Tutorial n°3 Solution

Solution 1 *Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by:*

$$f(x, y) = x^4 + y^4 + (x - y)^2.$$

1. - *The first order partial derivatives are*

$$\frac{\partial f}{\partial x}(x, y) = 4x^3 + 2(x - y), \quad \frac{\partial f}{\partial y}(x, y) = 4y^3 - 2(x - y).$$

   - *The second-order partial derivatives are :*

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 12x^2 + 2, \quad \frac{\partial^2 f}{\partial y^2}(x, y) = 12y^2 + 2, \quad \frac{\partial^2 f}{\partial x \partial y}(x, y) = -2.$$

   By Schwartz's theorem (symmetry of mixed partial derivatives), we deduce:

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y) = -2.$$

2. *The third-Order Partial Derivatives are :*

$$\frac{\partial^3 f}{\partial x^3} = 24x, \quad \frac{\partial^3 f}{\partial y^3} = 24y.$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = 0, \quad , \frac{\partial^3 f}{\partial y \partial x^2} = 0, \quad \frac{\partial^3 f}{\partial x \partial y \partial x} = 0.$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = 0, \quad \frac{\partial^3 f}{\partial y^2 \partial x} = 0, \quad \frac{\partial^3 f}{\partial y \partial x \partial y} = 0.$$

Solution 2 *Let the function $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by:*

$$g(x, y) = \begin{cases} \dfrac{x^3 y - y^3 x}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

1. *Continuity of g : at any point $(x, y) \neq (0, 0)$, the function g is continuous because it's a quotient of two polynomials. At the point $(0, 0)$, we analyse the limit as $(x, y) \to (0, 0)$. Using polar coordinates $x = r\cos\theta, y = r\sin\theta$, we have:*

$$g(r\cos\theta, r\sin\theta) = \frac{r^3 \cos^3\theta \sin\theta - r^3 \sin^3\theta \cos\theta}{r^2} = r^2(\cos^3\theta \sin\theta - \sin^3\theta \cos\theta).$$

   As $r \to 0$, $g(r\cos\theta, r\sin\theta) \to 0$.

   Therefore:

$$\lim_{(x,y)\to(0,0)} g(x, y) = 0 = g(0, 0).$$

   Conclusion: g is continuous on $\mathbb{R}^2$.

2. *Compute* $\nabla g(x,y)$

   For $(x,y)\neq(0,0)$, *we have :*

$$\frac{\partial g}{\partial x}(x,y)=\frac{(3x^2y-y^3)(x^2+y^2)-2x(x^3y-y^3x)}{(x^2+y^2)^2},$$

   and

$$\frac{\partial g}{\partial y}(x,y)=\frac{(x^3-3xy^2)(x^2+y^2)-2y(x^3y-y^3x)}{(x^2+y^2)^2}.$$

   At $(0,0)$, *using the definition, we obtain :*

$$\frac{\partial g}{\partial x}(0,0)=\lim_{h\to0}\frac{g(h,0)-0}{h}=0,\quad \frac{\partial g}{\partial y}(0,0)=\lim_{h\to0}\frac{g(0,h)-0}{h}=0.$$

   Then

$$\frac{\partial f}{\partial x}(x,y)=\begin{cases}\dfrac{y(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0),\\ \lim\limits_{x\to0}\dfrac{f(x,0)-f(0,0)}{x}=0 & \text{if } (x,y)=(0,0),\end{cases}$$

$$\frac{\partial g}{\partial y}(x,y)=\begin{cases}\dfrac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0),\\ \lim\limits_{y\to0}\dfrac{g(0,y)-g(0,0)}{y}=0 & \text{if } (x,y)=(0,0).\end{cases}$$

