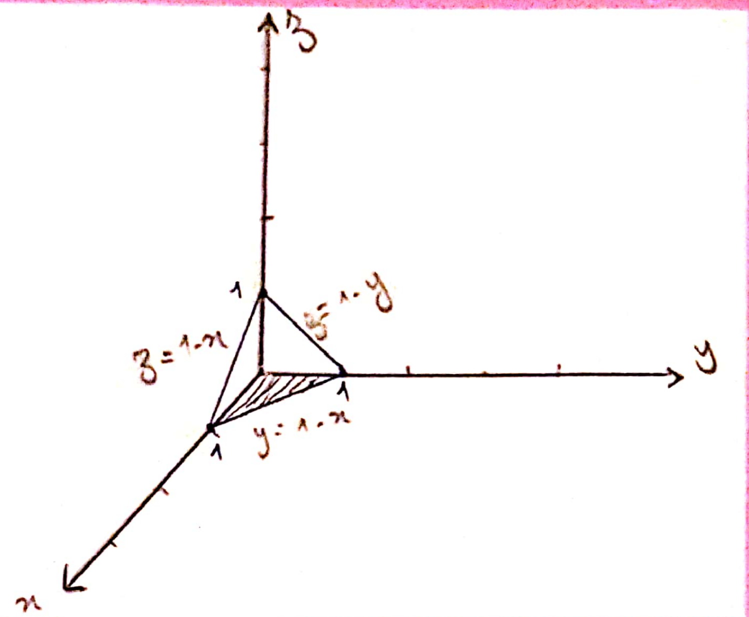
Les intégrales triples

Exercice 01 :

1/ Ω_1 : le domaine délimité par

$x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$



Exprimer $\iiint_{\Omega_1} f(x,y,z)\,dx\,dy\,dz$:

$$0 \le x \le 1$$
$$\phi_1(x,y) \le y \le \phi_2(x,y)$$
$$\mathcal{C}_1(x,y) \le z \le \mathcal{C}_2(x,y).$$

$\Rightarrow$
$$0 \le x \le 1$$
$$0 \le y \le 1-x.$$
$$0 \le z \le 1-x-y.$$

$$I_1 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x,y,z)\,dz\,dy\,dx.$$

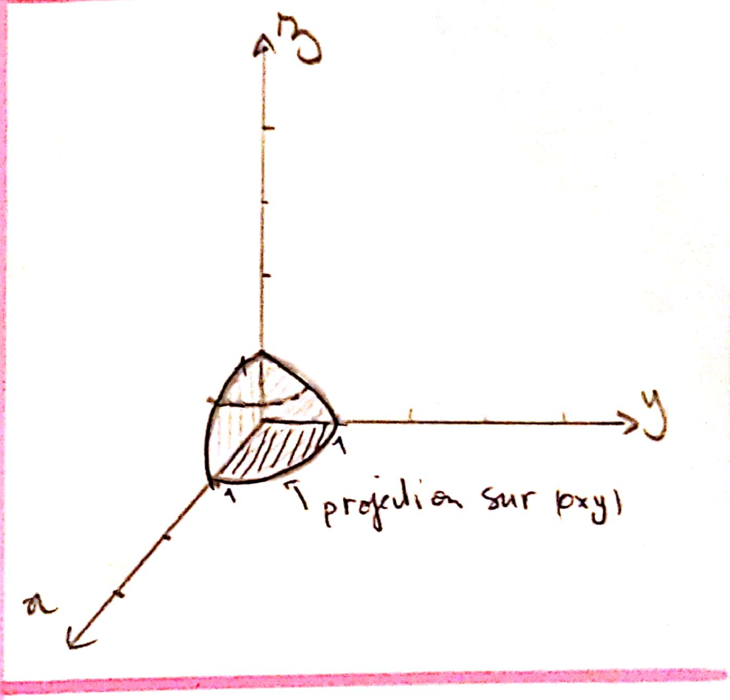
Ω_2 = le domaine délimité par les plans d'équations

$x = 0$, $y = 0$, $z = 0$, et la sphère de centre 0 et de rayon 1 dont les points ont des coordonnées positives.

