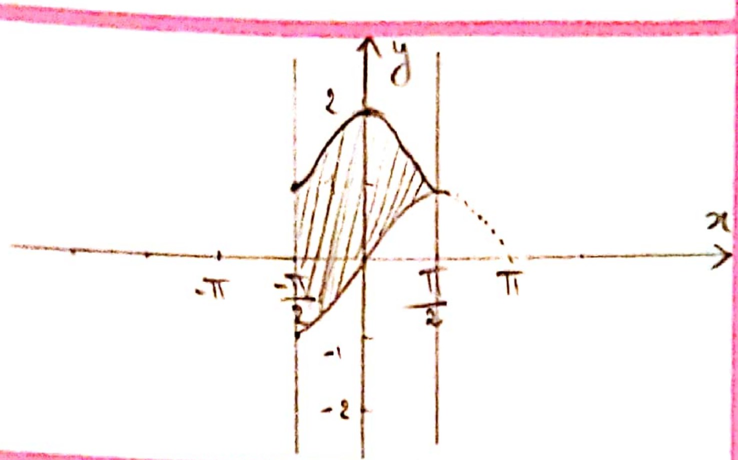
$$\phi_1(y) = \frac{-1-y}{2}$$

$$\phi_2(y) = \frac{1-y}{2}$$

$$I = \int_{-1}^{1} \int_{\frac{-1-y}{2}}^{\frac{1-y}{2}} f(x,y)\, dx\, dy.$$

$D_3 = \{(x,y) \in \mathbb{R}^2 / \ tq : y = \cos(x) + 1 \ , y = \sin(x)$

$x = -\frac{\pi}{2} \ , \ x = \frac{\pi}{2}$



D_3 est régulier selon x :

$D_3 = \{(x,y) \in \mathbb{R}^2 / -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \ , \ \mathcal{C}_1(x) \leq y \leq \mathcal{C}_2(x)\}$

$\mathcal{C}_1(x) = \sin(x) \quad , \ \mathcal{C}_2(x) = \cos(x) + 1 .$

Appliquant FUB 1 :

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\sin(x)}^{\cos(x)+1} f(x,y) \, dy \, dx .$$

Intervertir l'ordre d'integration :

$D_3 = \{(x,y) \in \mathbb{R}^2 / -1 \leq y \leq 2 \ , \ \phi_1(y) \leq x \leq \phi_2(y)\}$

$$\phi_1(y) = \begin{cases} -\frac{\pi}{2} & \text{si} \quad -1 \leq y \leq 1 \\ -\arccos(y-1) & \quad 1 \leq y \leq 2 . \end{cases}$$

$$\phi_2(y) = \begin{cases} \arcsin(y) & -1 \leq y \leq 1 \\ \arccos(y-1) & \ 1 \leq y \leq 2 \end{cases}$$

FUB 2 :

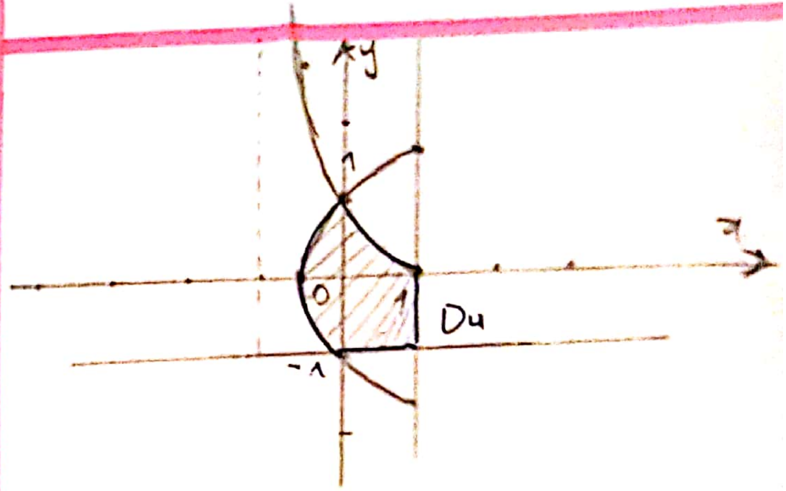
$$I = \int_{-1}^{1} \int_{-\frac{\pi}{2}}^{\arcsin(y)} f(x,y) \, dx \, dy + \int_{1}^{2} \int_{-\arccos(y-1)}^{\arccos(y-1)} f(x,y) \, dx \, dy .$$