3. *For any* $(x,y)\neq(0,0)$, g *admits all his second-order partial derivatives by direct computation (derivative of a fraction). At* $(0,0)$, *since first-order partial derivatives exist, we check second-order partial derivatives using limits :*

$$\frac{\partial^2 g}{\partial x^2}(x,y)=\begin{cases}\dfrac{4xy^3(x^2-3y^2)}{(x^2+y^2)^3} & \text{if } (x,y)\neq(0,0),\\ \lim\limits_{x\to0}\dfrac{\frac{\partial g}{\partial x}(x,0)-\frac{\partial g}{\partial x}(0,0)}{x}=0 & \text{if } (x,y)=(0,0),\end{cases}$$

$$\frac{\partial^2 g}{\partial x\partial y}(x,y)=\begin{cases}\dfrac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} & \text{if } (x,y)\neq(0,0),\\ \lim\limits_{x\to0}\dfrac{\frac{\partial g}{\partial y}(x,0)-\frac{\partial g}{\partial y}(0,0)}{x}=1 & \text{if } (x,y)=(0,0),\end{cases}$$

$$\frac{\partial^2 g}{\partial y\partial x}(x,y)=\begin{cases}\dfrac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} & \text{if } (x,y)\neq(0,0),\\ \lim\limits_{y\to0}\dfrac{\frac{\partial g}{\partial x}(0,y)-\frac{\partial g}{\partial x}(0,0)}{y}=-1 & \text{if } (x,y)=(0,0),\end{cases}$$

$$\frac{\partial^2 g}{\partial y^2}(x,y)=\begin{cases}\dfrac{4x^3y(3x^2-y^2)}{(x^2+y^2)^3} & \text{if } (x,y)\neq(0,0),\\ \lim\limits_{y\to0}\dfrac{\frac{\partial g}{\partial y}(0,y)-\frac{\partial g}{\partial y}(0,0)}{y}=0 & \text{if } (x,y)=(0,0).\end{cases}$$

4. *Since* $\frac{\partial^2 g}{\partial x\partial y}(0,0)\neq\frac{\partial^2 g}{\partial y\partial x}(0,0)$, *Shwarz theorem allows us to conclude that the mixed second-order partial derivatives are not continuous at the point* $(0,0)$.

Solution 3 *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of class $\mathbf{C}^2$ with respect to its two variables x and y.*

1) *Set $u = x - y$ and $v = x + y$. First, let's determine the expressions of x and y in terms of u and v. We have $x = \dfrac{u+v}{2}$ and $y = \dfrac{v-u}{2}$, so*
$\dfrac{\partial x}{\partial u} = 1/2$ *and* $\dfrac{\partial x}{\partial v} = 1/2$,
$\dfrac{\partial y}{\partial u} = -1/2$ *and* $\dfrac{\partial y}{\partial v} = 1/2$ *and we have:*

$$\frac{\partial f}{\partial u}(u,v) = \frac{\partial f}{\partial x}(x(u,v),y(u,v)) \times \frac{\partial x}{\partial u}(u,v) + \frac{\partial f}{\partial y}(x(u,v),y(u,v)) \times \frac{\partial y}{\partial u}(u,v),$$
$$\frac{\partial f}{\partial v}(u,v) = \frac{\partial f}{\partial x}(x(u,v),y(u,v)) \times \frac{\partial x}{\partial v}(u,v) + \frac{\partial f}{\partial y}(x(u,v),y(u,v)) \times \frac{\partial y}{\partial v}(u,v),$$

thus

$$\frac{\partial f}{\partial u}(u,v) = \frac{1}{2}\frac{\partial f}{\partial x}(x(u,v),y(u,v)) - \frac{1}{2}\frac{\partial f}{\partial y}(x(u,v),y(u,v)) \text{ ...}(*),$$
$$\frac{\partial f}{\partial v}(u,v) = \frac{1}{2}\frac{\partial f}{\partial x}(x(u,v),y(u,v)) + \frac{1}{2}\frac{\partial f}{\partial y}(x(u,v),y(u,v)) \text{ ...}(**),$$

