Université de batna2
Faculté de Technologie
Socle TC Technologie -2éme Année-
Mathématiques 3

Année 2019/2020

Intégrales Doubles et Triples
Solutions des exercices

4.1 $$\iint_D y\frac{e^{2x+y^2}}{1+e^x}\,dx\,dy = \int_0^1 \frac{\left(e^x\right)^2}{1+e^x}\,dx\int_0^2 y\,e^{y^2}\,dy = \int_1^e \frac{t}{1+t}\,dt\int_0^2 y\,e^{y^2}\,dy$$
$$= \left(t-\ln(1+t)\right)\Big|_1^e \left(\frac{e^{y^2}}{2}\right)\Bigg|_0^2 = \frac{1}{2}\left(e-1-\ln\frac{1+e}{2}\right)\left(e^4-1\right).$$

4.2 $$\iint_D (x^3+3x^2y+y^3)\,dx\,dy = \int_0^2 dx\int_0^1 (x^3+3x^2y+y^3)\,dy$$
$$= \int_0^2\left(x^3+\frac{3}{2}x^2+\frac{1}{4}\right)dy = \left(\frac{x^4}{4}+\frac{x^3}{2}+\frac{x}{4}\right)\Bigg|_0^2 = \frac{17}{2}.$$

4.3 $$\iint_D \frac{x\sin y}{1+x^2}\,dx\,dy = \int_0^1 \frac{x}{1+x^2}\,dx\int_0^{\frac{\pi}{2}} \sin y\,dy$$
$$= \frac{1}{2}\left(\ln(1+x^2)\right)\Big|_0^1 \left(-\cos y\right)\Big|_0^{\frac{\pi}{2}} = \frac{\ln 2}{2}.$$

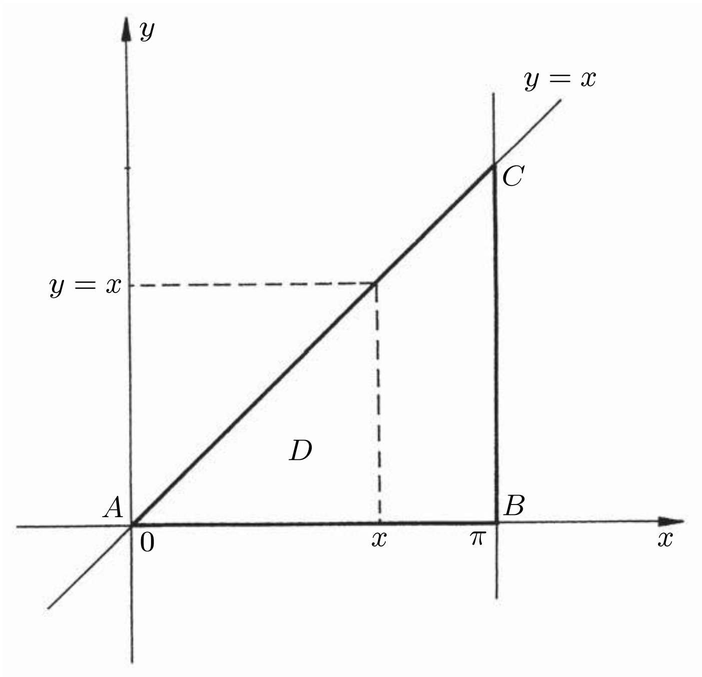
4.4 $$\iint_D x\sin xy\,dx\,dy = \int_0^\pi dx\int_0^1 x\sin xy\,dy = \int_0^\pi (1-\cos x)\,dx$$
$$= (x-\sin x)\big|_0^\pi = \pi.$$

4.5 $$\iint_D \frac{y}{x^2+y^2}\,dx\,dy = \int_1^2 dy\int_0^1 \frac{y}{x^2+y^2}\,dx = \int_1^2 \operatorname{Arctg}\frac{1}{y}\,dy$$
$$= y\operatorname{Arctg}\frac{1}{y}\Big|_1^2 + \int_1^2 \frac{y}{1+y^2}\,dy = 2\operatorname{Arctg}\frac{1}{2}-\frac{\pi}{4}+\frac{1}{2}\ln(1+y^2)\Big|_1^2$$
$$= 2\operatorname{Arctg}\frac{1}{2}-\frac{\pi}{4}+\frac{1}{2}\ln\frac{5}{2}.$$

4.6 $\displaystyle\iint_D x\cos(x+y)\,dx\,dy$

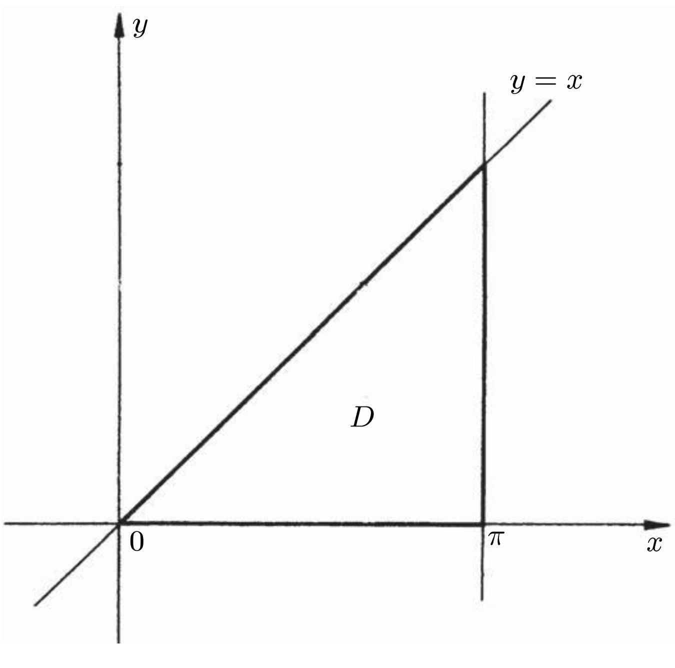
$$= \int_0^\pi dx \int_0^x x\cos(x+y)\,dy = \int_0^\pi x(\sin 2x - \sin x)\,dx$$

$$= \left(-\frac{x}{2}\cos 2x + \frac{\sin 2x}{4} + x\cos x - \sin x\right)\Big|_0^\pi = -\frac{3\pi}{2}\,.$$

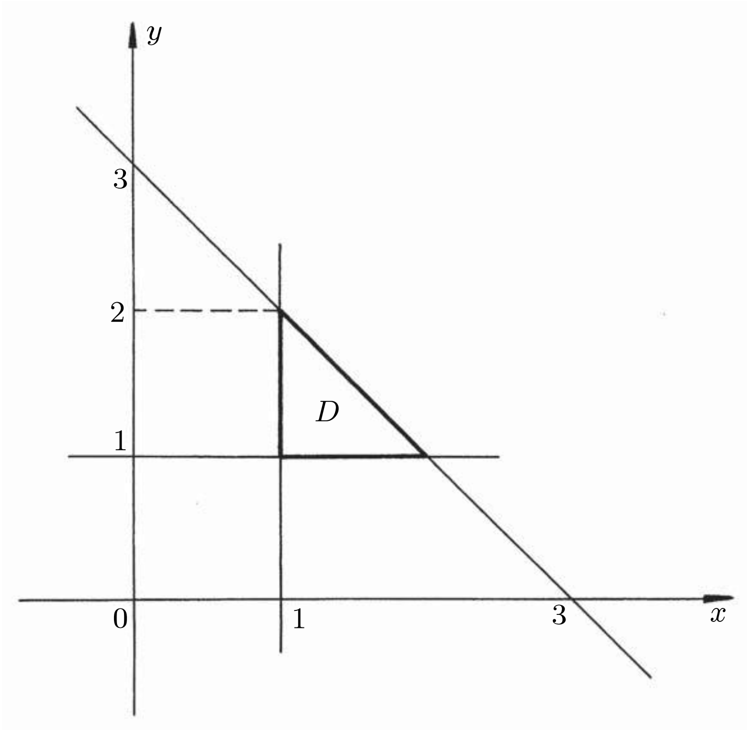


4.7 $\displaystyle\iint_D y^2 e^{xy}\,dx\,dy = \int_0^\pi dy \int_y^\pi y^2 e^{xy}\,dx = \int_0^\pi y\big(e^{\pi y} - e^{y^2}\big)\,dy$

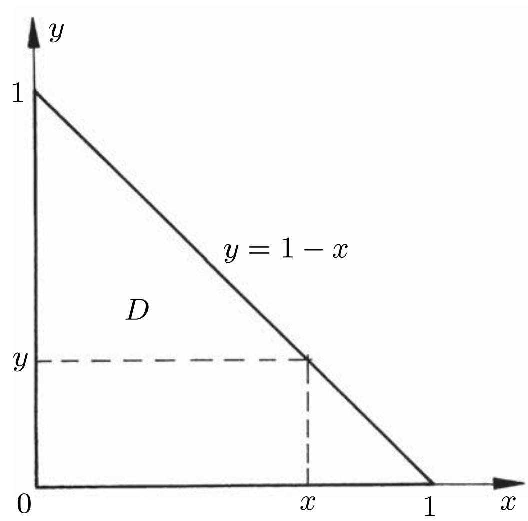
$$= \left(\frac{y}{\pi}e^{\pi y} - \frac{e^{\pi y}}{\pi^2} - \frac{e^{y^2}}{2}\right)\Big|_0^\pi = \left(\frac{1}{2} - \frac{1}{\pi^2}\right)e^{\pi^2} + \frac{1}{2} + \frac{1}{\pi^2}\,.$$



4.8 $$\iint_D \frac{dx\,dy}{(x+y)^3} = \int_1^2 dx \int_1^{3-x} \frac{dy}{(x+y)^3} = -\frac{1}{2}\int_1^2 \left(\frac{1}{9} - \frac{1}{(x+1)^2}\right) dx$$
$$= -\frac{1}{2}\left(\frac{x}{9} + \frac{1}{(x+1)}\right)\Bigg|_1^2 = \frac{1}{36}\,.$$



4.9 $$\iint_D 6^x 2^y dx\,dy = \int_0^1 dx \int_0^{1-x} 6^x 2^y\,dy = \frac{1}{\ln 2}\int_0^1 \left(2\,e^{x\ln 3} - e^{x\ln 6}\right) dx$$
$$= \frac{1}{\ln 2}\left(\frac{2}{\ln 3}\,e^{x\ln 3} - \frac{1}{\ln 6}\,e^{x\ln 6}\right)\Bigg|_0^1 = \frac{1}{\ln 2}\left(\frac{4}{\ln 3} - \frac{5}{\ln 6}\right).$$

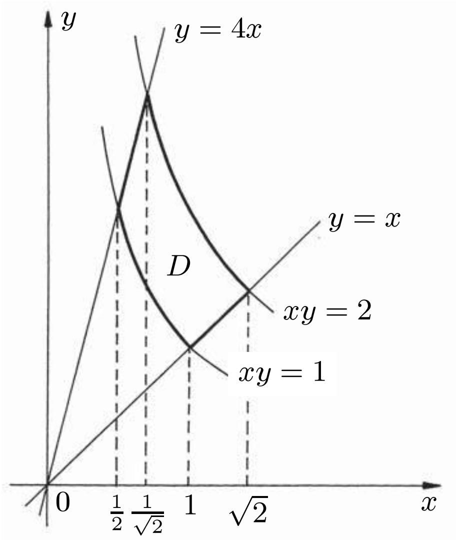


4.10 $\displaystyle\iint_D x^2y^2\,dx\,dy$

$$= \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} dx \int_{\frac{1}{x}}^{4x} x^2y^2\,dy + \int_{\frac{1}{\sqrt{2}}}^{1} dx \int_{\frac{1}{x}}^{\frac{2}{x}} x^2y^2\,dy + \int_{1}^{\sqrt{2}} dx \int_{x}^{\frac{2}{x}} x^2y^2\,dy$$

$$= \frac{1}{3}\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \left(64\,x^5 - \frac{1}{x}\right) dx + \frac{7}{3}\int_{\frac{1}{\sqrt{2}}}^{1} \frac{dx}{x} + \frac{1}{3}\int_{1}^{\sqrt{2}} \left(\frac{8}{x} - x^5\right) dx$$

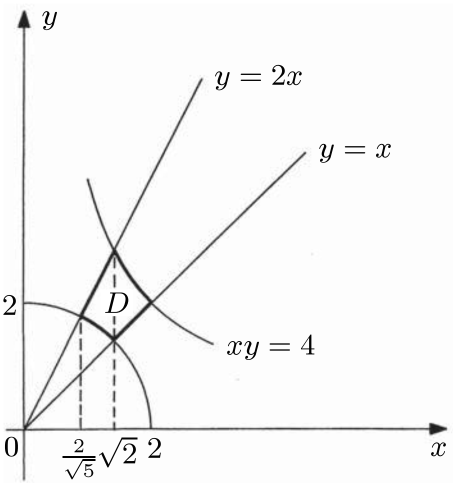
$$= \frac{1}{3}\left(\frac{32}{3}\,x^6 - \ln x\right)\Big|_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} + \frac{7}{3}\,\ln x\Big|_{\frac{1}{\sqrt{2}}}^{1} + \frac{1}{3}\left(8\ln x - \frac{x^6}{6}\right)\Big|_{1}^{\sqrt{2}} = \frac{7}{3}\,\ln 2\,.$$



4.11 $\displaystyle\iint_D x^2y\,dx\,dy = \int_{\frac{2}{\sqrt{5}}}^{\sqrt{2}} dx \int_{\sqrt{4-x^2}}^{2x} x^2y\,dy + \int_{\sqrt{2}}^{2} dx \int_{x}^{\frac{4}{x}} x^2y\,dy$

$$= \frac{1}{2}\int_{\frac{2}{\sqrt{5}}}^{\sqrt{2}} (5x^4 - 4x^2)\,dx + \frac{1}{2}\int_{\sqrt{2}}^{2} (16 - x^4)\,dx$$

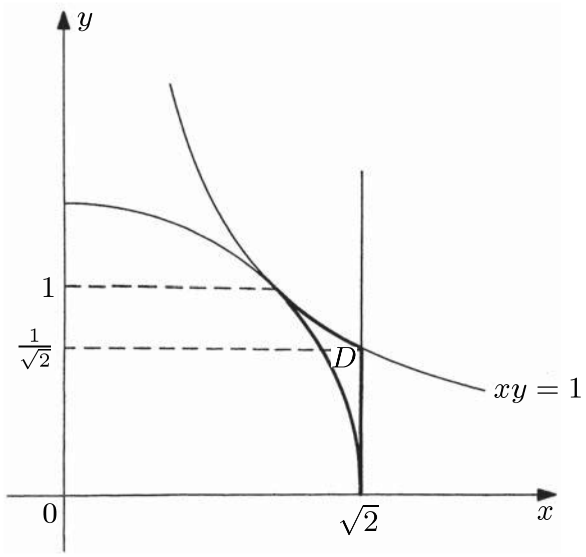
$$= \frac{1}{2}\,\left(x^5 - \frac{4}{3}\,x^3\right)\Big|_{\frac{2}{\sqrt{5}}}^{\sqrt{2}} + \frac{1}{2}\left(16\,x - \frac{x^5}{5}\right)\Big|_{\sqrt{2}}^{2} = -\frac{104}{15}\,\sqrt{2} + \frac{32}{375}\,\sqrt{5} + \frac{64}{5}\,.$$



4.12 $\displaystyle \iint_D xy^2\,dx\,dy = \int_0^{\frac{1}{\sqrt{2}}} dy \int_{\sqrt{2-y^2}}^{\sqrt{2}} xy^2\,dx + \int_{\frac{1}{\sqrt{2}}}^{1} dy \int_{\sqrt{2-y^2}}^{\frac{1}{y}} xy^2\,dx$

$$= \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} y^4\,dy + \frac{1}{2}\int_{\frac{1}{\sqrt{2}}}^{1} \left(y^2-1\right)^2 dy$$

$$= \frac{y^5}{10}\Big|_0^{\frac{1}{\sqrt{2}}} + \frac{1}{2}\left(\frac{y^5}{5} - \frac{2}{3}y^3 + y\right)\Big|_{\frac{1}{\sqrt{2}}}^{1} = \frac{1}{3}\left(\frac{4}{5} - \frac{1}{\sqrt{2}}\right).$$