D4 :

$y = -1 \ , x = 1 \ , y^2 = 2x+1 \ , xy + y + x = 1$

$y^2 = 2x+1 \implies y = \begin{cases} \sqrt{2x+1} \\ -\sqrt{2x+1} \end{cases}$

$x + y + xy = 1 \implies y = \frac{1-x}{1+x}$



D_4 est régulier selon x

$D_4 = \{(x,y) \in \mathbb{R}^2 / -\frac{1}{2} \leq x \leq 1 \ , \ \mathcal{C}_1(x) \leq y \leq \mathcal{C}_2(x)\}$

$$\mathcal{C}_1(x) = \begin{cases} -1 & \text{si} \quad 0 \leq x \leq 1 \\ -\sqrt{2x+1} & \text{si} \ -\frac{1}{2} \leq x \leq 0 \end{cases}$$

$$\mathcal{C}_2(x) = \begin{cases} \frac{1-x}{1+x} & \text{si} \quad 0 \leq x \leq 1 \\ \sqrt{2x+1} & \text{si} \ -\frac{1}{2} \leq x \leq 0 \end{cases}$$

FUB 1 :

$$I = \int_{-\frac{1}{2}}^{0} \int_{-\sqrt{2x+1}}^{\sqrt{2x+1}} f(x,y) \, dy \, dx + \int_{0}^{1} \int_{-1}^{\frac{1-x}{1+x}} f(x,y) \, dy \, dx .$$

Intervertir :

$D_4 = \{(x,y) \in \mathbb{R}^2 / -1 \leq y \leq 1 \ , \ \phi_1(y) \leq x \leq \phi_2(y)$

$$\varphi_1(y) = \left\{ \frac{y^2-1}{2} \right.$$

$$\varphi_2(y) = \begin{cases} 1 & -1 \leq y \leq 0 \\ \frac{1-y}{1+y} & 0 \leq y \leq 1 \end{cases}$$

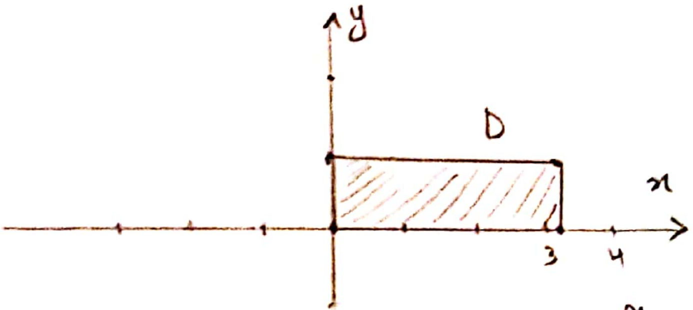
FUB2 :

$$I = \int_{-1}^{1} \int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\, dx\, dy .$$

Exercice 02 :

1) $\iint_{D_1} y \sin x \, dx\, dy$

D_1 : le rectangle de sommets O, A $(\pi, 0)$ B$(0,1)$, D$(\pi, 1)$.