$(*) + (**)$ *gives:*

$$\frac{\partial f}{\partial x}(x(u,v),y(u,v)) = \frac{\partial f}{\partial u}(u,v) + \frac{\partial f}{\partial v}(u,v) \text{ ...}(A),$$

and $(**)(*)$ *gives:*

$$\frac{\partial f}{\partial y}(x(u,v),y(u,v)) = \frac{\partial f}{\partial v}(u,v) - \frac{\partial f}{\partial u}(u,v) \text{ ...}(B),$$

now let's derivate (A) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}(x(u,v),y(u,v)) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x(u,v),y(u,v)) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}(u,v) + \frac{\partial f}{\partial v}(u,v)\right) \\ &= \left(\frac{\partial}{\partial u}(u,v) + \frac{\partial}{\partial v}(u,v)\right)\frac{\partial f}{\partial u}(u,v) + \left(\frac{\partial}{\partial u}(u,v) + \frac{\partial}{\partial v}(u,v)\right)\frac{\partial f}{\partial v}(u,v) \\ &= \frac{\partial^2 f}{\partial u^2}(u,v) + 2\frac{\partial^2 f}{\partial u \partial v}(u,v) + \frac{\partial^2 f}{\partial v^2}(u,v).\end{aligned}$$

Similarly, derivate (B) with respect to y we get:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2}(x(u,v),y(u,v)) &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x(u,v),y(u,v)) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}(u,v) - \frac{\partial f}{\partial u}(u,v)\right) \\ &= \left(\frac{\partial}{\partial v}(u,v) - \frac{\partial}{\partial u}(u,v)\right)\frac{\partial f}{\partial v}(u,v) - \left(\frac{\partial}{\partial v}(u,v) - \frac{\partial}{\partial u}(u,v)\right)\frac{\partial f}{\partial u}(u,v) \\ &= \frac{\partial^2 f}{\partial v^2}(u,v) - 2\frac{\partial^2 f}{\partial u \partial v}(u,v) + \frac{\partial^2 f}{\partial u^2}(u,v).\end{aligned}$$

Finally, we have

$$\frac{\partial^2 f}{\partial x^2}(x(u,v),y(u,v)) - \frac{\partial^2 f}{\partial y^2}(x(u,v),y(u,v)) = 4\frac{\partial^2 f}{\partial u \partial v}(u,v).$$

2) Now consider the polar coordinates r and θ where $x = r\cos\theta$ and $y = r\sin\theta$. Let's start with the first derivatives:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}, \ ...(4.1)$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}, \ ...(4.2).$$

Multiplying (4.1) by $\cos\theta$ *and (4.2) by* $\frac{\sin\theta}{r}$ *and adding the two results, we get:*

$$\frac{\partial f}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} \ ...(4.3),$$

on the other hand, multiplying (4.1) by $\sin\theta$ *and (4.2) by* $\frac{\cos\theta}{r}$ *and then adding the resulting equations, we obtain:*

$$\frac{\partial f}{\partial y} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta} \ ...(4.4).$$

Now, let's derivate both sides of (4.3) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\partial f}{\partial x}\right)\\ &= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\right)\\ &= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right)\\ &= \cos^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2} - \frac{\sin 2\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\sin 2\theta}{r^2}\frac{\partial f}{\partial\theta}.\end{aligned}$$

We do the same for (4.4), but derivate with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\partial f}{\partial y}\right)\\ &= \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right)\\ &= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}\right)\\ &= \sin^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\sin 2\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r} - \frac{\sin 2\theta}{r^2}\frac{\partial f}{\partial\theta}.\end{aligned}$$

The operator (called the Laplace operator) $\Delta = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ *can be written in terms of the partial derivatives of f with respect to r and θ as follows:*

$$\Delta = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}.$$

We can use also this method which is simpler than the previous one :