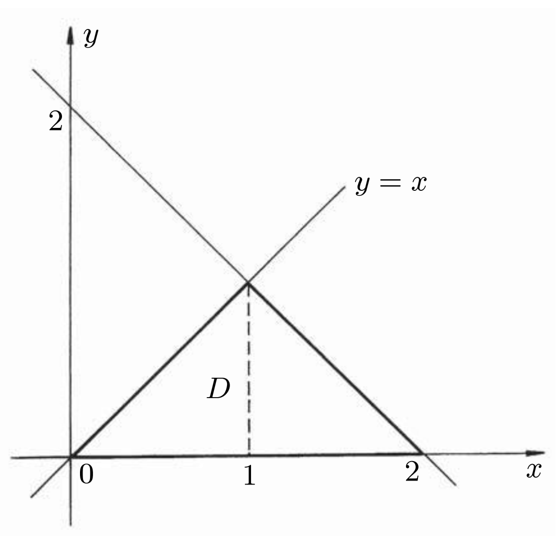


4.13 $\displaystyle \iint_D \left|(x-y)(x+y-2)\right| dx\,dy$

$$= \int_0^1 dx \int_0^x (x-y)(2-x-y)\,dy + \int_1^2 dx \int_0^{2-x} (x-y)(2-x-y)\,dy$$

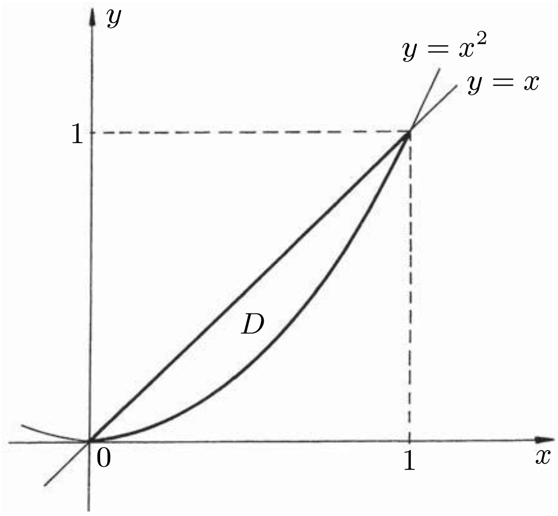
$$= \int_0^1 \left(x^2 - \frac{2}{3}x^3\right) dx + \int_1^2 \left(4x - 4x^2 + x^3 - (2-x)^2 + \frac{(2-x)^3}{3}\right) dx$$

$$= \left(\frac{x^3}{3} - \frac{x^4}{6}\right)\Big|_0^1 + \left(2x^2 - \frac{4}{3}x^3 + \frac{x^4}{4} + \frac{(2-x)^3}{3} - \frac{(2-x)^4}{12}\right)\Big|_0^2 = \frac{1}{3}.$$



4.14 $\displaystyle\iint_D x\sin y\,dx\,dy = \int_0^1 dx\int_{x^2}^{x} x\sin y\,dy = \int_0^1 x(\cos x^2 - \cos x)\,dx$

$\displaystyle = \left(\frac{\sin x^2}{2} - x\sin x - \cos x\right)\Big|_0^1 = 1 - \frac{\sin 1}{2} - \cos 1\,.$

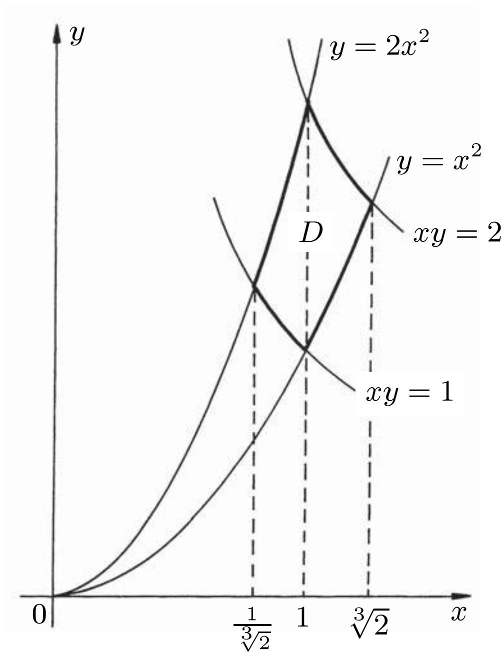


4.15 $\displaystyle\iint_D (x^3+y^3)\,dx\,dy$

$\displaystyle = \int_{\frac{1}{\sqrt[3]{2}}}^{1} dx\int_{\frac{1}{x}}^{2x^2}(x^3+y^3)\,dy + \int_1^{\sqrt[3]{2}} dx\int_{x^2}^{\frac{2}{x}}(x^3+y^3)\,dy$

$\displaystyle = \int_{\frac{1}{\sqrt[3]{2}}}^{1}\left(2x^5+4x^8-x^2-\frac{1}{4x^4}\right)dx + \int_1^{\sqrt[3]{2}}\left(2x^2+\frac{4}{x^4}-x^5-\frac{x^8}{4}\right)dx$

$\displaystyle = \left(\frac{x^6}{3}+\frac{4}{9}x^9-\frac{x^3}{3}+\frac{1}{12\,x^3}\right)\Big|_{\frac{1}{\sqrt[3]{2}}}^{1} + \left(\frac{2}{3}x^3-\frac{4}{3x^3}-\frac{x^6}{6}-\frac{x^9}{36}\right)\Big|_1^{\sqrt[3]{2}} = \frac{37}{36}\,.$



4.16 $$\iint_D \frac{dx\,dy}{\sqrt{xy}} = \int_1^{\frac{1+\sqrt5}{2}} dx \int_{\frac{4}{x}}^{\frac{8}{x}} \frac{dy}{\sqrt{xy}} + \int_{\frac{1+\sqrt5}{2}}^{2} dx \int_{4x-4}^{\frac{8}{x}} \frac{dy}{\sqrt{xy}}$$
$$= 4\left(\sqrt2-1\right)\int_1^{\frac{1+\sqrt5}{2}} \frac{dx}{x} + 4\int_{\frac{1+\sqrt5}{2}}^{2}\left(\frac{\sqrt2-\sqrt{x(x-1)}}{x}\right)dx$$
$$= 4\sqrt2\,\ln 2 - 4\ln\frac{1+\sqrt5}{2} - 4\int_{\frac{1+\sqrt5}{2}}^{2}\frac{\sqrt{x(x-1)}}{x}\,dx$$
$$= 4\sqrt2\,\ln 2 - 4\ln\frac{1+\sqrt5}{2} - 4\left(\sqrt{x(x-1)} - \ln\left(\sqrt{x}+\sqrt{x-1}\right)\right)\Big|_{\frac{1+\sqrt5}{2}}^{2}$$
$$= 4\left(\sqrt2\,\ln 2 - \ln\frac{1+\sqrt5}{2} - \sqrt2 + \ln\left(1+\sqrt2\right) + 1 - \ln\left(\sqrt{\frac{1+\sqrt5}{2}} + \sqrt{\frac{-1+\sqrt5}{2}}\right)\right).$$

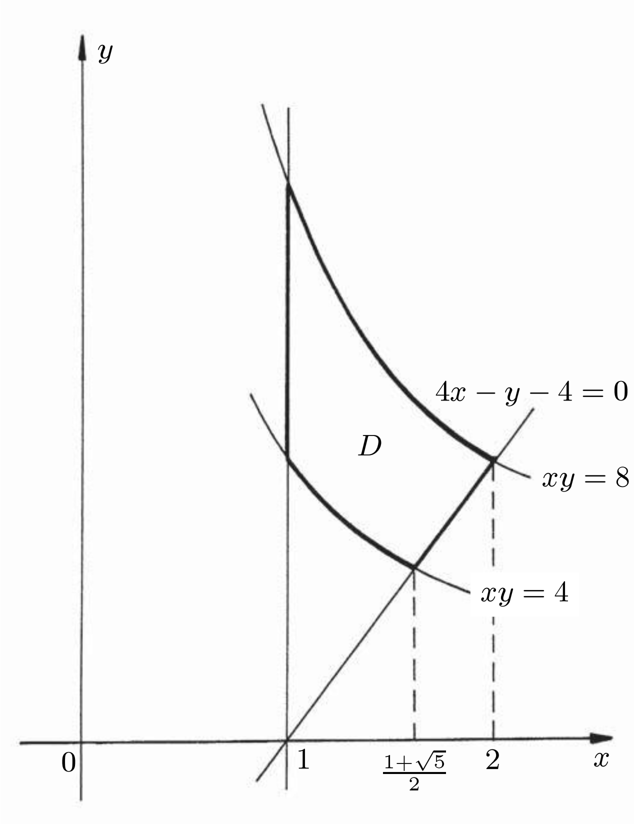


Fig. ex. 4.16

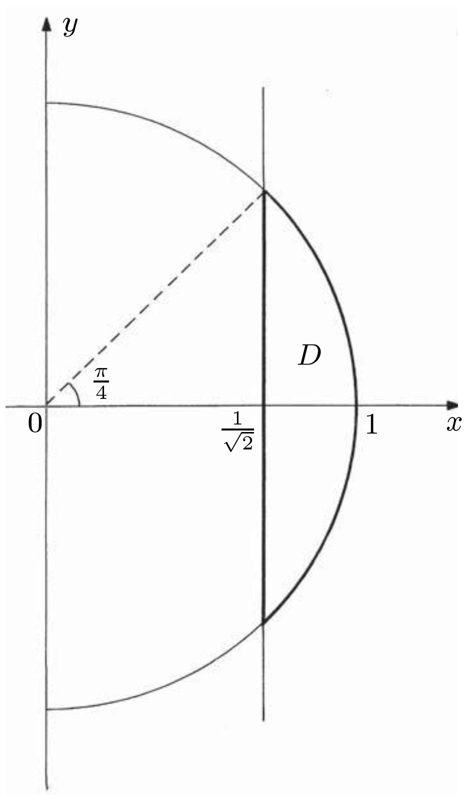
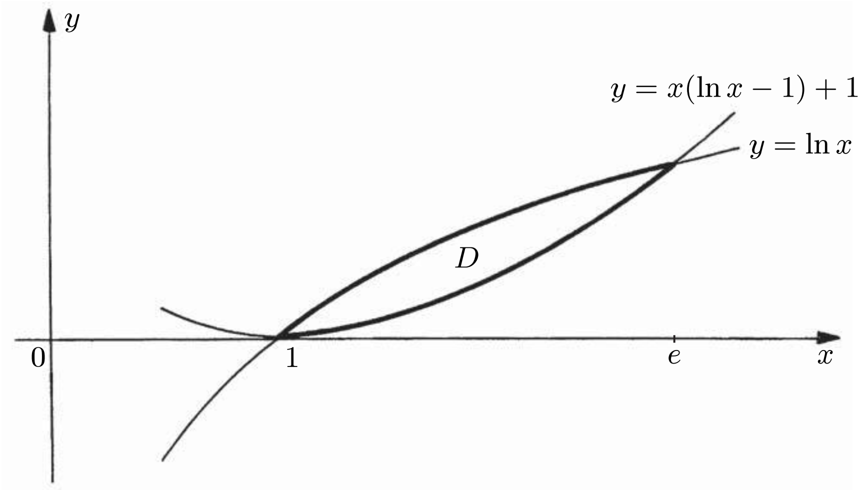


Fig. ex. 4.17

4.17 $$\iint_D (x-y)\,dx\,dy = \int_{\frac{1}{\sqrt2}}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x-y)\,dy = 2\int_{\frac{1}{\sqrt2}}^{1} x\sqrt{1-x^2}\,dx$$
$$= -\frac{2}{3}\left(1-x^2\right)^{\frac32}\Big|_{\frac{1}{\sqrt2}}^{1} = \frac{1}{3\sqrt2}.$$

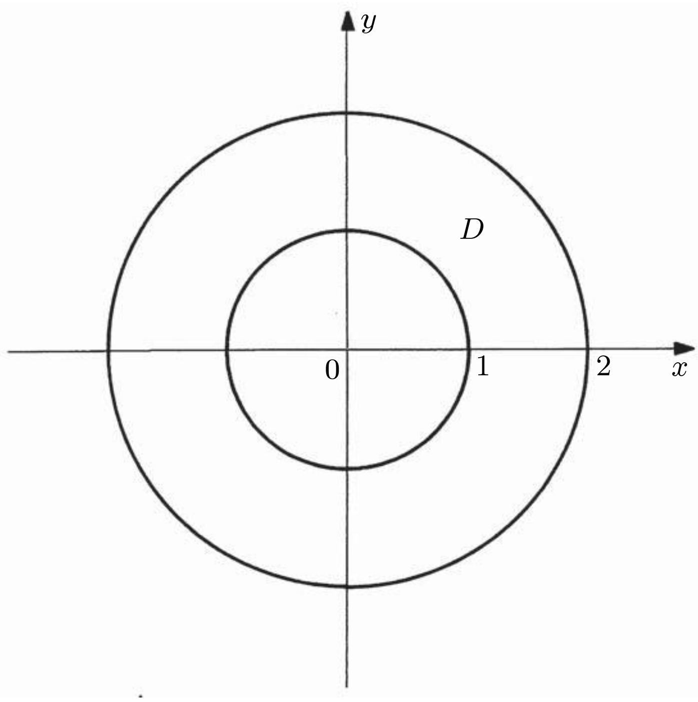
4.18 $\displaystyle\iint_D x\,dx\,dy = \int_1^e dx \int_{x(\ln x-1)+1}^{\ln x} x\,dy = \int_1^e (x\ln x - x^2\ln x + x^2 - x)\,dx$

$$= \left(\frac{x^2}{6}(3-2x)\ln x - \frac{3}{4}x^2 + \frac{4}{9}x^3\right)\Big|_1^e = \frac{e^3}{9} - \frac{e^2}{4} + \frac{11}{36}\,.$$



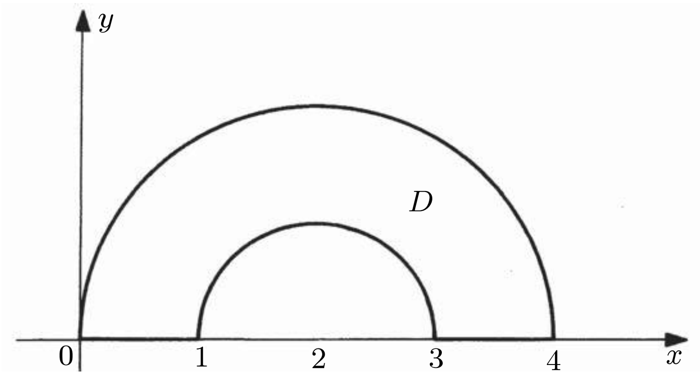
4.19 Puisque $D = \{(\rho\cos\theta,\ \rho\sin\theta) : 1 < \rho < 2\,,\ 0 \le \theta < 2\pi\}$, on a

$$\iint_D \frac{\sin(x^2+y^2)}{2+\cos(x^2+y^2)}\,dx\,dy = \int_0^{2\pi} d\theta \int_1^2 \frac{\sin\rho^2}{2+\cos\rho^2}\,\rho\,d\rho$$
$$= -\pi\ln\bigl(2+\cos\rho^2\bigr)\Big|_1^2 = \pi\ln\frac{2+\cos 1}{2+\cos 4}\,.$$