$D = \{(x,y) \in \mathbb{R}^2 \ 0 \leq x \leq \pi , \ 0 \leq y \leq 1\}$

Appliquons FUB 1

$$I_1 = \int_0^{\pi} \left(\int_0^1 y \sin(x)\, dy \right) dx .$$

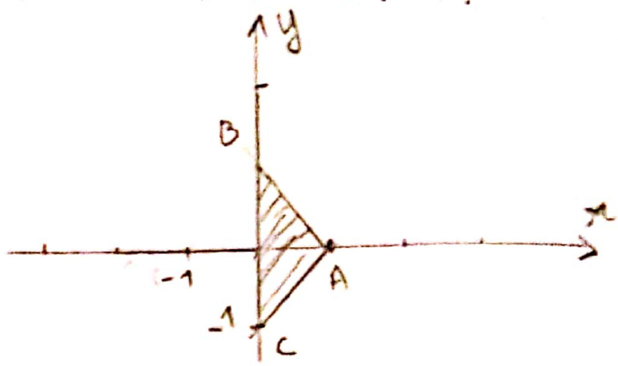
$$= \int_0^{\pi} \left(\sin(x) . \left[\frac{1}{2} y^2 \right]_0^1 \right) dx$$

$$= \frac{1}{2} \int_0^{\pi} \sin(x)\, dx$$

$$= \frac{1}{2} \left[-\cos(x) \right]_0^{\pi} = \frac{1}{2}(1+1) = 1$$

2) $\iint_{D_2} (x+2y)\, dx\, dy$

D : l'intérieur de triangle de sommets A$(1,0)$ B$(0,1)$ C$(0,-1)$.



D est régulier selon x

$D = \{(x,y) \in \mathbb{R}^2 / \ 0 \leq x \leq 1 , \ x-1 \leq y \leq 1-x \}$

$$D = \int_0^1 \int_{x-1}^{1-x} x + 2y \, dy\, dx$$

$$= \int_0^1 \left(x \int_{x-1}^{1-x} dy + 2 \int_{x-1}^{1-x} y \, dy \right) dx$$

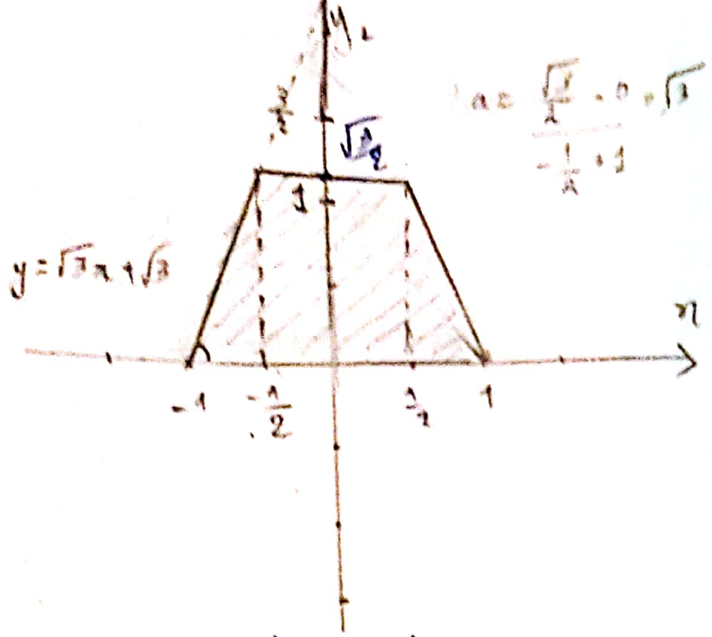
$$= \int_0^1 \left(x \left[y \right]_{x-1}^{1-x} + \left[y^2 \right]_{x-1}^{1-x} \right) dx$$

$$= \int_0^1 (2x + 0)\, dx$$

$$= 1$$

3) $\iint_{D_3} y \, dx\, dy$

D_3 : intérieur du trapèze dont la base est le segment de l'axe x et les abscisses comprises ent -1 et 1



D_3 est régulier selon y.