Laplacian in polar coordinates

Let f be C^2 over $\mathbb{R}^2$ and taking

$$u = f(x,y) = \tilde{u}(r,\theta), \quad r = \sqrt{x^2+y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right),$$

we have $f(x,y)=\tilde{f}(r\cos\theta, r\sin\theta)$*, where*

$$f(x,y)=\tilde{f}(r\cos\theta, r\sin\theta).$$

The first partial derivatives of f *are expressed by the following:*

$$\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}, \tag{1}$$

$$\frac{\partial}{\partial y}=\frac{\partial r}{\partial y}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}. \tag{2}$$

But:

$$r=\sqrt{x^2+y^2},\quad \theta=\arctan\left(\frac{y}{x}\right),$$

so:

$$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{x^2+y^2}},\quad \frac{\partial r}{\partial y}=\frac{y}{\sqrt{x^2+y^2}},$$

and:

$$\frac{\partial\theta}{\partial x}=\frac{-y}{x^2+y^2},\quad \frac{\partial\theta}{\partial y}=\frac{x}{x^2+y^2}.$$

Thus:

$$\frac{\partial f}{\partial x}=\frac{x}{r}\frac{\partial f}{\partial r}+\frac{\partial f}{\partial\theta}\left(-\frac{y}{r^2}\right), \tag{3}$$

$$\frac{\partial f}{\partial y}=\frac{y}{r}\frac{\partial f}{\partial r}+\frac{\partial f}{\partial\theta}\frac{x}{r^2}. \tag{4}$$

Also, the second derivatives of f *are given by derivation of (3) and (4) with respect to* x *and* y *respectively.*

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{x}{r}\frac{\partial f}{\partial r}+\left(-\frac{y}{r^2}\right)\frac{\partial f}{\partial\theta}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{x}{r}\right)\frac{\partial f}{\partial r}+\frac{x}{r}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial x}\left(-\frac{y}{r^2}\right)\frac{\partial f}{\partial\theta}+\left(-\frac{y}{r^2}\right)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial\theta}\right).$$

But

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)=\frac{y^2}{(x^2+y^2)^{3/2}},\quad \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right)=\frac{2xy}{(x^2+y^2)^2},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\right)=\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right)\frac{\partial r}{\partial x}+\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial r}\right)\frac{\partial\theta}{\partial x}=\frac{x}{r}\frac{\partial^2 f}{\partial r^2}-\frac{y}{r^2}\frac{\partial^2 f}{\partial r\partial\theta},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial\theta}\right)=\frac{x}{r}\frac{\partial^2 f}{\partial r\partial\theta}-\frac{y}{r^2}\frac{\partial^2 f}{\partial\theta^2}.$$

Thus

$$\frac{\partial^2 f}{\partial x^2}=\frac{y^2}{r^3}\frac{\partial f}{\partial r}+\frac{x^2}{r^2}\frac{\partial^2 f}{\partial r^2}-\frac{2xy}{r^3}\frac{\partial^2 f}{\partial r\partial\theta}+\frac{2xy}{r^4}\frac{\partial f}{\partial\theta}+\frac{y^2}{r^4}\frac{\partial^2 f}{\partial\theta^2} \tag{5}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{y}{r}\frac{\partial f}{\partial r}+\frac{x}{r^2}\frac{\partial f}{\partial\theta}\right),$$

$$= \frac{\partial}{\partial y}\left(\frac{y}{r}\right)\frac{\partial f}{\partial r} + \frac{y}{r}\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial y}\left(\frac{x}{r^2}\right)\frac{\partial f}{\partial \theta} + \frac{x}{r^2}\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{x^2}{r^3}\frac{\partial f}{\partial r} + \frac{y^2}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{2xy}{r^3}\frac{\partial^2 f}{\partial r \partial \theta} - \frac{2xy}{r^4}\frac{\partial f}{\partial \theta} + \frac{x^2}{r^4}\frac{\partial^2 f}{\partial \theta^2}$$