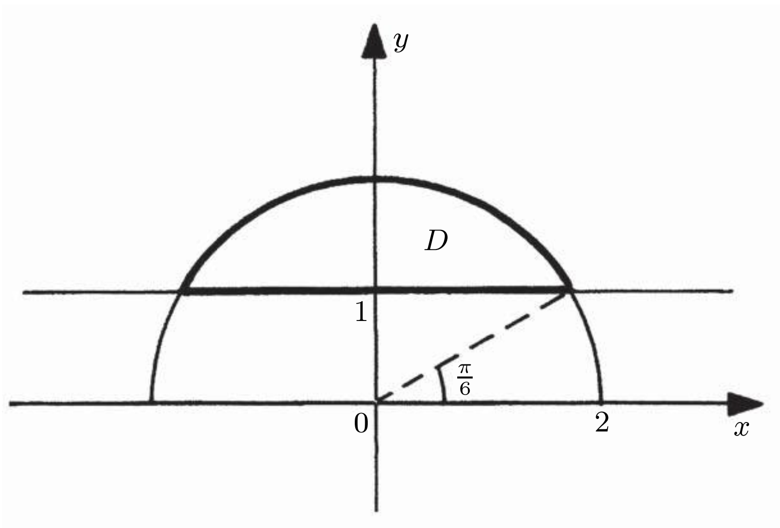
4.20 Puisque $D = \{(2+\rho\cos\theta,\ \rho\sin\theta) : 1 < \rho < 2\,,\ 0 < \theta < \pi\}$, on a

$$\iint_D \cos(x^2+y^2-4x+4)\,dx\,dy = \int_0^\pi d\theta \int_1^2 \rho\cos\rho^2\,d\rho$$
$$= \frac{\pi}{2}\left(\sin\rho^2\right)\Big|_1^2 = \frac{\pi}{2}\left(\sin 4 - \sin 1\right).$$



4.21 Puisque $D = \left\{(\rho\cos\theta,\ \rho\sin\theta) : \frac{1}{\sin\theta} < \rho < 2\,,\ \frac{\pi}{6} < \theta < \frac{5\pi}{6}\right\}$, on a

$$\iint_D \frac{y^2}{x^2+y^2}\,dx\,dy = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} d\theta \int_{\frac{1}{\sin\theta}}^{2} \rho\sin^2\theta\,d\rho = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(2\sin^2\theta - \frac{1}{2}\right) d\theta$$
$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\frac{1}{2} - \cos 2\theta\right) d\theta = \frac{1}{2}\left(\theta - \sin 2\theta\right)\Big| = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

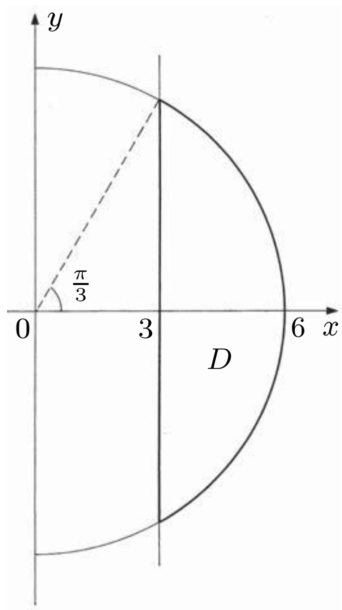


4.22 Puisque $D = \left\{(\rho\cos\theta,\ \rho\sin\theta) : \dfrac{3}{\cos\theta} < \rho < 6\,,\ |\theta| < \dfrac{\pi}{3}\right\}$, on a

$$\iint_D \frac{x}{\left(x^2+y^2\right)^2}\,dx\,dy = 2\int_0^{\frac{\pi}{3}} d\theta \int_{\frac{3}{\cos\theta}}^{6} \frac{\cos\theta}{\rho^2}\,d\rho$$

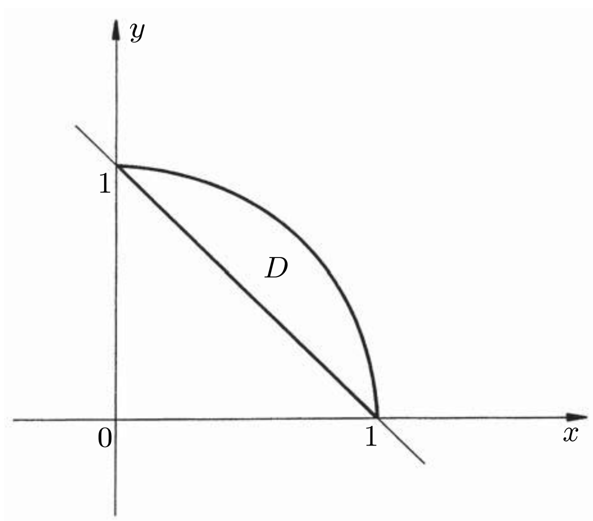
$$= 2\int_0^{\frac{\pi}{3}} \left(-\frac{\cos\theta}{6} + \frac{\cos^2\theta}{3}\right) d\theta = \frac{1}{3}\int_0^{\frac{\pi}{3}} (-\cos\theta + 1 + \cos 2\theta)\,d\theta$$

$$= \frac{1}{3}\left(-\sin\theta + \theta + \frac{\sin\theta}{2}\right)\Bigg|_0^{\frac{\pi}{3}} = -\frac{\sqrt{3}}{12} + \frac{\pi}{9}\,.$$



4.23 Puisque $D = \left\{(\rho\cos\theta,\ \rho\sin\theta) : \dfrac{1}{\sin\theta+\cos\theta} < \rho < 1\,,\ 0 < \theta < \dfrac{\pi}{2}\right\}$, on a

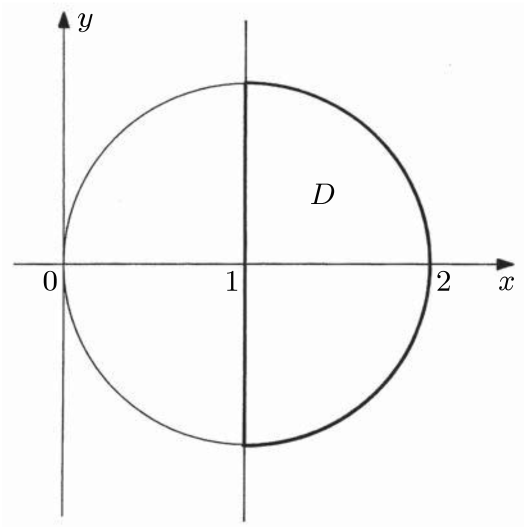
$$\iint_D \frac{dx\,dy}{\left(x^2+y^2\right)^2} = \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{1}{\sin\theta+\cos\theta}}^{1} \frac{d\rho}{\rho^3} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta = -\frac{1}{4}\,\cos 2\theta\Bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}\,.$$



4.25 Puisque $D = \left\{(\rho\cos\theta,\ \rho\sin\theta) : \dfrac{1}{\cos\theta} < \rho < 2\cos\theta\,,\ |\theta| < \dfrac{\pi}{4}\right\}$, on a

$$\iint_D \frac{dx\,dy}{\left(x^2+y^2\right)^2} = 2\int_0^{\frac{\pi}{4}} d\theta \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{d\rho}{\rho^3} = \int_0^{\frac{\pi}{4}} \left(-\frac{1}{4\cos^2\theta} + \cos^2\theta\right) d\theta$$

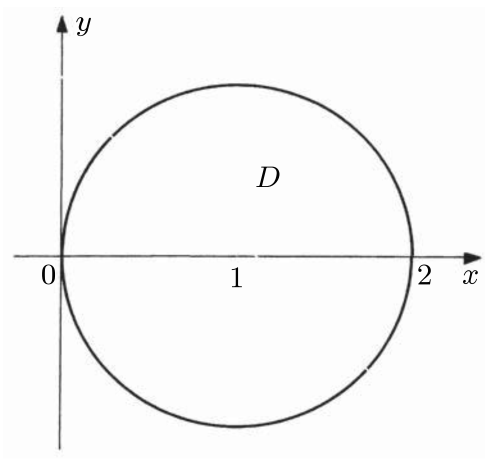
$$= -\frac{\operatorname{tg}\theta}{4} + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\frac{\pi}{4}} = \frac{\pi}{8}\,.$$



4.25 Puisque $D = \left\{(\rho\cos\theta,\ \rho\sin\theta) : 0 < \rho < 2\cos\theta\,,\ |\theta| < \dfrac{\pi}{2}\right\}$, on a

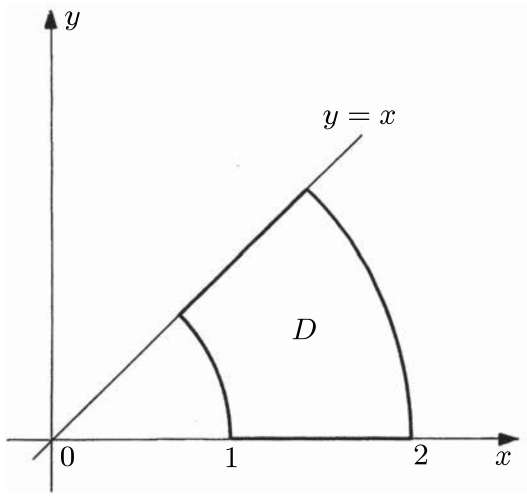
$$\iint_D xy(x^2+y^2)\,dx\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} (\rho^5 \sin\theta\cos\theta)\,d\rho$$

$$= \frac{32}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^7\theta\sin\theta\,d\theta = -\frac{4}{3}\,\cos^8\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0\,.$$



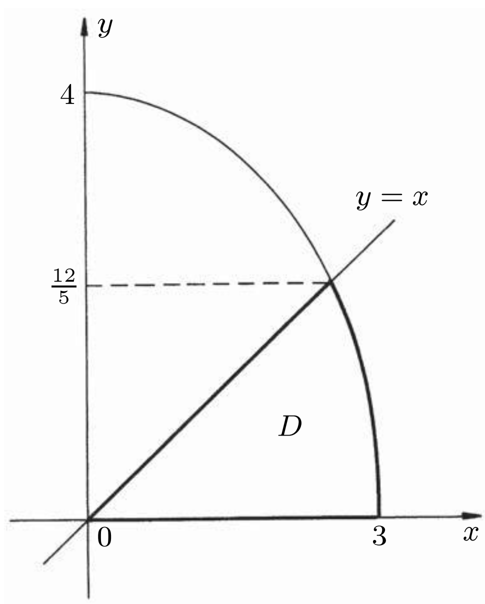
4.26 Puisque $D = \left\{(\rho\cos\theta,\ \rho\sin\theta) : 1 < \rho < 2\cos\theta\,,\ 0 < \theta < \frac{\pi}{4}\right\}$, on a

$$\iint_D \frac{y^2\cos(x^2+y^2)}{x^2}\,dx\,dy = \int_0^{\frac{\pi}{4}} \operatorname{tg}^2\theta\,d\theta \int_1^2 \rho\cos\rho^2\,d\rho = (\operatorname{tg}\theta - \theta)\Big|_0^{\frac{\pi}{4}}\ \frac{\sin\rho^2}{2}\Big|_1^2$$
$$= \frac{1}{2}\Big(1 - \frac{\pi}{4}\Big)\,(\sin 4 - \sin 1)\,.$$



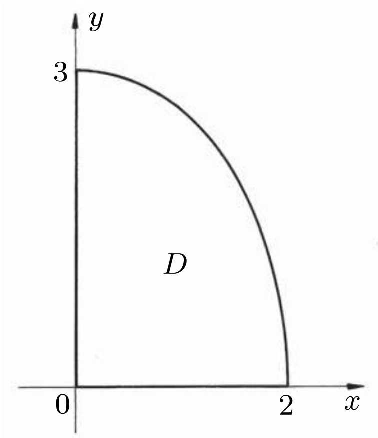
4.27 Puisque $D = \left\{(3\rho\cos\theta,\ 4\rho\sin\theta) : 0 < \rho < 1\,,\ 0 < \theta < \operatorname{Arctg}\frac{3}{4}\right\}$, on a

$$\iint_D x\,dx\,dy = \int_0^{\operatorname{Arctg}\frac{3}{4}} \cos\theta\,d\theta \int_0^1 36\,\rho^2\,d\rho = 12\,\sin\theta\Big|_0^{\operatorname{Arctg}\frac{3}{4}} = \frac{36}{5}\,.$$



4.28 Puisque $D = \left\{(2\rho\cos\theta,\, 3\rho\sin\theta) : 0 < \rho < 1\,,\ 0 < \theta < \frac{\pi}{2}\right\}$, on a

$$\iint_D x^2y^4\,dx\,dy$$
$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 (1944\,\rho^7\cos^2\theta\sin^4\theta)\,d\rho = 243\int_0^{\frac{\pi}{2}} (\cos^2\theta\sin^4\theta)\,d\theta$$
$$= \frac{243}{32}\left(2\theta - \frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{2} + \frac{\sin 6\theta}{6}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{243\pi}{32}\,.$$

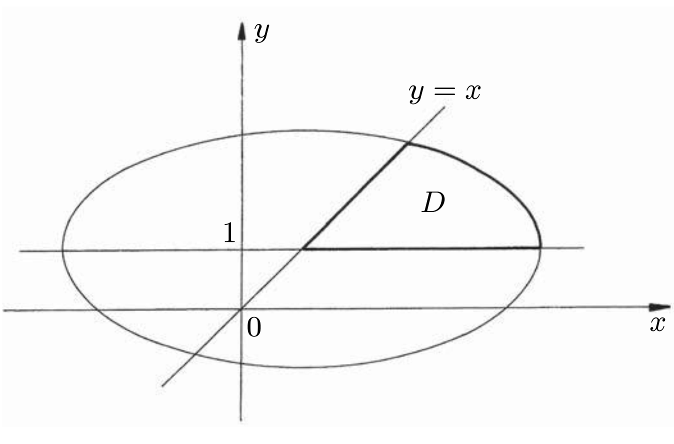


4.29 Puisque

$$D = \left\{(1 + 3\rho\cos\theta,\, 1 + 2\rho\sin\theta) : 0 < \rho < 1\,,\ 0 < \theta < \operatorname{Arctg}\frac{3}{2}\right\},$$

on a

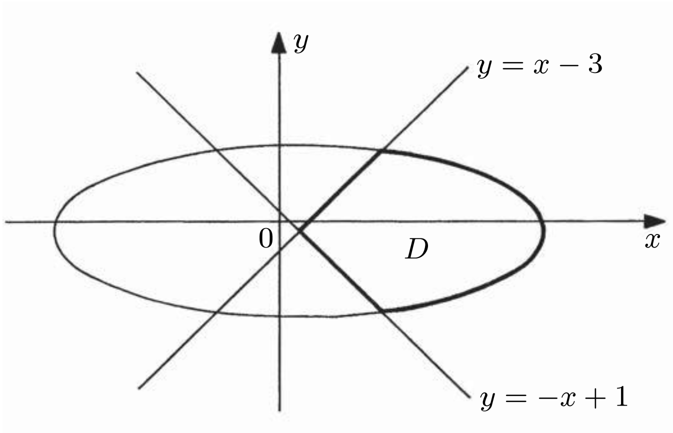
$$\iint_D xy\,dx\,dy = 6\int_0^{\operatorname{Arctg}\frac{3}{2}} d\theta \int_0^1 (\rho + 3\rho^2\cos\theta + 2\rho^2\sin\theta + 3\rho^3\sin 2\theta)\,d\rho$$
$$= 6\int_0^{\operatorname{Arctg}\frac{3}{2}} \left(\frac{1}{2} + \cos\theta + \frac{2}{3}\,\sin\theta + \frac{3}{4}\,\sin 2\theta\right) d\theta$$
$$= 6\left(\frac{\theta}{2} + \sin\theta - \frac{2}{3}\,\cos\theta - \frac{3}{8}\,\cos 2\theta\right)\Bigg|_0^{\operatorname{Arctg}\frac{3}{2}} = 3\operatorname{Arctg}\frac{3}{2} + \frac{10}{\sqrt{13}} + \frac{185}{26}\,.$$



4.30 Puisque $D = \left\{(2+5\rho\cos\theta,\ -1+3\rho\sin\theta) : 0<\rho<1\,,\ |\theta| < \text{Arctg}\,\frac{5}{3}\right\}$, on a