$D_3 = \{(x,y) \in \mathbb{R}^2 \,/\, 0 \le y \le \frac{\sqrt{3}}{2},\ C_1(y) \le x \le C_2(y)\}$

$C_1(y) = \frac{1}{\sqrt{3}}y - 1$

$C_2(y) = \frac{-1}{\sqrt{3}}y + 1$

Utilisons FUB 2 :

$$I = \int_0^{\frac{\sqrt{3}}{2}} \left(\int_{\frac{1}{\sqrt{3}}y+1}^{\frac{-1}{\sqrt{3}}y+1} \cancel{x}\, y \, dx \right) dy$$

$$\cancel{= \int_0^{\frac{\sqrt{3}}{2}} \left(\frac{y}{3} [x^3]_{\frac{1}{\sqrt{3}}y+1}^{\frac{-1}{\sqrt{3}}y+1} \right) dy}$$

$$= \int_0^{\frac{\sqrt{3}}{2}} \left(y \, [x]_{\frac{1}{\sqrt{3}}y+1}^{-\frac{1}{\sqrt{3}}y+1} \right) dy$$

$$= 2 \int_0^{\frac{\sqrt{3}}{2}} y \, [x]_0^{-\frac{1}{\sqrt{3}}y+1} \, dy$$

$$= 2 \int_0^{\frac{\sqrt{3}}{2}} y \left(\frac{-1}{\sqrt{3}}y + 1 \right) dy$$

$$= 2 \int_0^{\frac{\sqrt{3}}{2}} -\frac{1}{\sqrt{3}}y^2 + y \, dy$$

$$= \frac{-2}{\sqrt{3}} \times \frac{1}{3} [y^3]_0^{\frac{\sqrt{3}}{2}} + [y^2]_0$$

$$= \frac{-2}{3\sqrt{3}} \times \frac{3\sqrt{3}}{8} + \frac{3}{4}$$

$$= -\frac{1}{4} + \frac{3}{4} = \frac{1}{2}.$$

Exercice 03 :

1/ $\iint_{D_1} (x-y)^2 \, dx \, dy$

$D_1 = \{(x,y) \in \mathbb{R}^2 \,/\, \sqrt{x} + \sqrt{y} \ge 1$ et $\sqrt{1-x} + \sqrt{1-y} \ge 1\}$.

$$\begin{cases} \sqrt{x} + \sqrt{y} \ge 1 \\ x \ge 0 \text{ et } y \ge 0 \\ \sqrt{1-x} + \sqrt{1-y} \ge 1 \\ x \le 1 \text{ et } y \le 1 \end{cases}$$

$\Rightarrow \quad 0 \le x \le 1$

$\sqrt{y} \ge 1 - \sqrt{x} \Rightarrow y \ge (1-\sqrt{x})^2$

$\sqrt{1-y} \ge 1 - \sqrt{1-x}$

$1 - y \ge (1 - \sqrt{1-x})^2$

$y \le 1 - (1 - \sqrt{1-x})^2$

$(1-\sqrt{x})^2 \le y \le 1 - (1 - \sqrt{1-x})^2$

$(x,y) \in \mathbb{R}^2 \quad 0 \leq x \leq 1$ et

$(1-\sqrt{x})^2 \leq y \leq 1-(1-\sqrt{1-x})^2$

$$I = \int_0^1 \left(\int_{(1-\sqrt{x})^2}^{1-(1-\sqrt{1-x})^2} (x-y)^2 \, dy \right) dx$$

$$= \int_0^1 \left(\int_{(1-\sqrt{x})^2}^{1-(1-\sqrt{1-x})^2} x^2+y^2-2xy \, dy \right) dx$$

$$= \int_0^1 \left[\frac{1}{3}y^3 + x^2 y - xy^2 \right]_{(1-\sqrt{x})^2}^{1-(1-\sqrt{1-x})^2} dx$$