Similarly, for $\frac{\partial^2 f}{\partial y^2}$, we can obtain:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{y}{r}\frac{\partial f}{\partial r} + \frac{x}{r^2}\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{y}{r}\right)\frac{\partial f}{\partial r} + \frac{y}{r}\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial y}\left(\frac{x}{r^2}\right)\frac{\partial f}{\partial \theta} + \frac{x}{r^2}\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{x^2}{r^3}\frac{\partial f}{\partial r} + \frac{y^2}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{2xy}{r^3}\frac{\partial^2 f}{\partial r \partial \theta} - \frac{2xy}{r^4}\frac{\partial f}{\partial \theta} + \frac{x^2}{r^4}\frac{\partial^2 f}{\partial \theta^2}.$$

We get:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2}\frac{y^2}{r^2} + \frac{2}{r}\frac{\partial f}{\partial r}\frac{x^2}{r^2} + \frac{\partial^2 f}{\partial \theta^2}\frac{x^2}{r^4} - \frac{2x}{r^3}\frac{\partial f}{\partial \theta}, \tag{6}$$

The sum of (5) and (6) leads to:

$$\Delta f(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}$$

$$= \frac{\partial^2 f}{\partial r^2}\left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right) + \frac{\partial f}{\partial r}\left(\frac{2}{r}\frac{x^2}{r^2} + \frac{2}{r}\frac{y^2}{r^2}\right) + \frac{\partial^2 f}{\partial \theta^2}\left(\frac{y^2}{r^4} + \frac{x^2}{r^4}\right) + \frac{\partial f}{\partial \theta}\left(\frac{-2y}{r^3} + \frac{2x}{r^3}\right).$$

Thus, we finally get:

$$\boxed{\Delta f(x,y) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}}$$

Solution 4 *We consider the function: $f(x,y) = \sin(x+y)$, to write it's second-order Taylor expansion at the point $(x_0, y_0) = \left(0, \frac{\pi}{4}\right)$, we compute all it's partial derivatives of the first and second-order at this point :*

$$f(0,\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \frac{\partial f}{\partial x}(0,\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \frac{\partial f}{\partial y}(0,\frac{\pi}{4}) = \frac{\sqrt{2}}{2},$$

$$\frac{\partial^2 f}{\partial x^2}(0,\frac{\pi}{4}) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}, \quad \frac{\partial^2 f}{\partial y^2}(0,\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}, \quad \frac{\partial^2 f}{\partial xy}(0,\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}.$$

So

$$f(x,y) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left[x^2 + \left(y - \frac{\pi}{4}\right)^2 + 2x\left(y - \frac{\pi}{4}\right)\right] + \|(x,(y-\frac{\pi}{4}))\|_2^2\,\epsilon(x,(y-\frac{\pi}{4})).$$

Solution 5 *Let $g(x,y) = (x - \sin y)\ln(1 + 2x + 3y)$.*

1. *We use the Taylor expansions:*

$$\sin y = y - \frac{y^3}{6} + o(y^3), \quad \ln(1+u) = u - \frac{u^2}{2} + \epsilon(u^2),$$

where $u = 2x + 3y$. *Thus:*

$$x - \sin y = x - \left(y - \frac{y^3}{6}\right) = x - y + \frac{y^3}{6} + \epsilon(y^3),$$

and

$$\ln(1 + 2x + 3y) = 2x + 3y - \frac{(2x+3y)^2}{2} + \epsilon(x^2, xy, y^2).$$

Multiplying (keeping only terms up to second order):

$$(x-y)(2x+3y) = 2x^2 + 3xy - 2xy - 3y^2 = 2x^2 + xy - 3y^2,$$

and

$$-\frac{1}{2}(x-y)(2x+3y)^2$$

contributes only cubic or higher terms, so we ignore it.