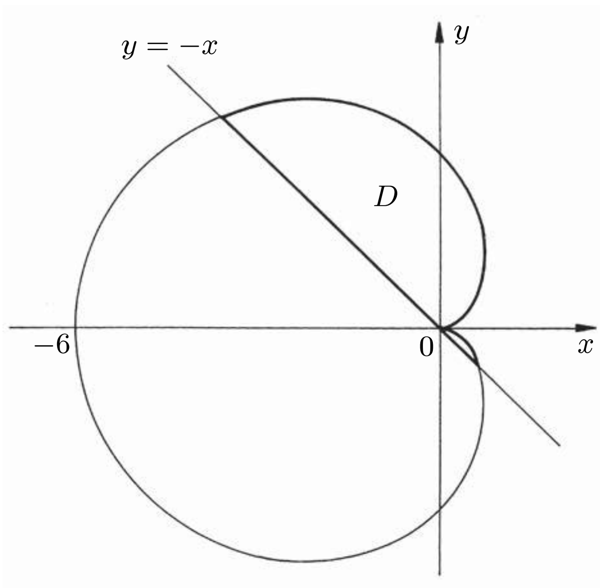
$$\iint_D (x-2)(y+1)^2\,dx\,dy = \int_{-\text{Arctg}\,\frac{5}{3}}^{\text{Arctg}\,\frac{5}{3}} d\theta \int_0^1 (675\,\rho^4 \sin^2\theta\cos\theta)\,d\rho$$

$$= 135 \int_{-\text{Arctg}\,\frac{5}{3}}^{\text{Arctg}\,\frac{5}{3}} \sin^2\theta\cos\theta\,d\theta = 45\,\sin^3\theta\Big|_{-\text{Arctg}\,\frac{5}{3}}^{\text{Arctg}\,\frac{5}{3}} = \frac{11\,250}{34\sqrt{34}}\,.$$



4.31 Puisque $D = \left\{(\rho\cos\theta,\ \rho\sin\theta) : 0<\rho<3(1-\cos\theta)\,,\ -\frac{\pi}{4} < \theta < \frac{3\pi}{4}\right\}$, on a

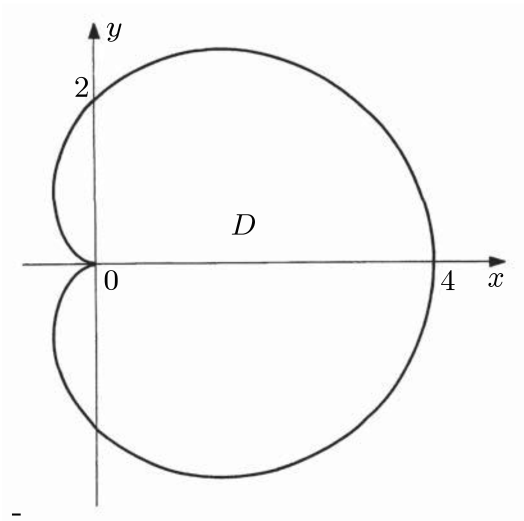
$$\iint_D \frac{dx\,dy}{\sqrt{x^2+y^2}} = 2\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^{3(1-\cos\theta)} d\rho = 3\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (1-\cos\theta)\,d\theta = 3(\pi-\sqrt{2}\,)\,.$$



4.32 Puisque $D = \big\{(\rho\cos\theta,\ \rho\sin\theta) : 0 < \rho < 2(1+\cos\theta)\,,\ 0 \le \theta < 2\pi\big\}$, on a

$$\iint_D \frac{dx\,dy}{\sqrt[4]{\left(x^2+y^2\right)^3}} = \int_0^{2\pi} d\theta \int_0^{2(1+\cos\theta)} \frac{d\rho}{\sqrt{\rho}} = 2\int_0^{2\pi} \sqrt{2(1+\cos\theta)}\,d\theta$$

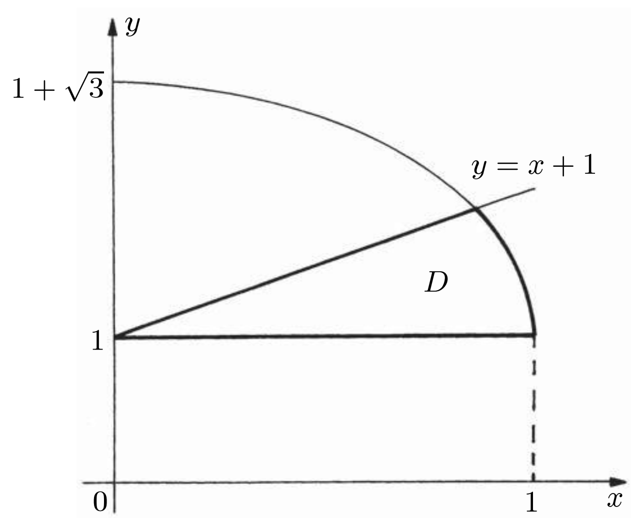
$$= 4\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta = 4\left(\int_0^{\pi} \cos\frac{\theta}{2}\,d\theta - \int_{\pi}^{2\pi} \cos\frac{\theta}{2}\,d\theta\right) = 16\,.$$



4.33 Puisque $D = \left\{(\rho\cos\theta,\ 1+\sqrt{3}\rho\sin\theta) : 0 < \rho < 1\,,\ 0 < \theta < \frac{\pi}{6}\right\}$, on a

$$\iint_D x(y-1)^2 dx\,dy = \int_0^{\frac{\pi}{6}} d\theta \int_0^1 \left(3\sqrt{3}\,\rho^4 \sin^2\theta\cos\theta\right) d\rho$$

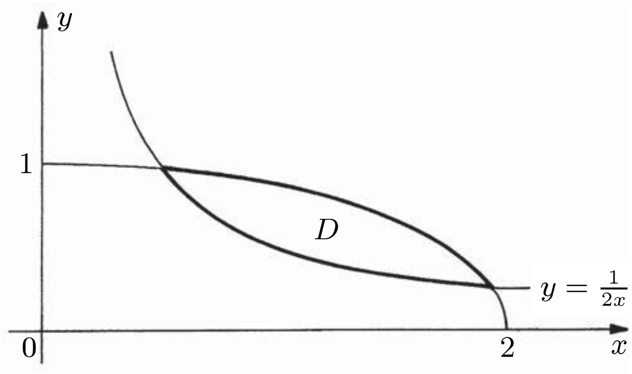
$$= \frac{3\sqrt{3}}{5}\int_0^{\frac{\pi}{6}} \sin^2\theta\cos\theta\,d\theta = \frac{\sqrt{3}}{5}\sin^3\theta\Big|_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{40}\,.$$



4.34 Puisque $D = \left\{(2\rho\cos\theta,\ \rho\sin\theta) : \dfrac{1}{\sqrt{2\sin 2\theta}} < \rho < 1\,,\ \dfrac{\pi}{12} < \theta < \dfrac{5\pi}{12}\right\}$, on a

$$\iint_D (x^2+4y^2)\,dx\,dy = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} d\theta \int_{\frac{1}{\sqrt{2\sin 2\theta}}}^{1} 8\rho^3\,d\rho$$

$$= 2\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \left(1-\frac{1}{4\sin^2 2\theta}\right) d\theta = \frac{2\pi}{3} + \frac{1}{4}\,\operatorname{cotg} 2\theta\Big|_{\frac{\pi}{12}}^{\frac{5\pi}{12}} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\,.$$



4.35 Puisque $D = \left\{(\rho\cos\theta,\ \rho\sin\theta) : 0 < \rho < 1\,,\ 0 < \theta < \dfrac{\pi}{4}\right\}$, on a

$$\iint_D (x^2+y^2)\,\operatorname{Arctg}\frac{y}{x}\,dx\,dy = \int_0^{\frac{\pi}{4}} \theta\,d\theta \int_0^1 \rho^3\,d\rho = \frac{\pi^2}{128}\,.$$

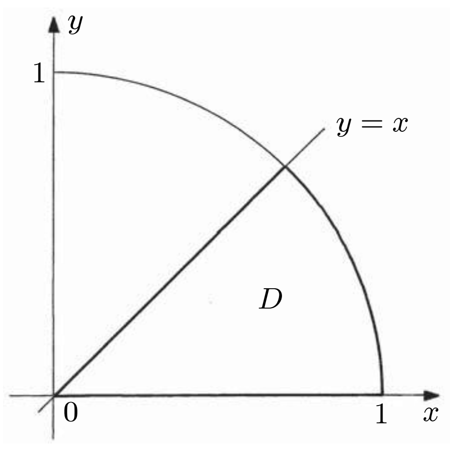


Fig. ex. 4.35 et **Fig. ex. 4.36**

4.36 Puisque

$$D = \left\{(\rho\cos\theta,\ 3\rho\sin\theta) : 0 < \rho < \frac{1}{\sqrt{1+8\sin^2\theta}}\,,\ 0 < \theta < \operatorname{Arctg}\frac{1}{3}\right\},$$

on a

$$\iint_D \left(x^2+\frac{y^2}{9}\right)\sin\!\left(2\operatorname{Arctg}\frac{y}{3x}\right)dx\,dy$$
$$=\int_0^{\operatorname{Arctg}\frac13} d\theta\int_0^{\frac{1}{\sqrt{1+8\sin^2\theta}}}(3\rho^3\sin 2\theta)\,d\rho$$
$$=\frac34\int_0^{\operatorname{Arctg}\frac13}\frac{\sin 2\theta}{\left(1+8\sin^2\theta\right)^2}\,d\theta=-\frac{3}{32}\,\frac{1}{1+8\sin^2\theta}\Bigg|_0^{\operatorname{Arctg}\frac13}=\frac{1}{24}\,.$$

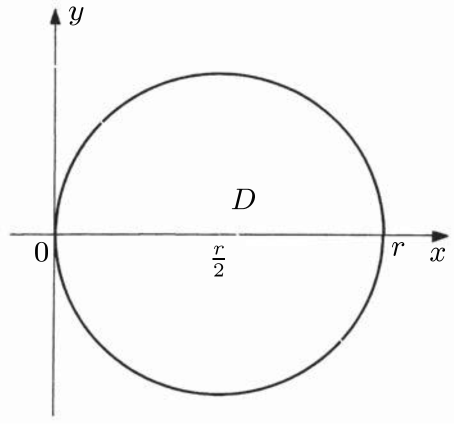
4.37 Soit $\gamma:\left[0,\frac{\pi}{2}\right]\to\mathbb{R}$ la fonction définie par $\gamma(\theta)=\begin{cases}\dfrac{\sqrt2}{\cos\theta} & \text{si } 0<\theta\le\dfrac{\pi}{6}\\[2ex] \dfrac{\sqrt{\frac23}}{\sin\theta} & \text{si } \dfrac{\pi}{6}<\theta\le\dfrac{\pi}{2}\end{cases}$

et posons $D=\left\{(\rho\cos\theta,\ \rho\sin\theta):0<\rho<\gamma(\theta)\,,\ 0<\theta<\frac{\pi}{2}\right\}$. D'où

$$\iint_D \frac{dx\,dy}{\sqrt{\left(\dfrac13+x^2+y^2\right)^3}}$$
$$=\int_0^{\frac{\pi}{6}}d\theta\int_0^{\frac{\sqrt2}{\cos\theta}}\frac{\rho}{\sqrt{\left(\dfrac13+\rho^2\right)^3}}\,d\rho+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}d\theta\int_0^{\frac{\sqrt{\frac23}}{\sin\theta}}\frac{\rho}{\sqrt{\left(\dfrac13+\rho^2\right)^3}}d\rho$$
$$=\int_0^{\frac{\pi}{6}}\left(\sqrt3-\frac{1}{\sqrt{\dfrac13+\dfrac{2}{\cos^2\theta}}}\right)d\theta+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(\sqrt3-\frac{1}{\sqrt{\dfrac13+\dfrac{2}{3\sin^2\theta}}}\right)d\theta$$
$$=\frac{\sqrt3}{2}\pi-\sqrt3\int_0^{\frac{\pi}{6}}\frac{\cos\theta}{\sqrt{7-\sin^2\theta}}\,d\theta-\sqrt3\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\sin\theta}{\sqrt{3-\cos^2\theta}}\,d\theta$$
$$=\frac{\sqrt3}{2}\pi-\sqrt3\,\operatorname{Arcsin}\frac{\sin\theta}{\sqrt7}\Bigg|_0^{\frac{\pi}{6}}+\sqrt3\,\operatorname{Arcsin}\frac{\cos\theta}{\sqrt3}\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}}=\frac{\pi}{\sqrt3}-\sqrt3\operatorname{Arcsin}\frac{1}{2\sqrt7}\,.$$

4.38 Puisque $D=\left\{(\rho\cos\theta,\ \rho\sin\theta):0<\rho<r\cos\theta\,,\ |\theta|<\dfrac{\pi}{2}\right\}$, on a

$$\iint_D\sqrt{r^2-x^2-y^2}\,dx\,dy=2\int_0^{\frac{\pi}{2}}d\theta\int_0^{r\cos\theta}\sqrt{r^2-\rho^2}\,\rho\,d\rho$$
$$=\frac{2r^3}{3}\int_0^{\frac{\pi}{2}}(1-\sin^3\theta)\,d\theta=\frac{2r^3}{3}\left(\theta+\cos\theta-\frac{\cos^3\theta}{3}\right)\Bigg|_0^{\frac{\pi}{2}}=\frac{r^3}{9}\,(3\pi-4)\,.$$



4.39 1) En effet, pour tout $r > 0$: $B(\mathbf{0}, r) \subset D_r =]-r, r[\times]-r, r[\subset B(\mathbf{0}, 2r)$ et

$$\iint_{B(\mathbf{0},r)} e^{-(x^2+y^2)}\,dx\,dy \leq \iint_{D_r} e^{-(x^2+y^2)}\,dx\,dy = 4\left(\int_0^r e^{-t^2}\,dt\right)^2 \leq \iint_{B(\mathbf{0},2r)} e^{-(x^2+y^2)}\,dx\,dy\,.$$

2) Ainsi, puisque pour tout $r > 0$:

$$\iint_{B(\mathbf{0},r)} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi} d\theta \int_0^r e^{-\rho^2}\rho\,d\rho = \pi\big(1-e^{-r^2}\big)$$

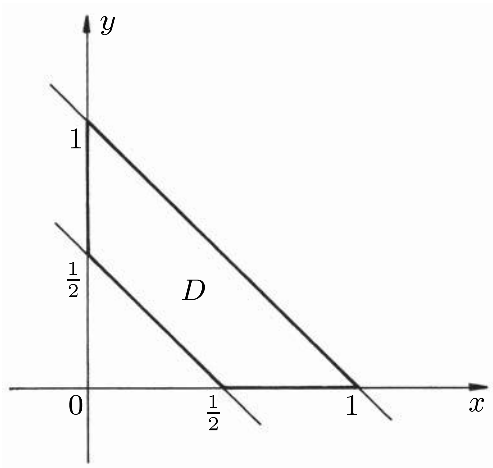
et

$$\iint_{B(\mathbf{0},2r)} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi} d\theta \int_0^{2r} e^{-\rho^2}\rho\,d\rho = \pi\big(1-e^{-4r^2}\big)\,,$$

on peut écrire $\sqrt{\frac{\pi}{4}\left(1-e^{-r^2}\right)} \leq \int_0^r e^{-t^2}\,dt \leq \sqrt{\frac{\pi}{4}\left(1-e^{-4r^2}\right)}$ ou encore, par passage à la limite,

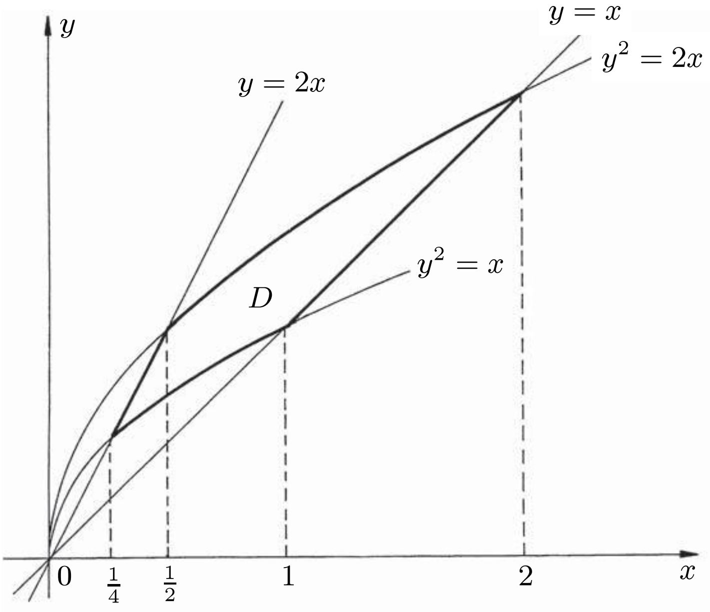
$$\int_0^{+\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\,.$$