!! bzf calcul XD

2/ $\iint_{D_2} xy \, dx\,dy$

$D_2 = \{(x,y) \in \mathbb{R}^2 \;/\; x \geq 0, \; y \geq 0, \; xy+x+y \leq 1\}$

$xy \geq 0$ et $x+y \geq 0$

~~$\Rightarrow x+y \leq 1 \quad x \leq 1-y$~~

~~$0 \leq y \leq 1$~~

$y(x+1)+x \leq 1$

$\Rightarrow \left\{ 0 \leq y \leq \frac{1-x}{1+x} \right\}$

$x+y \leq 1 \quad \Rightarrow \quad x \leq 1$

$D_2 = \{(x,y) \in \mathbb{R}^2 \;/\; 0 \leq x \leq 1,$
$\quad 0 \leq y \leq \frac{1-x}{1+x}\}$

$$I_2 = \int_0^1 \left(\int_0^{\frac{1-x}{1+x}} xy \, dy \right) dx$$

$$= \int_0^1 \frac{1}{2} x \left[y^2 \right]_0^{\frac{1-x}{1+x}} dx$$

$$= \int_0^1 \frac{1}{2} x \frac{(1-x)^2}{(1+x)^2} dx$$

$$= \int_0^1 \frac{1}{2} x \left(\frac{-(1+x)+2}{1+x} \right)^2 dx$$

$$= \int_0^1 \frac{1}{2} x \left(-1 + \frac{2}{1+x} \right)^2 dx$$

$$= \frac{1}{2} \int_0^1 x \left(1 + \frac{4}{(1+x)^2} - \frac{4}{1+x} \right) dx$$

$$= \frac{1}{2} \int_0^1 \left(x + \frac{4x}{(1+x)^2} - \frac{4}{1+x} \right) dx$$

$$= \frac{1}{2} \int_0^1 x\,dx + \int_0^1 \frac{2x+2}{(1+x)^2} dx$$

$$- \int_0^1 \frac{+2}{(1+x)^2} dx - 2 \int_0^1 \frac{1}{1+x} dx$$

$$= \frac{1}{2} \times \cancel{\frac{1}{2}} \left[x^2 \right]_0^1 + \left[\ln(1+x)^2 \right]_0^1$$

$$- 2 \cdot \left[\frac{-1}{1+x} \right]_0^1 - 2 \left[\ln(1+x) \right]_0^1$$

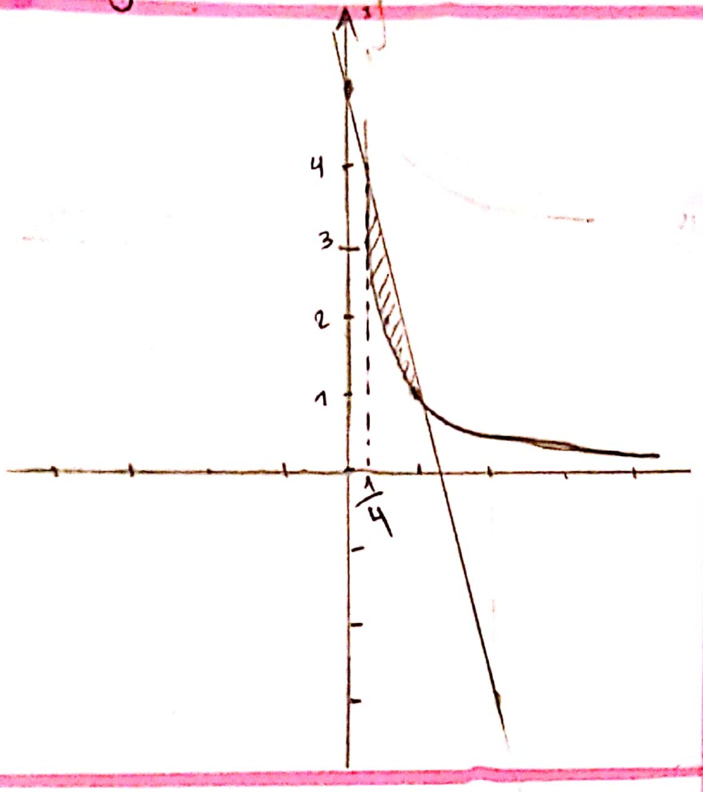
$$= \frac{1}{4} + \cancel{\ln 4} - 1 - \cancel{2 \ln 2}$$