Ω_3 : L'equation de la sphère

$$x^2 + y^2 + z^2 = 1$$

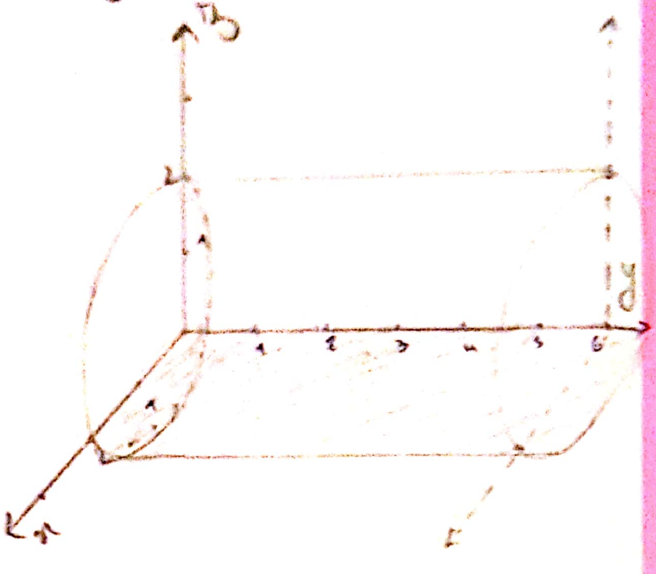
$$\Omega_2 = \{(x,y,z) \in \mathbb{R}^3,\ x \ge 0,\ y \ge 0,\ z \ge 0,\ \text{et}\ x^2 + y^2 + z^2 \le 1\}.$$



$$0 \le x \le 1$$
$$0 \le y \le \sqrt{1-x^2}$$
$$0 \le z \le \sqrt{1-x^2-y^2}$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx.$$

Ω_3 délimité par $y=0$ et $y=6$ et le cylindre $x^2+z^2=4$



$$0 \le x \le 2$$

$$z^2 = 4 - x^2$$

$$-\sqrt{4-x^2} \le z \le \sqrt{4-x^2}$$

$$0 \le y \le 6.$$

$$I_3 = \int_0^2 \int_0^6 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y,z)\, dz\, dy\, dx$$

Exercice 02 :

1/ $\iiint_{\Omega_1} e^x \, dx\, dy\, dz$

$\Omega_1 = \{(x,y,z) \in \mathbb{R}^3 / 0 \le y \le 1,$

$0 \le x \le y, \; 0 \le z \le x+y\}$

soit

$D = \{(x,y) \in \mathbb{R}^2 / \; 0 \le y \le 1, \; 0 \le x \le y\}$

$$\cancel{I = \iint_D \left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^x \, dz \right)}$$

$$= \iint_D \left(\cancel{2\sqrt{4-x^2}} \right) dx\, dy$$

$$= \iint_D \left(\int_0^{x+y} e^x \, dz \right) dx\, dy$$

$$= \iint_D \left(e^x (x+y) \right) dx\, dy$$

$$= \int_0^1 \int_0^y e^x (x+y)\, dx\, dy.$$

$$= \int_0^1 \left[e^x (x+y) + e^x \right]_0^y dy$$

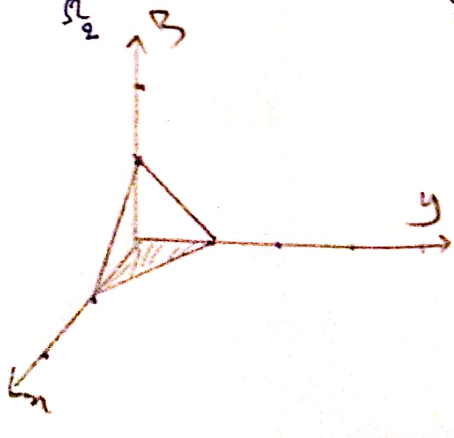
$$= \int_0^1 e^y (2y+1) - y + 1 \, dy$$

$$= \left[e^y (2y+1) + 2e^y \right]_0^1 - \frac{1}{2} \left[y^2 \right]_0^1 + \left[y \right]_0^1$$

$$= 5e - 2 \cdot - \frac{1}{2} + 1$$

$$= 5e - \frac{3}{2}.$$

2/ $\iiint_{\Omega_2} e^{x+y+z} \, dx\, dy\, dz$



$0 \leq x \leq 1$

$0 \leq y \leq 1-x$

$0 \leq z \leq 1-x-y$

$$I_2 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^{x+y+z} \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} e - e^{x+y} \, dy \, dx$$