4.40 $$\iint_D e^{\left(\frac{x-y}{x+y}\right)}\,dx\,dy = \frac{1}{2}\int_{\frac{1}{2}}^1 du \int_{-u}^{u} e^{\frac{v}{u}}\,dv = \operatorname{sh} 1 \int_{\frac{1}{2}}^1 u\,du = \frac{3}{8}\operatorname{sh} 1\,.$$



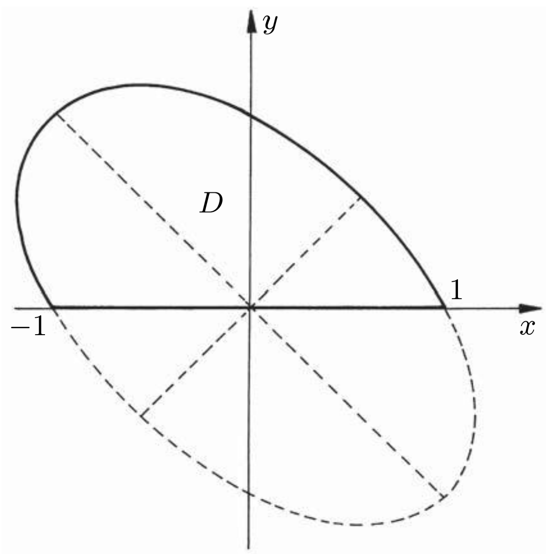
4.41 $$\iint_D \frac{dx\,dy}{(1+x)(1+xy^2)} = 2\int_0^1 du \int_0^u \frac{dv}{(1+u^2)(1+v^2)}$$
$$= 2\int_0^1 \frac{\operatorname{Arctg} u}{1+u^2}\,du = \operatorname{Arctg}^2 u\Big|_0^1 = \frac{\pi^2}{16}\,.$$

4.42 $$\iint_D \frac{y}{x}\,dx\,dy = \int_{\frac{1}{2}}^1 u\,du \int_1^2 v\,dv = \frac{9}{16}\,.$$



4.43 Posons $E = \{(u,v) \in \mathbb{R}^2 : u^2+v^2 < 1\,,\ v > 0\}$. Alors,

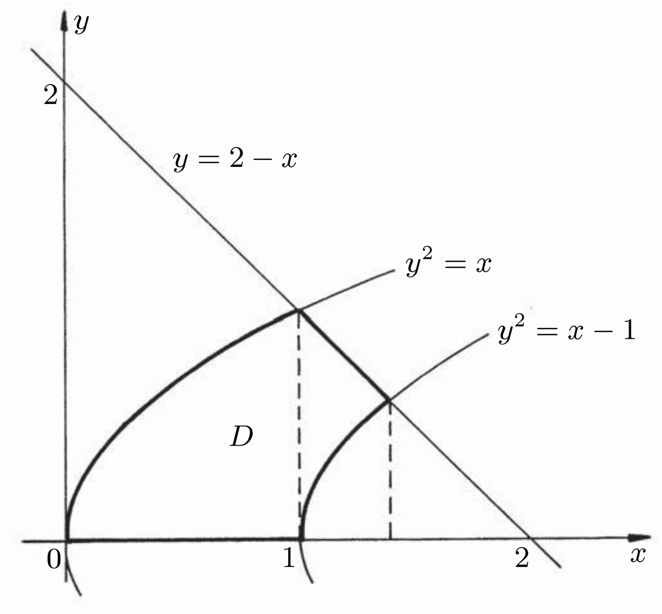
$$\iint_D e^{x^{2+xy+y^2}}\,dx\,dy = \frac{2}{\sqrt{3}}\iint_E e^{u^2+v^2}\,du\,dv$$
$$= \frac{2}{\sqrt{3}}\int_0^\pi d\theta \int_0^1 \rho\, e^{\rho^2}\,d\rho = \frac{\pi}{\sqrt{3}}\,(e-1)\,.$$



4.44 Posons $E =]0,1[\times]-1,0[$. Alors,

$$\iint_E u^2\,du\,dv = \iint_D \frac{y^2(2+2y^2-x)}{(2-x)^4}\,dx\,dy\,.$$

D'où $\displaystyle\iint_D \frac{y^2(2+2y^2-x)}{(2-x)^4}\,dx\,dy = \int_0^1 u^2\,du = \frac{1}{3}$.

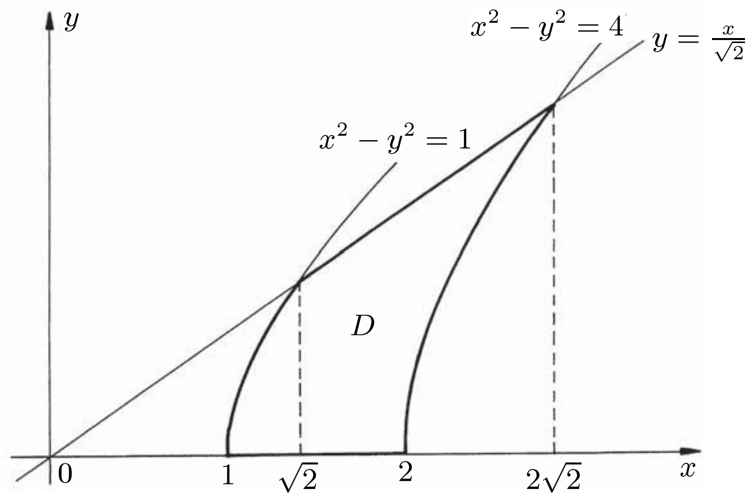


4.45 Posons $E =]1,4[\times]0,\frac{1}{\sqrt{2}}[$. Alors,

$$\iint_E \frac{v}{2u}\,\sin\pi(1-v^2)\,du\,dv = \iint_D \frac{y}{x^3}\,\sin\pi\left(1-\frac{y^2}{x^2}\right)dx\,dy\,.$$

D'où

$$\begin{aligned}\iint_D \frac{y}{x^3}\,\sin\pi\left(1-\frac{y^2}{x^2}\right)dx\,dy &= \int_1^4 du\int_0^{\frac{1}{\sqrt{2}}}\left(\frac{v}{2u}\,\sin\pi(1-v^2)\right)dv\\ &= \frac{1}{4\pi}\int_1^4\frac{du}{u} = \frac{\ln 2}{2\pi}\,.\end{aligned}$$



4.46 Posons, pour $0 < t < 1$:

$$D_t =]t,1[\times]0,1[\text{ et } E_t = \left\{(x,t) \in \mathbb{R}^2 : t < \mu < 1\,,\ 0 < \theta < \operatorname{Arctg}\frac{1}{\mu}\right\}.$$

Alors,

$$\int_t^1 \left(\int_0^1 \frac{x^2-y^2}{\left(x^2+y^2\right)^2}\,dy\right)dx = \iint_{D_t} \frac{x^2-y^2}{\left(x^2+y^2\right)^2}\,dx\,dy = \iint_{E_t} \frac{\cos 2\theta}{\mu}\,d\mu\,d\theta$$
$$= \int_t^1 d\mu \int_0^{\operatorname{Arctg}\frac{1}{\mu}} \frac{\cos 2\theta}{\mu}\,d\theta = \int_t^1 \frac{d\mu}{1+\mu^2} = \frac{\pi}{4} - \operatorname{Arctg} t\,.$$

D'où $\displaystyle\lim_{t\to 0+} \int_t^1 \left(\int_0^1 \frac{x^2-y^2}{\left(x^2+y^2\right)^2}\,dy\right)dx = \lim_{t\to 0+}\left(\frac{\pi}{4} - \operatorname{Arctg} t\right) = \frac{\pi}{4}.$

4.47 Posons $D = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1\,,\ \sqrt{x} < y < \sqrt[3]{x}\right\}$. Alors,

$$\int_0^1 dx \int_{\sqrt{x}}^{\sqrt[3]{x}} (1+y^6)\,dy = \iint_D (1+y^6)\,dx\,dy = \int_0^1 dy \int_{y^3}^{y^2} (1+y^6)\,dx$$
$$= \int_0^1 (y^2-y^3)(1+y^6)\,dy = \left(\frac{y^3}{3} - \frac{y^4}{4} + \frac{y^9}{9} - \frac{y^{10}}{10}\right)\Bigg|_0^1 = \frac{17}{180}\,.$$

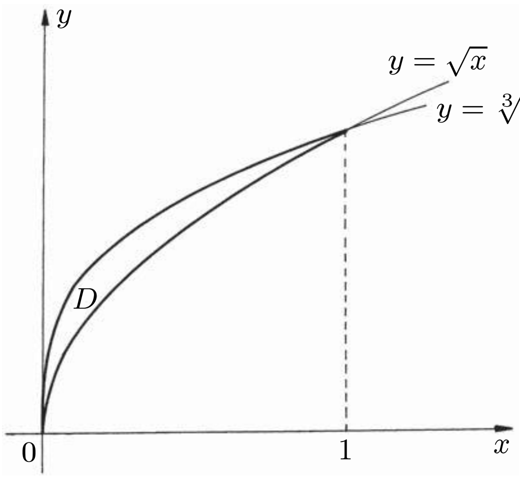


Fig. ex. 4.47

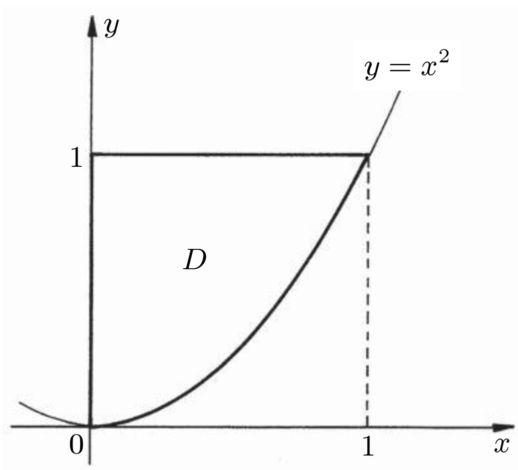


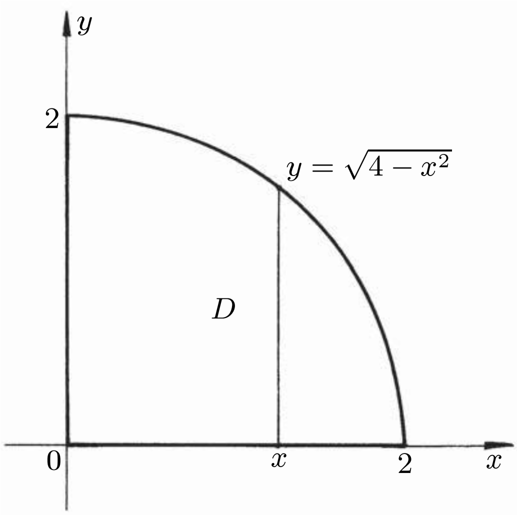
Fig. ex. 4.48

4.48 Posons $D = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1\,,\ x^2 < y < 1\right\}$. Alors,

$$\int_0^1 dx \int_{x^2}^1 xy\,e^{y^3}\,dy = \iint_D xy\,e^{y^3}\,dx\,dy = \int_0^1 dy \int_0^{\sqrt{y}} xy\,e^{y^3}\,dx$$
$$= \frac{1}{2}\int_0^1 y^2\,e^{y^3}\,dy = \frac{e^{y^3}}{6}\Bigg|_0^1 = \frac{e-1}{6}\,.$$

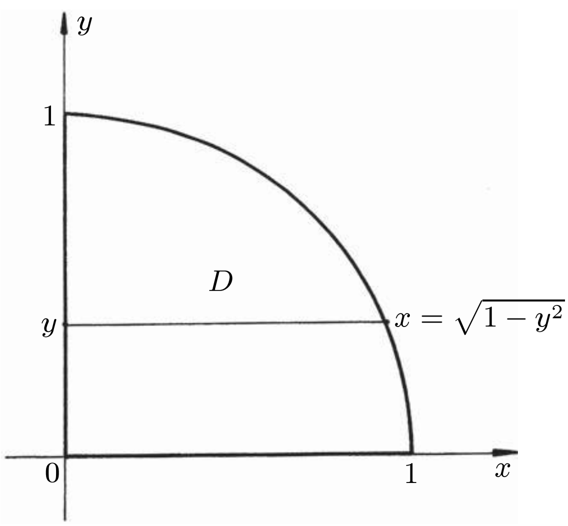
4.49 Posons $D = \{(x, y) \in \mathbb{R}^2 : x,\, y > 0\,,\ x^2 + y^2 < 4\}$. Alors,

$$\begin{aligned}\int_0^2 dx \int_0^{\sqrt{1-x^2}} \sqrt{\left(4-y^2\right)^3}\, dy &= \iint_D \sqrt{\left(4-y^2\right)^3}\, dx\, dy \\ &= \int_0^2 dy \int_0^{\sqrt{4-y^2}} \sqrt{\left(4-y^2\right)^3}\, dx = \int_0^2 \left(4-y^2\right)^2 dy \\ &= \left(16\, y - \frac{8}{3}\, y^3 + \frac{y^5}{5}\right)\Big|_0^2 = \frac{256}{15}\,.\end{aligned}$$



4.50 Posons $D = \{(x, y) \in \mathbb{R}^2 : x,\, y > 0\,,\ x^2 + y^2 < 1\}$. Alors,

$$\begin{aligned}\int_0^1 dy \int_0^{\sqrt{1-y^2}} \frac{y}{\sqrt{1+x^2+y^2}}\, dx &= \iint_D \frac{y}{\sqrt{1+x^2+y^2}}\, dx\, dy \\ &= \int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{y}{\sqrt{1+x^2+y^2}}\, dy = \int_0^1 \left(\sqrt{2} - \sqrt{1+x^2}\right) dx \\ &= \left(\sqrt{2}\, x - \frac{x}{2}\sqrt{1+x^2} - \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right)\right)\Big|_0^1 = \frac{1}{2}\left(\sqrt{2} - \ln\left(1+\sqrt{2}\right)\right).\end{aligned}$$



4.51 Posons $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\,,\ 0 < y < x\}$. Alors, en faisant le changement de variables $x = \rho\cos\theta$ et $y = \rho\sin\theta$ avec $0 < \rho < 1$ et $0 < \theta < \frac{\pi}{4}$, on peut écrire

$$\int_0^{\frac{1}{\sqrt{2}}} dy \int_y^{\sqrt{1-y^2}} \ln(1 + x^2 + y^2)\, dx = \iint_D \ln(1 + x^2 + y^2)\, dx\, dy$$
$$= \int_0^{\frac{\pi}{4}} d\theta \int_0^1 \ln(1 + \rho^2)\rho\, d\rho = \frac{\pi}{8} \int_1^2 \ln t\, dt = \frac{\pi}{8}(2\ln 2 - 1)\,.$$