$$= \cancel{\ln 4} \quad -\frac{3}{4}$$

$$\iint_{D_3} x^2 y \, dx\, dy$$

D_3 le plan limité par $y = \frac{1}{x}$

et $y = -4x+5$



$D_3 = \{(x,y) \in \mathbb{R}^2 / \frac{1}{4} \leq x \leq 1 ,\ \frac{1}{x} \leq y \leq -4x+5\}$

$$I_3 = \int_{\frac{1}{4}}^{1} \int_{\frac{1}{x}}^{-4x+1} x^2 y \cdot dy\, dx$$

$$= \int_{\frac{1}{4}}^{1} x^2 \times \frac{1}{2} \left[y^2\right]_{\frac{1}{x}}^{-4x+1} dx$$

$$= \int_{\frac{1}{4}}^{1} \frac{x^2}{2} \times \left((-4x+1)^2 - \frac{1}{x^2} \right) dx$$

$$= \int_{\frac{1}{4}}^{1} \frac{(x^2(-4x+1))^2}{2} - \frac{1}{2}\, dx.$$

$$= \int_{\frac{1}{4}}^{1} \frac{x^2}{2}(16x^2+1-8x) - \frac{1}{2} dx$$

$$= \int_{\frac{1}{4}}^{1} 8x^4 + \frac{x^2}{2} - 4x^3 - \frac{1}{2}$$

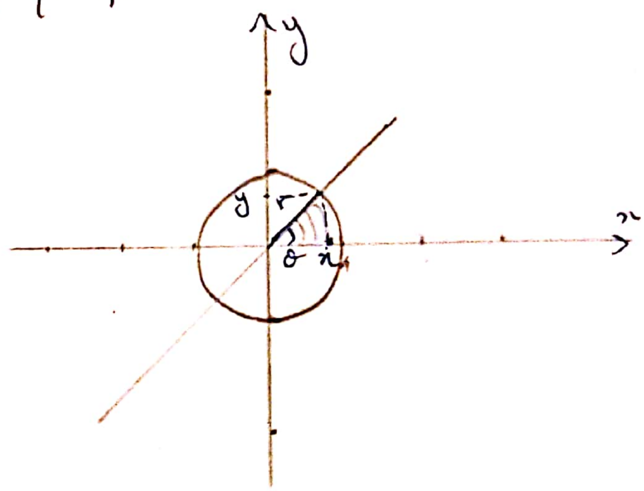
$$= -\frac{1}{2}\left[x\right]_{\frac{1}{4}}^{1} + \frac{1}{2} \times \frac{1}{3}\left[x^3\right]_{\frac{1}{4}}^{1} - 4 \times \frac{1}{4} \times \left[x^4\right]_{\frac{1}{4}}^{1} + 8 \times \frac{1}{5} \times \left[x^5\right]_{\frac{1}{4}}^{1}$$

$$= -\frac{1}{2}\left(1 - \frac{1}{4}\right) + \frac{1}{6}\left(1 - \frac{1}{64}\right) - \left(1 - \frac{1}{256}\right) + \frac{8}{5}\left(1 - \frac{1}{1024}\right)$$

Exercice 04 :

1/ $\iint_{D_1} (2x-y)\, dx\, dy$

$D_1 = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 \leq 1 ,\ 0 \leq y \leq x\}$



Les coordonnées polaires

$x = r\cos(\theta)$ $\qquad y = r\sin\theta$

$(r,\theta) \in \mathbb{R}_+ \times [0, 2\pi]$

$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(r,\theta) \longrightarrow (x,y) = (r\cos\theta, r\sin\theta)$

$|\det J\varphi(r,\theta)| = r.$

$D' = \{(r,\theta) \in \mathbb{R}_+^* \times [0, 2\pi],$

$\cos^2(\theta) + \sin^2(\theta) \leq 1 \quad , 0 \leq \sin\theta \leq \cos\theta \}$