$$= e \int_0^1 \int_0^{1-x} dy \, dx - \iint_{0\,0}^{1\,1-x} e^{x+y} \, dy \, dx$$

$$= e \int_0^1 1-x \, dx - \int_0^1 e - 1 \, dx$$

$$= e[x]_0^1 - \frac{e}{2}[x^2]_0^1 + (1-e)[x]_0^1$$

$$= \cancel{e} - \frac{e}{2} + 1 - \cancel{e} = 1 - \frac{e}{2}$$

$$I_3 = \iiint_{\Omega_3} x^2 y \, dx \, dy \, dz$$

$\Omega_3 = \{(x,y,z) \in \mathbb{R}^3 \quad 0 \leq y \leq 1-x^2$

$|x+y+z| \leq 1\}$.

$0 \leq y$ et $x+y+z \leq 1$

$-1 \leq x+y+z \leq 1$

$0 \leq y$ donc $x+z \leq 1$

$0 \leq 1-x^2 \Rightarrow -1 \leq x \leq 1$

$\Omega_3 \{(x,y,z) \in \mathbb{R}^3 \,.\, -1 \leq x \leq 1$

$0 \leq y \leq 1-x^2 \,.\, \cancel{z \ldots}$

$-1-x-y \leq z \leq 1-x-y$

$$I_3 = \int_{-1}^1 \int_0^{1-x^2} \int_{-1-x-y}^{1-x-y} x^2 y \, dz \, dy \, dx$$

$$= \int_{-1}^1 \int_0^{1-x^2} x^2 y \, [z]_{-1-x-y}^{1-x-y} \, dy \, dx$$

$$= \int_{-1}^1 \int_0^{1-x^2} 2x^2 y \, dy \, dx$$

$$= \int_{-1}^1 \frac{2x^2}{2} [y^2]_0^{1-x^2} \, dx$$

$$= \int_{-1}^1 x^2 (1-x^2)^2 \, dx$$

$$= \int_{-1}^1 x^2 (1 + x^4 - 2x^2) \, dx$$

$$= \int_{-1}^1 x^2 + x^6 - 2x^4 \, dx$$

$$= \frac{1}{3}[x^3]_{-1}^1 + \frac{1}{7}[x^7]_{-1}^1 - \frac{2}{5}[x^5]_{-1}^1$$

$$= \frac{2}{3} + \frac{2}{7} - \frac{2}{5}$$

4/ $$I_4 = \iiint_{\Omega_4} x+y+z \, dx \, dy \, dz$$

$\Omega_4 = \{(x,y,z) \in \mathbb{R}^3 \; \cancel{A} \; 0 \leq z \leq x^2 + y^2$

$0 \leq y \leq x \leq 1\}$.

$D = \{(x,y) \in \mathbb{R}^2 / 0 \leq y \leq x$ et $0 \leq x \leq 1\}$

$$I_4 = \int_0^1 \int_0^x \int_0^{x^2+y^2} x+y+z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^x \frac{1}{2}\left[z^2\right]_0^{x^2+y^2} dy \, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^x (x^2+y^2)^2 \, dy \, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^x x^4+y^4+2x^2y^2 \, dy \, dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{5}y^5 + \frac{2x^3}{3}\left[y^3\right]\right]_0^x dx$$

$$= \frac{1}{2}\int_0^1 \frac{1}{5}x^5 + \frac{2}{3}x^6 \, dx$$

$$= \frac{1}{2}\left(\frac{1}{5}\left[\frac{1}{6}x^6\right]_0^1 + \frac{2}{3} \times \frac{1}{7}\left[x^7\right]_0^1 \cancel{dx}\right)$$

$$= \frac{1}{2}\left(\frac{1}{6} + \frac{2}{21}\right) = \frac{1}{12} + \frac{1}{21}$$