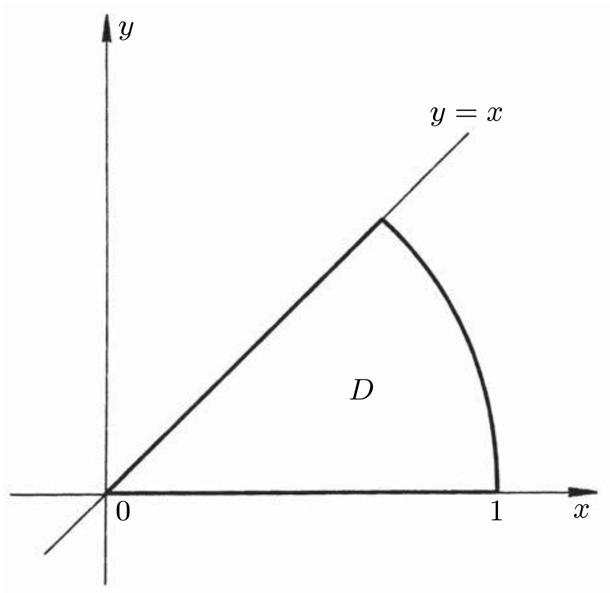


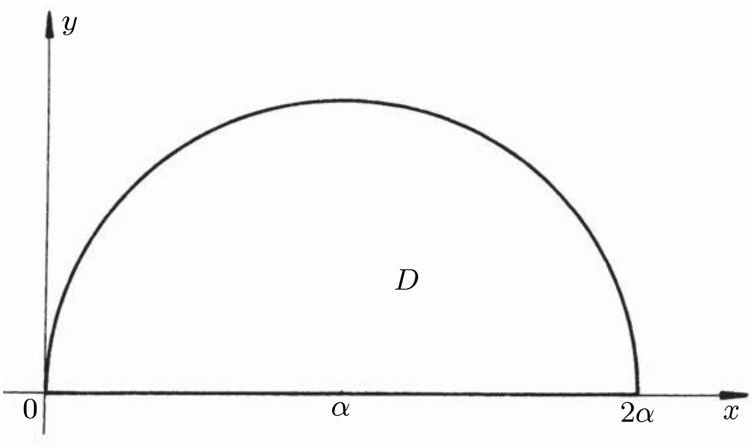
Fig. ex. 4.51 et **Fig. ex. 4.52**

4.52 Posons $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\,,\ 0 < y < x\}$. Alors, en faisant le changement de variables $x = \rho\cos\theta$ et $y = \rho\sin\theta$ avec $0 < \rho < 1$ et $0 < \theta < \frac{\pi}{4}$, on peut écrire

$$\int_0^{\frac{1}{\sqrt{2}}} dy \int_y^{\sqrt{1-y^2}} \sqrt{1 - x^2}\, dx$$
$$= \iint_D \sqrt{1 - x^2}\, dx\, dy = \int_0^{\frac{\pi}{4}} d\theta \int_0^1 \sqrt{1 - \rho^2\cos^2\theta}\, \rho\, d\rho = \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{1 - \sin^3\theta}{\cos^2\theta}\, d\theta$$
$$= \frac{1}{3}\operatorname{tg}\theta\Big|_0^{\frac{\pi}{4}} - \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{1 - \cos^2\theta}{\cos^2\theta}\sin\theta\, d\theta = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{\cos\theta} + \cos\theta\right)\Big|_0^{\frac{\pi}{4}} = 1 - \frac{1}{\sqrt{2}}\,.$$

4.53 Posons $D = \{(x,y) \in \mathbb{R}^2 : (x - \alpha)^2 + y^2 < \alpha^2\,,\ y > 0\}$. Alors, en faisant le changement de variables $x = \alpha + \rho\cos\theta$ et $y = \rho\sin\theta$ avec $0 < \rho < \alpha$ et $0 < \theta < \pi$, on peut écrire

$$\int_0^{2\alpha} dx \int_0^{\sqrt{2\alpha x - x^2}} (x^2 + y^2)\, dx\, dy = \iint_D (x^2 + y^2)\, dx\, dy$$
$$= \int_0^{\pi} d\theta \int_0^{\alpha} (\alpha^2 + \rho^2 + 2\alpha\rho\cos\theta)\rho\, d\rho = \alpha^4 \int_0^{\pi} \left(\frac{3}{4} + \frac{2}{3}\cos\theta\right) d\theta = \frac{3}{4}\,\alpha^4\pi\,.$$

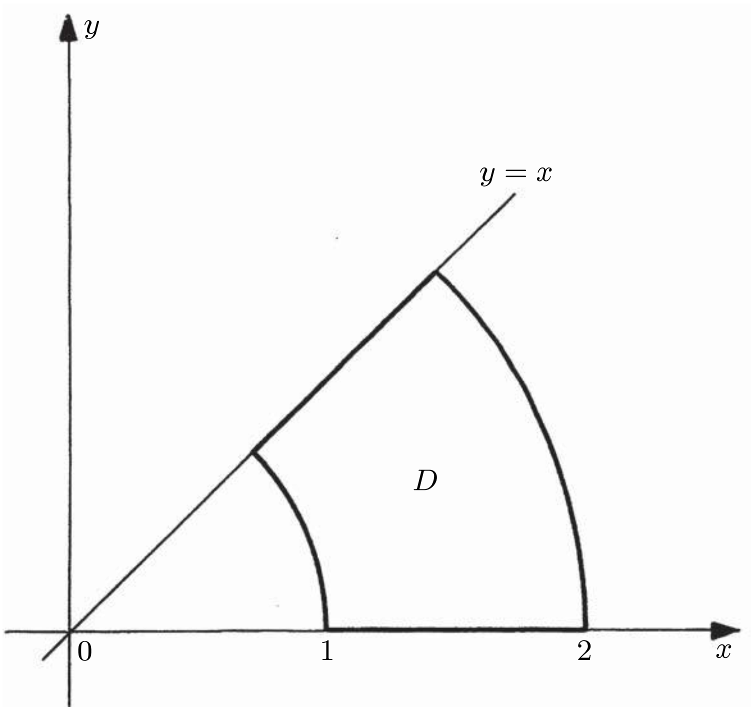


4.54 Posons $D = \{(x,y) \in \mathbb{R}^2 : 1 < x^2+y^2 < 4,\ 0 < y < x\}$. Alors, en faisant le changement de variables $x = \rho\cos\theta$ et $y = \rho\sin\theta$ avec $1 < \rho < 2$ et $0 < \theta < \frac{\pi}{4}$, on peut écrire

$$\int_{\frac{1}{\sqrt{2}}}^{1} dx \int_{\sqrt{1-x^2}}^{x} \frac{x^2-y^2}{x^2+y^2}\, dy + \int_{1}^{\sqrt{2}} dx \int_{0}^{x} \frac{x^2-y^2}{x^2+y^2}\, dy$$

$$+ \int_{\sqrt{2}}^{2} dx \int_{0}^{\sqrt{4-x^2}} \frac{x^2-y^2}{x^2+y^2}\, dy$$

$$= \iint_D \frac{x^2-y^2}{x^2+y^2}\, dx\, dy = \int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta \int_1^2 \rho\, d\rho = \frac{3}{4}\,.$$



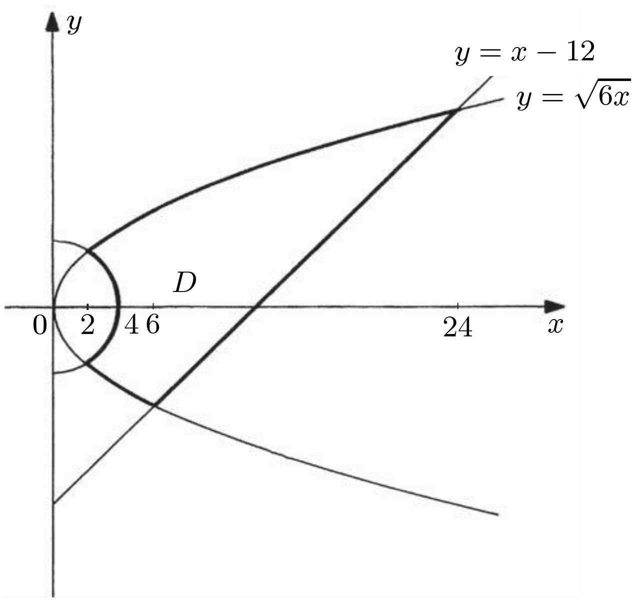
4.55 Aire$(D) = \iint_D dx\,dy$

$$= 2\int_2^4 dx \int_{\sqrt{16-x^2}}^{\sqrt{6x}} dy + 2\int_4^6 dx \int_0^{\sqrt{6x}} dy + \int_6^{24} dx \int_{x-12}^{\sqrt{6x}} dy$$

$$= 2\int_2^4 \left(\sqrt{6x} - \sqrt{16-x^2}\right) dx + 2\int_4^6 \sqrt{6x}\,dx + \int_6^{24} \left(\sqrt{6x} - (x-12)\right) dx$$

$$= \left(\frac{4}{3}\,x\sqrt{6x} - x\sqrt{16-x^2} - 16\ \text{Arcsin}\,\frac{x}{4}\right)\Bigg|_2^4$$

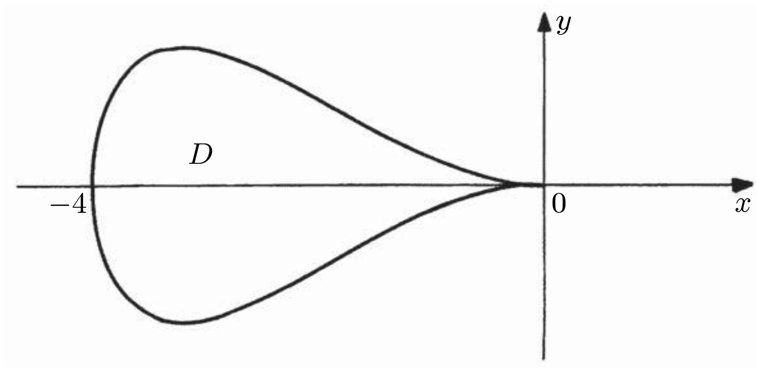
$$+\frac{4}{3}\,x\sqrt{6x}\Bigg|_4^6 + \left(\frac{2}{3}\,x\sqrt{6x} - \frac{1}{2}\,(x-12)^2\right)\Bigg|_6^{24} = 162 - \frac{16\,\pi}{3} - \frac{4}{\sqrt{3}}\,.$$



4.56 En faisant le chagement de variable $x = t^2 - 4$ avec $t \geq 0$, on peut écrire

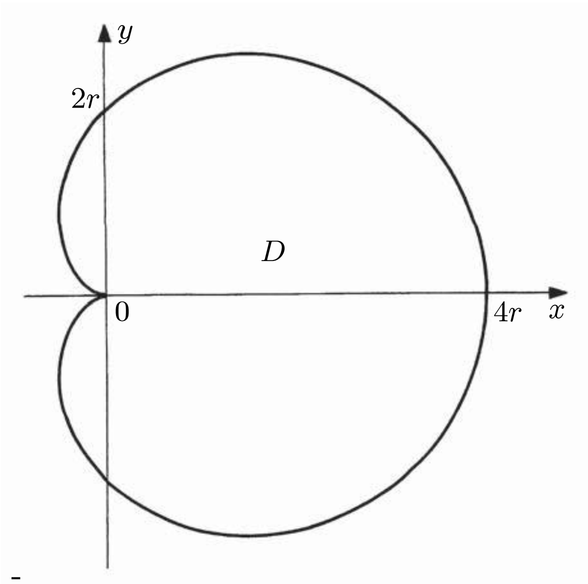
$$\text{Aire}(D) = \iint_D dx\,dy = \int_{-4}^0 dx \int_{-x^2\sqrt{x+4}}^{x^2\sqrt{x+4}} dy = 2\int_{-4}^0 x^2\sqrt{x+4}\,dx$$

$$= 4\int_0^2 t^2\left(t^2-4\right)^2 dt = 4\left(\frac{t^7}{7} - \frac{8}{5}\,t^5 + \frac{16}{3}\,t^3\right)\Bigg|_0^2 = \frac{2^{12}}{105}\,.$$



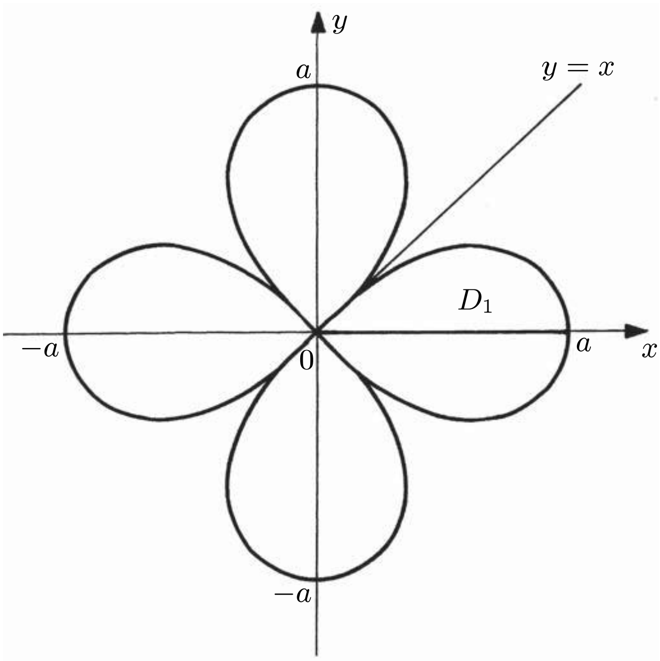
4.57 Puisque $D = \{(\rho\cos\theta,\ \rho\sin\theta) : 0 < \rho < 2r(1+\cos\theta)\,,\ 0 \le \theta < 2\pi\}$, on a

$$\text{Aire}(D) = \iint_D dx\,dy = \int_0^{2\pi} d\theta \int_0^{2r(1+\cos\theta)} \rho\,d\rho = 2r^2 \int_0^{2\pi} (1+\cos\theta)^2\,d\theta$$
$$= 2r^2 \left(\frac{3}{2}\,\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi} = 6\pi r^2\,.$$



4.58 Posons $D_1 = \left\{(\rho\cos\theta,\ \rho\sin\theta) : 0 < \rho < a\sqrt{\cos 2\theta}\,,\ 0 < \theta < \frac{\pi}{4}\right\}$. Alors,

$$\text{Aire}(D) = \iint_D dx\,dy = 8\iint_{D_1} dx\,dy = 8\int_0^{\frac{\pi}{4}} d\theta \int_0^{a\sqrt{\cos 2\theta}} \rho\,d\rho$$
$$= 4a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta\,d\theta = 2a^2\,.$$



4.59 Puisque D admet la première bissectrice $y = x$ pour axe de symétrie, on peut écrire

$$\iint_D \frac{f(x)}{f(x)+f(y)}\, dx\, dy = \iint_D \frac{f(y)}{f(x)+f(y)}\, dx\, dy \; ;$$

ce qui entraîne, entre autres, que

$$\text{Aire}(D) = \iint_D dx\, dy = \iint_D \frac{f(x)+f(y)}{f(x)+f(y)}\, dx\, dy = 2 \iint_D \frac{f(x)}{f(x)+f(y)}\, dx\, dy$$

ou encore $\displaystyle\iint_D \frac{f(x)}{f(x)+f(y)}\, dx\, dy = \frac{\text{Aire}(D)}{2} = 2\pi.$

Par conséquent $\displaystyle\iint_D \frac{2f(x)+5f(y)}{f(x)+f(y)}\, dx\, dy = 14\,\pi.$

4.60 Pour commencer, on va supposer que les deux fonctions f et g sont linéairement dépendantes et que $g \neq 0$ (sinon l'égalité est évidence). Alors, il existe un nombre réel λ tel que $f = \lambda g$ et l'on a

$$\begin{aligned}\left(\iint_D f(x,y)g(x,y)\, dx\, dy\right)^2 &= \left(\iint_D \lambda g^2(x,y)\, dx\, dy\right)^2 \\ &= \left(\iint_D f^2(x,y)\, dx\, dy\right)\left(\iint_D g^2(x,y)\, dx\, dy\right).\end{aligned}$$

Montrons à présent la réciproque. Pour cela, considérons la fonction auxiliaire $F : \mathbb{R} \to \mathbb{R}$ définie par

$$\begin{aligned}F(t) &= \iint_D (f+tg)^2(x,y)\, dx\, dy \\ &= t^2 \iint_D g^2(x,y)\, dx\, dy + 2t \iint_D fg(x,y)\, dx\, dy + \iint_D f^2(x,y)\, dx\, dy\end{aligned}$$

et supposons à nouveau pour la même raison que $g \neq 0$. Alors,

$$\iint_D g^2(x,y) dx\, dy > 0$$

et, en posant

$$\alpha = -\frac{\displaystyle\iint_D fg(x,y)\, dx\, dy}{\displaystyle\iint_D g^2(x,y)\, dx\, dy},$$

on obtient $F(\alpha) = 0$; ce qui entraîne que $f + \alpha g = 0$ ou encore que les deux fonctions f et g sont linéairement dépendantes.

4.61 *Rappel* : $\forall\, a,\, b \geq 0 : ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$ (ex. 5.233 vol. 1).