Therefore, the second-order expansion is:

$$g(x,y) = 2x^2 + xy - 3y^2 + \|(x,y)\|_2^2\epsilon(x,y).$$

2. *From the general second-order Taylor formula:*

$$g(x,y) = g(0,0) + \frac{\partial g}{\partial x}(0,0)x + \frac{\partial g}{\partial y}(0,0)y + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(0,0)x^2 + \frac{\partial^2 g}{\partial xy}(0,0)xy + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(0,0)y^2 + \|(x,y)\|_2^2\epsilon(x,y),$$

we match:
so, there is no term involving x *or* y, *we can deduce that* $\frac{\partial g}{\partial x}(0,0) = 0$, *and* $\frac{\partial g}{\partial y}(0,0) = 0$, *also, we have*

$$\begin{aligned} 2x^2 &\implies \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(0,0) = 2 \implies \frac{\partial^2 g}{\partial x^2}(0,0) = 4, \\ xy &\implies \frac{\partial^2 g}{\partial xy}(0,0) = 1, \\ -3y^2 &\implies \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(0,0) = -3 \implies \frac{\partial^2 g}{\partial y^2}(0,0) = -6. \end{aligned}$$

Solution 6 *Let* $f(x,y) = \dfrac{1+x+y}{1+x-y}$ *be a function defined on* $D = \{(x,y) \in \mathbb{R}^2 : 0 \neq 1 + x - y\}$

1) *The Taylor expansion of the function* $1/(1+u)$ *of the order* 2 *is given by :*

$$\frac{1}{1+u} = 1 - u + u^2 + (u)^2\epsilon(u,)$$

If we take $t = (x-y)$, *so* $\frac{1}{1+(x-y)} = 1 - (x-y) + (x-y)^2 + (x-y)^2\epsilon(x-y)$; *so we have :*

$$\begin{aligned} f(x,y) &= (1+x+y)[1 - (x-y) + (x-y)^2 + (x-y)^2\epsilon(x-y)] \\ &= 1 + 2y - 2xy + 2y^2 + (x^2+y^2)\epsilon(x,y) \end{aligned}$$

.

2) By matching with theoretical formula, we get :

$$\frac{\partial f}{\partial x}(0,0)=0,\ \frac{\partial f}{\partial y}(0,0)=2,\ \frac{\partial^2 f}{\partial x^2}(0,0)=0,\ \frac{\partial^2 f}{\partial x\partial y}(0,0)=-2,\ \frac{\partial^2 f}{\partial y^2}(0,0)=4.$$

Solution 7 *1) The function f is of class $\mathcal{C}^2$ on $D/\{(x,y)\in\mathbb{R}^2 : 3x-y=0\}$, and for all $(x,y)\in D$ such that $3x-y\neq 0$, we have:*

$$\frac{\partial f}{\partial x}(x,y)=\frac{3}{2\sqrt{3x-y}},\quad \frac{\partial f}{\partial y}(x,y)=\frac{-1}{2\sqrt{3x-y}},$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y)=\frac{3}{4(3x-y)\sqrt{3x-y}}=\frac{\partial^2 f}{\partial y\partial x}(x,y),$$

(this is due to Schwarz's theorem since f is of class $\mathcal{C}^2$),

$$\frac{\partial^2 f}{\partial x^2}(x,y)=\frac{-9}{4(3x-y)\sqrt{3x-y}},\quad \frac{\partial^2 f}{\partial y^2}(x,y)=\frac{-1}{4(3x-y)\sqrt{3x-y}}.$$

2) The first-order Taylor expansion of f near the point $(1,2)$:

$$f(x,y)=f(1,2)+(x-1)\frac{\partial f}{\partial x}(1,2)+(y-2)\frac{\partial f}{\partial y}(1,2)+\|((x-1),(y-2))\|\,\epsilon((x-1),(y-2)),$$

with

$$\lim_{(x,y)\to(1,2)}\epsilon((x-1),(y-2))=0,$$

thus

$$f(x,y)=1+\frac{3}{2}(x-1)-\frac{1}{2}(y-2)+\|((x-1),(y-2))\|\,\epsilon((x-1),(y-2)).$$