Exercice 03 :

$$I \iiint_{\Omega_1} |xyz| \, dx \, dy \, dz$$

$\Omega_1 = \{(x,y,z) \in \mathbb{R}^3, \; x^2+y^2 \leq z^2$

$0 \leq z \leq 1\}$

$x^2+y^2 \leq 1$

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$$

$r^2 \leq z^2$

$-0 \leq r \leq z$

$\theta \in [0, 2\pi]$

$\Omega_1 = \{(r,\theta,z) \in \mathbb{R} \times [0,2\pi] \times \mathbb{R} /$

$0 \leq r \leq z, \; 0 \leq \theta \leq 2\pi, \; 0 \leq z \leq 1\}$

$$I_1 = \int_0^1 \int_{-0}^{z} \int_0^{2\pi} r\,|r^2 \cos\theta . \sin\theta . z| \, d\theta \, dr \, dz$$

$$= \int_0^1 \int_{-0}^{z} \int_0^{2\pi} \frac{r}{2}|r^2 \sin(2\theta) z| \, d\theta \, dr \, dz$$

$$= \frac{1}{2}\int_0^1 \int_0^z \int_0^{2\pi} r^3 z\,|\sin(2\theta)|.1 \, d\theta \, dr \, dz$$

$$= \frac{1}{2}\int_0^1 \int_0^z r^3 z \int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta \, dr \, dz$$

$$= \frac{1}{2}\int_0^1 z \int_0^z r^3 \left[2\cos(2\theta)\right]_0^{\frac{\pi}{2}} dr \, dz$$

$$= \frac{1}{4}\int_0^1 z \int_0^z r^3(1+1) \, dr \, dz$$

$$= 2\int_0^1 z \times \frac{1}{4}\left[r^4\right]_0^z dz$$

$$= \frac{1}{2}\int_0^1 z^5 \, dz$$

$$= \frac{1}{12}\left[z^6\right]_0^1 = \frac{1}{12}.$$

$$\iiint_{\Omega_2} x \, dx\,dy\,dz$$

$\Omega_2 = \{(x,y,z) \in \mathbb{R}^3 \,/\, x^2+y^2+z^2 \leq 4$

$x^2+z^2 \leq x\}$.

soit $\begin{cases} x = r\cos(\theta) \\ y = y & \theta \in [0, 2\pi]. \\ z = r\sin(\theta). & r \in \mathbb{R}_+ \end{cases}$

$y^2 \leq 4 - x^2$

$-\sqrt{4-r^2} \leq y \leq \sqrt{4-r^2}$

$0 \leq r^2 \leq x \quad \cancel{\Rightarrow}$

$0 \leq r^2 \leq r\cos(\theta) \Rightarrow \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$0 \leq r \leq \cos\theta$.

soit D transformé de Ω par les C.E.

$D = \{(r,\theta,z) \in \mathbb{R}_+ \times [0,2\pi] \times \mathbb{R} \,/$

$0 \leq r \leq \cos(\theta),\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$

$-\sqrt{4-r^2} \leq y \leq \sqrt{4-r^2}\}$.

$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\cos\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2\cos(\theta)\,.dy\,dr\,d\theta\,.$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\,.\int_0^{\cos\theta} r^2\,.\,[y]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}}\,dr\,d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\,.\underbrace{\int_0^{\cos\theta} r^2\,.\sqrt{4-r^2}\,dr}_{I_2'}\,d\theta$$

posons

$\frac{r}{2} = \cos(\alpha)$

$r = 2\cos(\alpha)$

$\frac{dr}{d\alpha} = -2\sin(\alpha)$

$dr = -2\sin(\alpha)\,.\,d\alpha$

$$I_2' = \int_0^{\cos\theta} 2r^2\sqrt{1-\left(\frac{r}{2}\right)^2}\,dr$$

on a

$\alpha = \arccos\left(\frac{r}{2}\right)$.

$r \in [0, \cos\theta]$.

$\Rightarrow \alpha \in \left[\frac{\pi}{2}, \arccos\left(\frac{\cos\theta}{2}\right)\right]$.

$$I_2' = \int_{\frac{\pi}{2}}^{\arccos\left(\frac{\cos\theta}{2}\right)} -2\,.\times 4\,.\cos^2(\alpha)\,.\sin^2(\alpha)\,d\alpha$$

$$= -2\int_{\frac{\pi}{2}}^{\arccos\left(\frac{\cos\theta}{2}\right)} \left(2\cos(\alpha)\,.\sin(\alpha)\right)^2 d\alpha$$

$$= -2\int_{\frac{\pi}{2}}^{\arccos\left(\frac{\cos\theta}{2}\right)} \sin^2(2\alpha)\,.\,d\alpha\,.$$

$$= -2\int_{\frac{\pi}{2}}^{\arccos\left(\frac{\cos\theta}{2}\right)} \frac{1}{2} - \frac{\cos(4\alpha)}{2}\,d\alpha$$