$D' = \{(r,\theta) \in \mathbb{R}_+^* \times [0, 2\pi], \ / \ 0 \leq r \leq 1$

$0 \leq \theta \leq \frac{\pi}{4} \}$.

$$I = \int_0^1 \int_0^{\frac{\pi}{4}} (2r\cos\theta - r\sin\theta) r \, d\theta \, dr$$

$$= \int_0^1 \int_0^{\frac{\pi}{4}} r^2 [2\sin\theta + \cos\theta]_0^{\frac{\pi}{4}} \, dr$$

$$= \int_0^1 r^2 (\sqrt{2} - 1) \, dr$$

$$= \frac{1}{3}(\sqrt{2} - 1)$$

2) $\iint_D \sqrt{x^2+y^2} \, dx \, dy$

$D_2 = \{(x,y) \in \mathbb{R}^2 \ / x^2 + y^2 - 2x \leq 0\}$

On pose

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$D_2' = \{(r,\theta) \times \mathbb{R}_+ \times [0, 2\pi]$,

$r^2 - 2r\cos\theta \leq 0$

$\Leftrightarrow \quad (r,\theta) \in \mathbb{R}_+ \times [0, 2\pi]$

$r(r - 2\cos\theta) \leq 0$

or $r \in \mathbb{R}_+$ donc $r > 0$

$\Rightarrow \quad r - 2\cos\theta \leq 0$

$\Rightarrow 0 \leq r \leq 2\cos\theta$

$\cos\theta \geq 0 \Rightarrow \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r \times r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3} [r^3]_0^{2\cos\theta} \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3} (8\cos^3\theta) \, d\theta$$

$$= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta (1 - \sin^2\theta) \, d\theta$$

$$\cancel{= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) (1}$$

soit $u = \sin(\theta)$

$du = +\cos\theta \, d\theta$

$$J = +\frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - u^2) \, du$$

$$= +\frac{8}{3} [u]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{8}{3} \times \frac{1}{3} [u^3]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= +\frac{8}{3} [\sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{8}{9} [\sin^3\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{+16}{3} - \frac{16}{9} = \frac{+32}{9}$$

$D_3 =$ l'intérieur de l'ellipse d'equation $\frac{x^2}{2} + y^2 = 1$

$$I_3 = \iint_{D_3} (x^2 + y^2)\, dx\, dy$$

$$\frac{x^2}{2} + y^2 < 1$$

Coordonnés polaires :

$(r, \theta) \in [\mathbb{R}_+ \times [0, 2\pi]$.

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{r^2\cos^2(\theta)}{2} + r^2\sin^2(\theta) < 1$$

$$\frac{r^2}{\cancel{2}}\left(\frac{\cos^2(\theta)}{2} + \frac{\sin^2(\theta)}{2}\right) + \frac{r^2}{2}\sin\theta^2 < 1$$

$$\frac{r^2}{2}\left(\frac{1}{\cancel{2}} + \sin^2(\theta)\right) < 1$$

$$0 \leq r^2 < \frac{2}{1+\sin^2\theta} \qquad r \in \mathbb{R}_+$$

$$0 \leq r < \frac{\sqrt{2}}{\sqrt{1+\sin^2\theta}}$$

$\theta \in [0, 2\pi]$

$D_3 = \{(r, \theta) \in \mathbb{R}_+ \times [0, 2\pi] /$

$0 \leq \theta \leq 2\pi$, $0 \leq r \leq \frac{\sqrt{2}}{\sqrt{1+\sin^2\theta}}\}$

$$I_3 = \int_0^{2\pi} \int_0^{\frac{\sqrt{2}}{\sqrt{1+\sin^2\theta}}} r^3\, dr\, d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} \frac{4}{(1+\sin^2\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{(1+\sin^2\theta)^2}\, d\theta$$