3) The second-order Taylor expansion of f near the point $(1,2)$ is written as:

$$\begin{aligned}f(x,y)=&f(1,2)+(x-1)\frac{\partial f}{\partial x}(1,2)+(y-2)\frac{\partial f}{\partial y}(1,2)\\&+\frac{1}{2}\left((x-1)^2\frac{\partial^2 f}{\partial x^2}(1,2)+(y-2)^2\frac{\partial^2 f}{\partial y^2}(1,2)+2(x-1)(y-2)\frac{\partial^2 f}{\partial x\partial y}(1,2)\right)\\&+\|((x-1),(y-2))\|_2^2\,\epsilon((x-1),(y-2)),\end{aligned}$$

with

$$\lim_{(x,y)\to(1,2)}\epsilon((x-1),(y-2))=0,$$

thus

$$\begin{aligned}f(x,y)=1&+\frac{3}{2}(x-1)-\frac{1}{2}(y-2)-\frac{9}{4}(x-1)^2-\frac{1}{4}(y-2)^2\\&+\frac{3}{4}(x-1)(y-2)+\|((x-1),(y-2))\|_2^2\,\epsilon((x-1),(y-2)).\end{aligned}$$

4) By definition, the differential of f exists at the point $(2,2)$ if

$$\lim_{(h,k)\to 0}\frac{f(2+h,2+k)-f(2,2)-h\cdot\frac{\partial f}{\partial x}(2,2)-k\cdot\frac{\partial f}{\partial y}(2,2)}{\sqrt{h^2+k^2}}=0,$$

we have:

$$f(2+h,2+k)=\sqrt{4-3h-k},\quad f(2,2)=2,\quad \frac{\partial f}{\partial x}(2,2)=\frac{3}{4},\quad \frac{\partial f}{\partial y}(2,2)=-\frac{1}{4},$$

so the above limit becomes:

$$\lim_{(h,k)\to 0}\frac{\sqrt{4-3h-k}-2-\frac{3h}{4}+\frac{k}{4}}{\sqrt{h^2+k^2}},$$

multiplying by the conjugate $\sqrt{4-3h-k}+\left(2+\frac{3h}{4}-\frac{k}{4}\right)$ *(then passing to polar coordinates), we obtain:*

$$\lim_{(h,k)\to 0}\frac{-\frac{9h^2}{16}-\frac{k^2}{16}+\frac{3}{8}hk}{\left(\sqrt{4-3h-k}+\left(2+\frac{3h}{4}-\frac{k}{4}\right)\right)\sqrt{h^2+k^2}}=0,$$

therefore f *is totally differentiable at the point* $(2,2)$ *and it's differential is giving by :*

$$Df_{(2,2)}(h,k)=h\cdot\frac{\partial f}{\partial x}(2,2)+k\cdot\frac{\partial f}{\partial y}(2,2).$$

5) The equation of the tangent plane at the point $(2,2)$ *is:*

$$z=f(2,2)+(x-2)\frac{\partial f}{\partial x}(2,2)+(y-2)\frac{\partial f}{\partial y}(2,2)=2+\frac{3}{4}(x-2)-\frac{1}{4}(y-2).$$

6) From question 5) and since $(2.01,1.99)$ *is in the neighbourhood of* $(2,2)$*, we deduce that the approximate value of* $f(2.01,1.99)$ *is:*

$$f(2.01,1.99)\simeq 2+\frac{3}{4}(2.01-2)-\frac{1}{4}(1.99-2)\simeq 2.01.$$

The exact value obtained directly from the expression of f *is* $f(2.01,1.99)=\sqrt{4.2.01-1.99}=2.009975$*, so the difference between the approximate and exact values is nearly zero (*$2.01-2.009975=25\times 10^{-6}$*).*