Puisque pour $f = 0$ ou $g = 0$ le résultat est évident, on fera l'hypothèse supplémentaire que $f \neq 0$ et $g \neq 0$. Alors, $I_D\big(|f|^p\big) > 0$ et $I_D\big(|g|^q\big) > 0$. Ainsi, puisque pour tout $(x, y) \in D$:

$$\frac{|fg|(x,y)}{\big(I_D\big(|f|^p\big)\big)^{\frac{1}{p}}\,\big(I_D\big(|g|^q\big)\big)^{\frac{1}{q}}} \leq \frac{\big|f(x,y)\big|^p}{p\,\big(I_D\big(|f|^p\big)\big)} + \frac{\big|g(x,y)\big|^q}{q\,\big(I_D\big(|g|^q\big)\big)}\,,$$

on a

$$\begin{aligned}
&\frac{\displaystyle\iint_D |fg|(x,y)\,dx\,dy}{\big(I_D\big(|f|^p\big)\big)^{\frac{1}{p}}\,\big(I_D\big(|g|^q\big)\big)^{\frac{1}{q}}}\\
&\qquad\leq \frac{\displaystyle\iint_D \big|f(x,y)\big|^p\,dx\,dy}{p\,\big(I_D\big(|f|^p\big)\big)} + \frac{\displaystyle\iint_D \big|g(x,y)\big|^q\,dx\,dy}{q\,\big(I_D\big(|g|^q\big)\big)} = \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}$$

ou encore $\displaystyle\iint_D |fg|(x,y)\,dx\,dy \leq \big(I_D\big(|f|^p\big)\big)^{\frac{1}{p}}\,\big(I_D\big(|g|^q\big)\big)^{\frac{1}{q}}$.

4.62 Puisque pour $p = 1$ ou $|f + g| = 0$ le résultat est évident on fera les hypothèses supplémentaires que $p > 1$ et $|f + g| \neq 0$. Ainsi, en constatant que pour $q = \frac{p}{p-1}$: $\frac{1}{p} + \frac{1}{q} = 1$ et

$$|f + g|^p = |f + g|\,|f + g|^{p-1} \leq |f|\,|f + g|^{p-1} + |g|\,|f + g|^{p-1}$$

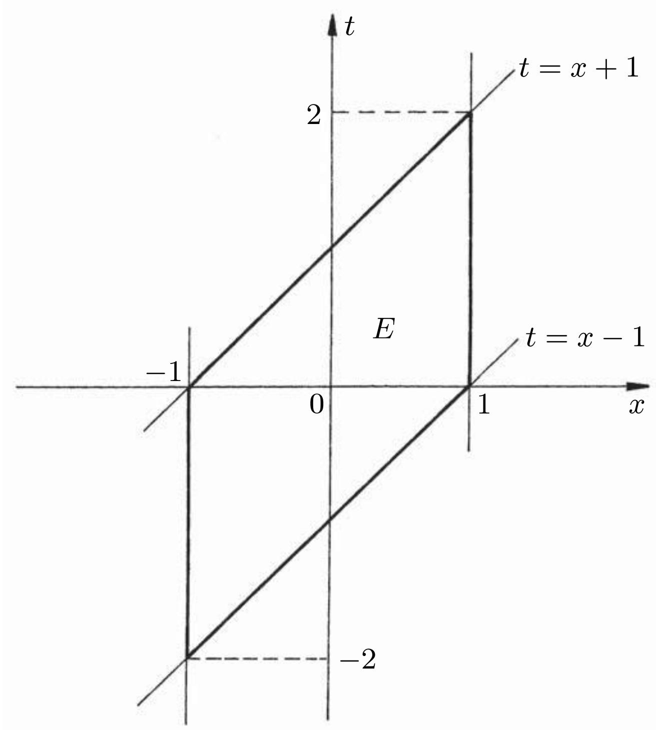
on peut écrire, en utilisant l'inégalité de Hölder (exercice précédent), que

$$\begin{aligned}
I_D\big(|f + g|^p\big) &\leq I_D\big(|f|\,|f + g|^{p-1}\big) + I_D\big(|g|\,|f + g|^{p-1}\big)\\
&\leq \big(I_D\big(|f|^p\big)\big)^{\frac{1}{p}}\,\big(I_D\big(|f + g|^p\big)\big)^{\frac{p-1}{p}} + \big(I_D\big(|g|^p\big)\big)^{\frac{1}{p}}\,\big(I_D\big(|f + g|^p\big)\big)^{\frac{p-1}{p}}
\end{aligned}$$

ou encore $\big(I_D\big(|f + g|^p\big)\big)^{\frac{1}{p}} \leq \big(I_D\big(|f|^p\big)\big)^{\frac{1}{p}} + \big(I_D\big(|g|^p\big)\big)^{\frac{1}{p}}$.

4.63 En effet, en posant $E = \big\{(x, t) \in \mathbb{R}^2 : |x| < 1,\ x - 1 < t < x + 1\big\}$, on obtient, puisque la fonction est paire, que

$$\begin{aligned}
\iint_D f(x - y)\,dx\,dy &= \int_{-1}^{1} dx \int_{-1}^{1} f(x - y)\,dy = \int_{-1}^{1} dx \int_{x-1}^{x+1} f(t)\,dt\\
&= \iint_E f(t)\,dx\,dt = \int_{-2}^{0} dt \int_{-1}^{1+t} f(t)\,dx + \int_{0}^{2} dt \int_{t-1}^{1} f(t)\,dx\\
&= \int_{-2}^{0} (2 + t) f(t)\,dt + \int_{0}^{2} (2 - t) f(t)\,dt = 2\int_{0}^{2} (2 - t) f(t)\,dt\,.
\end{aligned}$$



4.64 Soit $g : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}$ la fonction définie par

$$g(r,y) = \int_0^y f(r,s)ds\,.$$

Alors, pour tout $(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$:

$$\frac{\partial F}{\partial x}\,(x,y) = g(x,y) = \int_0^y f(x,s)\,ds \text{ et } \frac{\partial^2 F}{\partial x \partial y}\,(x,y) = f(x,y)\,.$$

De même, pour tout $(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$:

$$\frac{\partial F}{\partial y}\,(x,y) = \int_0^x \frac{\partial g}{\partial y}\,(r,y)\,dr = \int_0^x f(r,y)\,dr \text{ et } \frac{\partial^2 F}{\partial y \partial x}\,(x,y) = f(x,y)\,.$$

4.65 En faisant le changement de variable $t = 1+\rho^2$, on peut écrire que pour tout $k \in \mathbb{N}^*$:

$$\iint_{B(\mathbf{0},k)} \frac{\ln(1+x^2+y^2)}{\left(1+x^2+y^2\right)^2}\,dx\,dy = \int_0^{2\pi} d\theta \int_0^k \frac{\ln(1+\rho^2)}{\left(1+\rho^2\right)^2}\,\rho\,d\rho = \pi \int_1^{1+k^2} \frac{\ln t}{t^2}\,dt$$
$$= -\pi\left(\frac{\ln(1+k^2)}{1+k^2} + \frac{1}{1+k^2} - 1\right)\ ;$$

ce qui donne, par passage à la limite,

$$\iint_{\mathbb{R}^2} \frac{\ln(1+x^2+y^2)}{\left(1+x^2+y^2\right)^2}\,dx\,dy = \lim_{k\to+\infty} \iint_{B(\mathbf{0},k)} \frac{\ln(1+x^2+y^2)}{\left(1+x^2+y^2\right)^2}\,dx\,dy = \pi\,.$$

4.66 Posons $D_k =]-k,k[\times]-k,k[$ avec $k \in \mathbb{N}^*$. Ainsi, puisque pour tout $k > 0$:

$$\iint_{D_k} \frac{dx\,dy}{(1+x^2)(1+y^2)} = \left(\int_{-k}^{k} \frac{dt}{1+t^2}\right)^2 = 4\left(\int_0^k \frac{dt}{1+t^2}\right)^2 = 4\,\mathrm{Arctg}^2\,k\,,$$

on obtient, par passage à la limite,

$$\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+x^2)(1+y^2)} = 4 \lim_{k\to+\infty} \mathrm{Arctg}^2\,k = \pi^2\,.$$

4.67 Posons $D_k =]-k,k[\times]-k,k[$ avec $k \in \mathbb{N}^*$. Ainsi, puisque pour tout $k > 0$:

$$\iint_{D_k} \frac{dx\,dy}{(1+x^4)(1+y^4)} = \left(\int_{-k}^{k} \frac{dt}{1+t^4}\right)^2 = 4\left(\int_0^k \frac{dt}{1+t^4}\right)^2\,,$$

on obtient, par passage à la limite (ex. 6.81 vol. 1),

$$\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+x^4)(1+y^4)} = \lim_{k\to+\infty} \iint_{D_k} \frac{dx\,dy}{(1+x^4)(1+y^4)} = \frac{\pi^2}{2}\,.$$

4.68 Posons $D_k =]-k,k[\times]-k,k[$ avec $k \in \mathbb{N}^*$. Ainsi, puisque pour tout $k > 0$:

$$\iint_{D_k} \frac{e^{-x^2}}{1+y^2}\,dx\,dy = \int_{-k}^{k} dx \int_{-k}^{k} \frac{e^{-x^2}}{1+y^2}\,dy = 4\,\mathrm{Arctg}\,k \int_0^k e^{-x^2}\,dx\,,$$

on obtient, par passage à la limite (ex. 4.39),

$$\iint_{\mathbb{R}^2} \frac{e^{-x^2}}{1+y^2}\,dx\,dy = \lim_{k\to+\infty} \iint_{D_k} \frac{e^{-x^2}}{1+y^2}\,dx\,dy = \pi\sqrt{\pi}\,.$$

4.69 Puisque pour tout $(x,y) \in \mathbb{R}^2$: $\left|e^{-(x^2+y^2)}\cos(x^2+y^2)\right| \leq e^{-(x^2+y^2)}$ et

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \lim_{k\to+\infty} \int_0^{2\pi} d\theta \int_0^k e^{-\rho^2}\rho\,d\rho = \pi \lim_{k\to+\infty} \left(1 - e^{-k^2}\right) = \pi\,,$$

on a, d'après le critère de comparaison, que

$$\iint_{\mathbb{R}^2} \left|e^{-(x^2+y^2)}\cos(x^2+y^2)\right|\,dx\,dy < +\infty\,;$$

ce qui nous permet d'écrire, en constatant que pour tout $k \in \mathbb{N}^*$:

$$\iint_{B(\mathbf{0},k)} e^{-(x^2+y^2)} \cos(x^2+y^2)\, dx\, dy = \int_0^{2\pi} d\theta \int_0^k e^{-\rho^2} \cos \rho^2 \rho\, d\rho$$

$$= \pi \int_0^{k^2} e^{-t} \cos t\, dt = \frac{\pi}{2} e^{-t}(\sin t - \cos t)\Big|_0^{k^2} = \frac{\pi}{2}\left(e^{-k^2}(\sin k^2 - \cos k^2) + 1\right)$$

que

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \cos(x^2+y^2)\, dx\, dy$$

$$= \lim_{k \to +\infty} \iint_{B(\mathbf{0},k)} e^{-(x^2+y^2)} \cos(x^2+y^2)\, dx\, dy = \frac{\pi}{2}\,.$$

4.70 Soit $f : \mathbb{R}^2 \to \,]0, +\infty[$ la fonction définie par

$$f(x,y) = \frac{1+y^2}{(1+y^4)\left(1+\left(1+y^2\right)^2 x^2\right)}\,.$$

1) Posons $D_k = \,]-k,k[\, \times \,]-k,k[$ avec $k \in \mathbb{N}^*$. Ainsi, puisque pour tout $k > 0$:

$$\iint_{D_k} f(x,y) dx\, dy = \int_{-k}^{k} dy \int_{-k}^{k} \frac{1+y^2}{(1+y^4)\left(1+\left(1+y^2\right)^2 x^2\right)}\, dx$$

$$= 2\int_{-k}^{k} \frac{\operatorname{Arctg}\left(k(1+y^2)\right)}{1+y^4}\, dy < 2\pi \int_0^k \frac{dy}{1+y^4} < 2\pi \int_0^{+\infty} \frac{dy}{1+y^4}\,,$$

on a $\displaystyle\iint_{\mathbb{R}^2} f(x,y)\, dx\, dy < +\infty$.

2) Soient $\alpha > 0$ et $g_\alpha : \mathbb{R} \to \mathbb{R}$ la fonction définie par

$$g_\alpha(x) = \frac{1+\alpha^2}{1+x^2}\,.$$

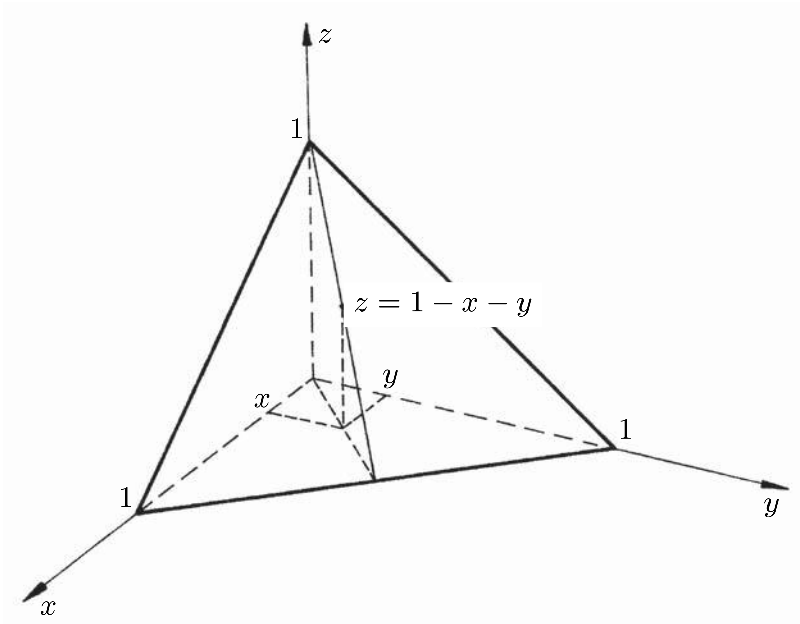
Alors, pour tout $|y| \leq \alpha$ et tout $x \in \mathbb{R}$: $0 < f(x,y) \leq g_\alpha(x,y)$ et $\displaystyle\int_{-\infty}^{+\infty} g_\alpha(x)\, dx$ $= \pi(1+\alpha^2)$. Ainsi, les hypothèses de la proposition 4.15 étant vérifiées, on peut écrire (ex. 6.81 vol. 1)

$$\iint_{\mathbb{R}^2} f(x,y)\, dx\, dy = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} f(x,y)\, dx$$

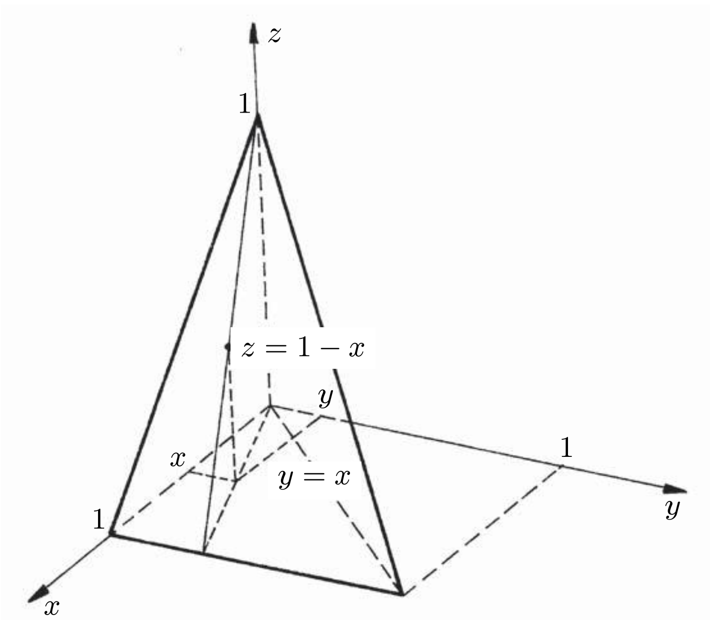
$$= \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} \frac{1+y^2}{(1+y^4)\left(1+\left(1+y^2\right)^2 x^2\right)}\, dx = 2\pi \int_0^{+\infty} \frac{dy}{1+y^4}$$

$$= \frac{\pi}{\sqrt{2}}\left(\frac{1}{2} \ln \frac{y^2+\sqrt{2}y+1}{y^2-\sqrt{2}y+1} + \operatorname{Arctg}\left(\sqrt{2}y-1\right) + \operatorname{Arctg}\left(\sqrt{2}y+1\right)\right)\Bigg|_0^{+\infty} = \frac{\pi^2}{\sqrt{2}}\,.$$