$$= -2.\left[\frac{1}{2}\alpha - \frac{1}{8}\sin(4\alpha)\right]_{\frac{\pi}{2}}^{\arccos\left(\frac{\cos\theta}{2}\right)}$$

$$= -2\left(\frac{1}{2}\arccos\left(\frac{\cos\theta}{2}\right) - \frac{1}{8}\sin\left(4\arccos\left(\frac{\cos\theta}{2}\right)\right)\right)$$

$$-2\left(-\frac{\pi}{4}\right)$$

$$= \cancel{\frac{2}{2}}\left(-\arccos\left(\frac{\cos\theta}{2}\right) + \frac{\pi}{2} + \frac{1}{4}\sin\left(4\arccos\left(\frac{\cos(\theta)}{2}\right)\right)\right)$$

$\arccos(2x) = \arcsin(x)$.

$\arccos\left(2\left(\frac{\cos\theta}{4}\right)\right) = \arcsin\left(\frac{\cos\theta}{4}\right)$.

$I_2' = \frac{\pi}{2} - \arccos\left(\frac{\cos(\theta)}{2}\right) +$

$\frac{1}{4} \sin\left(4 \arcsin\left(\frac{\cos\theta}{4}\right)\right)$

Exo4 :

$$\iiint_{\Omega_1} \sqrt{R^2-(x^2+y^2+z^2)}\, dx\,dy\,dz$$

$\Omega_1 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq R^2\}$.

utilisons les C.S :

$$\begin{cases} x = r \sin\alpha \cos\theta. & r \in \mathbb{R}_+^* . \\ y = r \sin\alpha . \sin\theta. & \theta \in]0, 2\pi[\\ z = r\cos\alpha . & \alpha \in]0, \pi[\end{cases}$$

$\det(J\varphi) = r^2 . \sin\alpha$.

$r = \sqrt{x^2+y^2+z^2}$

$x^2+y^2+z^2 \leq R^2$

$\Rightarrow . \quad r^2 \leq R^2 \Rightarrow 0 < r < R.$

$\Omega_1' = \{(r,\theta,\alpha) \in \mathbb{R}_+^* \times]0,2\pi[\times]0,\pi[$

$/ \ 0 \leq \theta \leq 2\pi , \ 0 < \alpha < \pi$

$0 < r < R\}$.

$$\iiint_{\Omega_1} \sqrt{R^2-(x^2+y^2+z^2)}\, dx\,dy\,dz$$

$$= \iiint_{\Omega_1'} \sqrt{R^2 - r^2} . r^2 \sin\alpha \, d\alpha \dots$$

$$= \underbrace{\int_0^{\pi} \sin\alpha \, d\alpha}_{I_1} \times \int_0^{2\pi} \underbrace{\int_0^R r^2 \sqrt{R^2-r^2}\, dr\, d\theta}_{I_2}$$

$I_1 = [-\cos\alpha]_0^{\pi} = 1+1 = 2$

$I_2 = \int_0^{2\pi} \int_0^R R r^2 \sqrt{1 - \left(\frac{r}{R}\right)^2}\, dr\, d\theta$

$\frac{r}{R} = \cos(\gamma) \Rightarrow r = R\cos(\gamma)$

$\frac{d\cos(\gamma)}{d\gamma} = -\sin(\gamma)$.

$\gamma = \arccos\left(\frac{r}{R}\right)$

$\frac{dr}{d\gamma} = -R\sin(\gamma)$.

$dr = -R\sin(\gamma) . d\gamma$

$I_2 = \int_0^{2\pi} \int_0^R -R^4 \cos^2(\gamma) \times \sin^2(\gamma)\, d\gamma . d\theta$

$= \frac{-R^4}{4} \int_0^{2\pi} \int_0^R \sin^2(2\gamma)\, d\gamma\, d\theta$.

$r \in]0, R[$

$\arccos\left(\frac{r}{R}\right) \in \cancel{]-\frac{\pi}{2}, 0[} \ ]0, \frac{\pi}{2}[$.

$I_2 = \frac{-R^4}{4} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin^2(2\gamma)\, d\gamma\, d\theta$.

$= \int_0^{2\pi} \frac{-R^4}{4} \left[\frac{1}{2}\gamma - \frac{1}{8}\sin(4\gamma)\right]_0^{\frac{\pi}{2}} d\theta$