4.71 $$\iiint_D \frac{dx\,dy\,dz}{(x+y+z+1)^2} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(x+y+z+1)^2}$$
$$= \int_0^1 dx \int_0^{1-x} \left(-\frac{1}{2} + \frac{1}{1+x+y}\right) dy$$
$$= \int_0^1 \left(\frac{x-1}{2} + \ln 2 - \ln(1+x)\right) dx$$
$$= \left(\frac{(x-1)^2}{4} + x(1+\ln 2) - (x+1)\ln(1+x)\right)\Bigg|_0^1 = \frac{3}{4} - \ln 2\,.$$



4.72 $$\iiint_D (x^2+y^2)\,dx\,dy\,dz = \int_0^1 dx \int_0^x dy \int_0^{1-x} (x^2+y^2)\,dz$$
$$= \int_0^1 dx \int_0^x (1-x)(x^2+y^2)\,dy = \frac{4}{3}\int_0^1 (1-x)x^3\,dx = \frac{1}{15}\,.$$



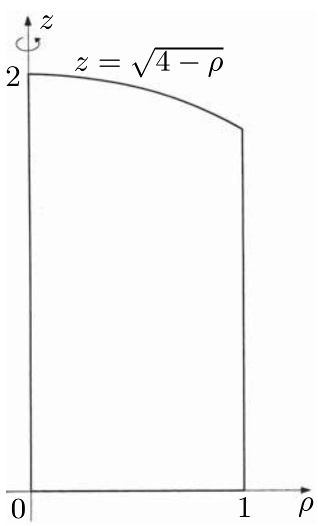
4.73 Puisque

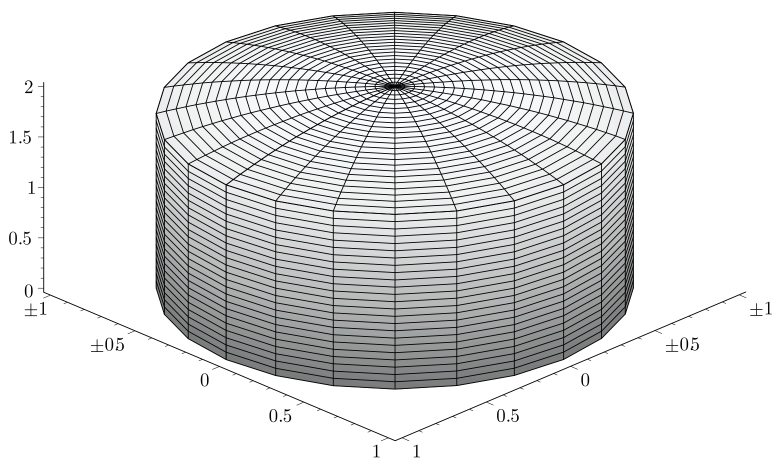
$$D = \big\{(\rho \sin\gamma \cos\theta, \rho \sin\gamma \sin\theta, \rho\cos\gamma) : 1 < \rho < 2\,,\ 0 \le \theta < 2\pi\,,\ 0 \le \gamma \le \pi\big\}\,,$$

on a $\displaystyle\iint_D \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}} = \int_0^{2\pi} d\theta \int_0^{\pi} d\gamma \int_1^2 \rho \sin\gamma \, d\rho = 6\pi.$

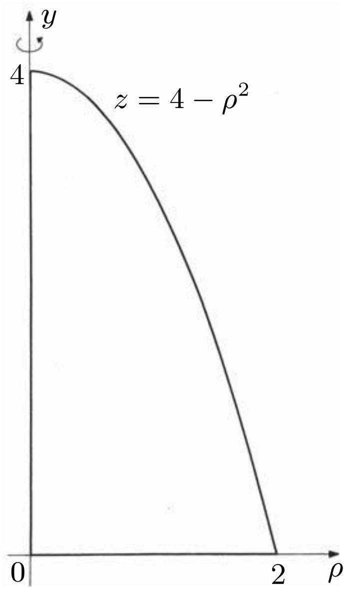
4.74 $$\iiint_D \frac{dx\,dy\,dz}{\sqrt{(x-2)^2+y^2+z^2}} = \int_{-1}^{1} dx \iint_{B(\mathbf{0},\sqrt{1-x^2})} \frac{dy\,dz}{\sqrt{(x-2)^2+y^2+z^2}}$$
$$= \int_{-1}^{1} dx \int_0^{2\pi} d\theta \int_0^{\sqrt{1-x^2}} \frac{\rho}{\sqrt{(x-2)^2+\rho^2}}\, d\rho$$
$$= 2\pi \int_{-1}^{1} \big(\sqrt{5-4x} + x - 2\big)\, dx$$
$$= 2\pi \left(-\frac{\sqrt{(5-4x)^3}}{6} + \frac{x^2}{2} - 2x\right)\Bigg|_{-1}^{1} = \frac{2\pi}{3}\,.$$

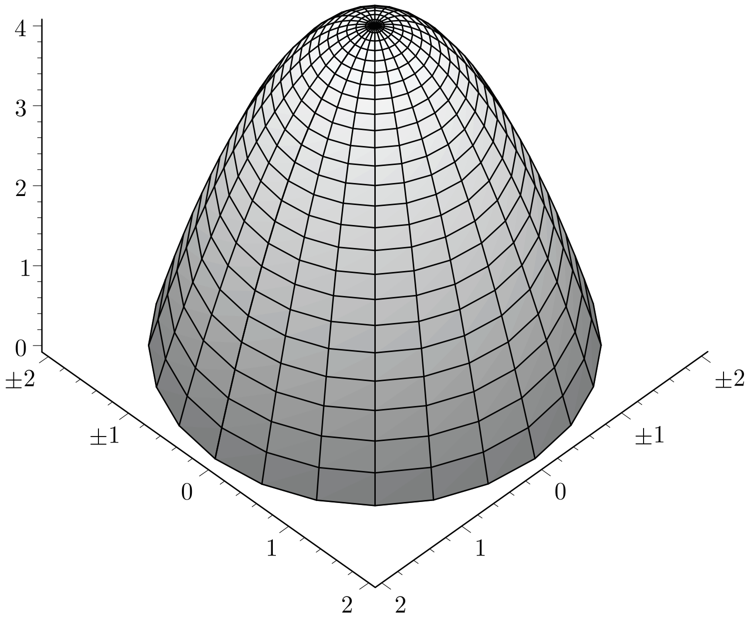
4.75 $$\iiint_D z\,dx\,dy\,dz = \iint_{B(\mathbf{0},1)} \left(\int_0^{\sqrt{4-x^2-y^2}} z\,dz\right) dx\,dy$$
$$= \frac{1}{2}\iint_{B(\mathbf{0},1)} (4 - x^2 - y^2)\,dx\,dy = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^1 (4-\rho^2)\rho\,d\rho = \frac{7\pi}{4}\,.$$



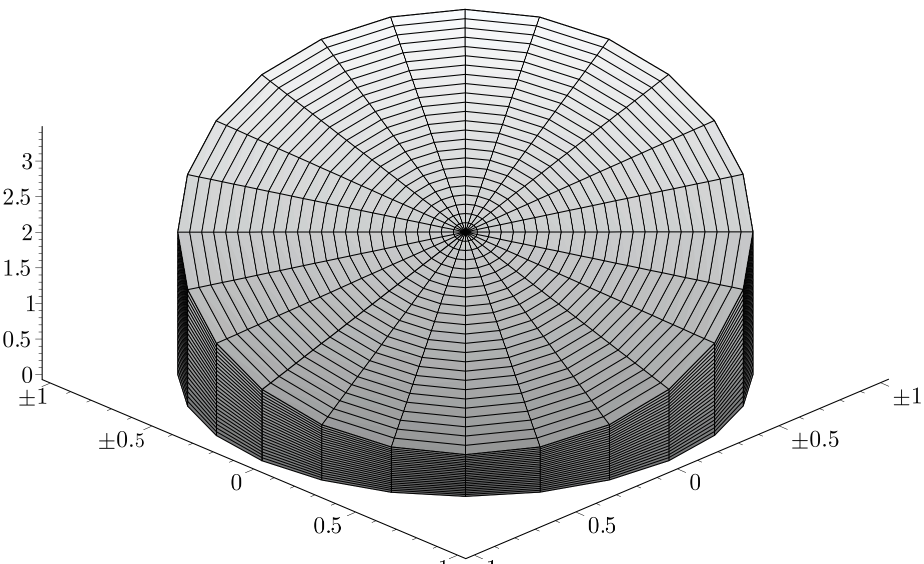


4.76 $$\iiint_D z\,dx\,dy\,dz = \iint_{B(\mathbf{0},2)} \left(\int_0^{4-x^2-y^2} z\,dz\right) dx\,dy$$
$$= \frac{1}{2}\iint_{B(\mathbf{0},2)} \left(4-x^2-y^2\right)^2 dx\,dy = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^2 \left(4-\rho^2\right)^2 \rho\,d\rho = \frac{32\,\pi}{3}\,.$$



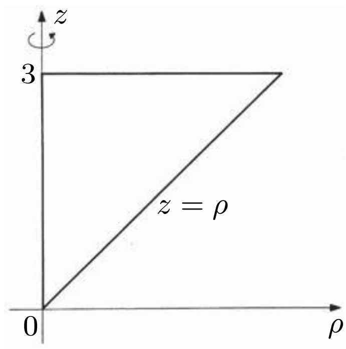


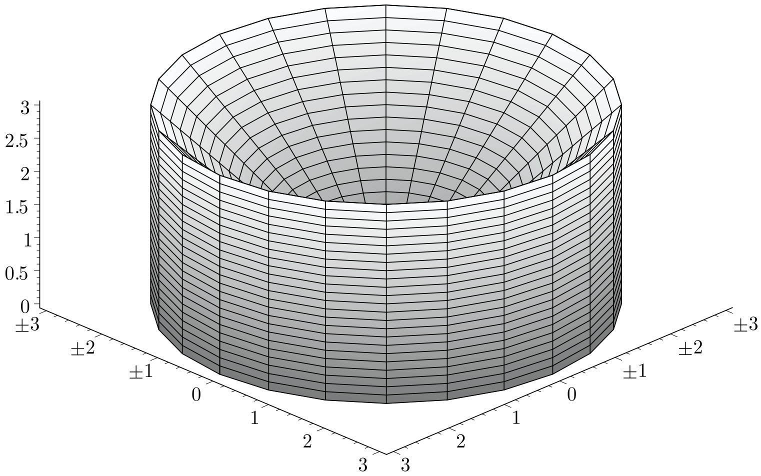
4.77 $$\iiint_D \sqrt{x^2+y^2}\,dx\,dy\,dz = \iint_{B(\mathbf{0},1)} \left(\int_0^{2-x-y} \sqrt{x^2+y^2}\,dz\right) dx\,dy$$
$$= \iint_{B(\mathbf{0},1)} (2-x-y)\sqrt{x^2+y^2}\,dx\,dy$$
$$= \int_0^{2\pi} d\theta \int_0^1 (2-\rho\cos\theta-\rho\sin\theta)\rho^2\,d\rho$$
$$= \int_0^{2\pi} \left(\frac{2}{3} - \frac{\cos\theta+\sin\theta}{4}\right) d\theta = \frac{4\pi}{3}\,.$$



4.78 $\displaystyle\iiint_D z(x^2+y^2)\,dx\,dy\,dz = \int_0^3 dz \iint_{B(\mathbf{0},z)} z(x^2+y^2)\,dx\,dy$

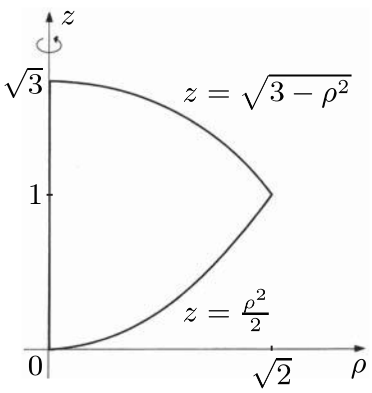
$$= \int_0^3 dz \int_0^{2\pi} d\theta \int_0^z z\rho^3\,d\rho = \frac{\pi}{2}\int_0^3 z^5 dz = \frac{243\,\pi}{4}\,.$$

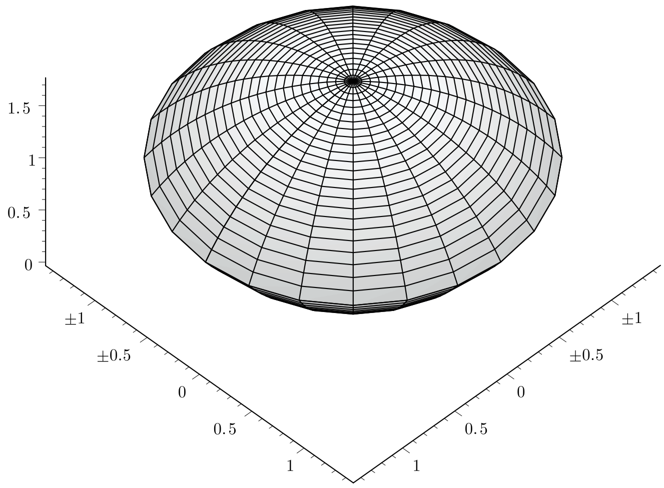




4.79 Puisque

$$D = \left\{(\rho\cos\theta, \rho\sin\theta, z) : 0 < \rho < \sqrt{2},\ 0 \le \theta < 2\pi\,,\ \frac{\rho^2}{2} < z < \sqrt{3-\rho^2}\right\},$$





on a

$$\iiint_D (x+y+z)^2\,dx\,dy\,dz = \int_0^{\sqrt{2}} d\rho \int_{\frac{\rho^2}{2}}^{\sqrt{3-\rho^2}} dz \int_0^{2\pi} (\rho\cos\theta+\rho\sin\theta+z)^2 \rho\,d\theta$$

$$= 2\pi \int_0^{\sqrt{2}} d\rho \int_{\frac{\rho^2}{2}}^{\sqrt{3-\rho^2}} \rho(\rho^2+z^2)\,dz$$

$$= 2\pi \int_0^{\sqrt{2}} \left(\left(\rho + \frac{2}{3}\rho^3\right)\sqrt{3-\rho^2} - \frac{\rho^5}{2} - \frac{\rho^7}{24}\right) d\rho$$

$$= 2\pi \left(\sqrt{(3-\rho^2)}\left(-\frac{9}{5} + \frac{\rho^2}{5} + \frac{2}{15}\rho^4\right) - \frac{\rho^6}{12} - \frac{\rho^8}{192}\right)\Bigg|_0^{\sqrt{2}} = \frac{2\pi}{5}\left(-\frac{97}{12} + 9\sqrt{3}\right).$$

4.80 Posons

$$E = \left\{ (\rho \cos\theta, \rho \sin\theta) : \sqrt{2} < \rho < 2\sqrt{2}\,(\cos\theta + \sin\theta)\,,\ 0 < \theta < \frac{\pi}{2} \right\}.$$

Alors

$$\iiint_D \frac{dx\,dy\,dz}{x^2+y^2} = \iint_E \frac{x+y}{x^2+y^2}\,dx\,dy = \int_0^{\frac{\pi}{2}} d\theta \int_{\sqrt{2}}^{2\sqrt{2}\,(\cos\theta+\sin\theta)} (\cos\theta + \sin\theta)\,d\rho$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \big(2(\cos\theta+\sin\theta)-1\big)(\cos\theta+\sin\theta)\,d\theta = \pi\sqrt{2}\,.$$