Correction TD n° 1.

Sol. Exo 1.

a) $\mathcal{L}[f_1(t)] = \int_0^{+\infty} a e^{-pt} dt = a\left(-\frac{1}{p} e^{-pt}\Big|_0^{+\infty}\right) = \frac{a}{p}$ (où $a = cte$)

b) $\mathcal{L}[f_2(t)] = \mathcal{L}[at] = \int_0^{+\infty} at\, e^{-pt} dt$ on remarque que

$f_2'(t) = f_1(t) \Rightarrow \mathcal{L}[f_2'(t)] = \mathcal{L}[f_1(t)]$ — ①

Selon le Théorème de dérivation : $\mathcal{L}[f_2'(t)] = p\mathcal{L}[f_2(t)] - \cancel{f_2(0)}^{\,0}$ — ②

① = ② $\Rightarrow p\mathcal{L}[f_2(t)] = \mathcal{L}[f_1(t)] \Rightarrow \boxed{\mathcal{L}[f_2(t)] = \frac{a}{p^2}}$

c) $\mathcal{L}[f_3(t)] = \mathcal{L}[at^2]$ de la même manière que (b) on a :

$f_3'(t) = 2at = 2f_2(t) \Rightarrow \mathcal{L}[f_3'(t)] = 2\mathcal{L}[f_2(t)]$

$\Rightarrow p\mathcal{L}[f_3(t)] - \cancel{f_3(0)}^{\,0} = 2\mathcal{L}[f_2(t)] \Rightarrow \boxed{\mathcal{L}[f_3(t)] = \frac{2a}{p^3}}$

d) $\mathcal{L}[f_5(t)] = \mathcal{L}[at^n] = a\frac{n!}{p^{n+1}}$

e) $\mathcal{L}[f_4(t)] = \mathcal{L}[\delta(t)] = ?$ on a $\delta(t) = \frac{du(t)}{dt}$ avec $u(0) = 0$.

$\Rightarrow \mathcal{L}[\delta(t)] = \mathcal{L}\left[\frac{du(t)}{dt}\right] = p.\mathcal{L}[u(t)] = p.\frac{1}{p} = 1.$

f) $\mathcal{L}[f_6(t)] = \mathcal{L}[3e^{-t} + e^{-3t}] = 2\mathcal{L}[e^{-t}] + \mathcal{L}[e^{-3t}] = 2\frac{1}{p+1} + \frac{1}{p+3}$

g) $\mathcal{L}[f_7(t)] = ?$ on selon Euler : $e^{j\omega t} = \cos\omega t + j\sin\omega t$

alors $\mathcal{L}[e^{j\omega t}] = \mathcal{L}[\cos\omega t] + j\mathcal{L}[\sin\omega t]$ — ①

$\mathcal{L}[e^{j\omega t}] = \int_0^{+\infty} e^{j\omega t}. e^{-pt} dt = \int_0^{+\infty} e^{-(p-j\omega)t} dt = -\frac{1}{p-j\omega} e^{-(p-j\omega)t}\Big|_0^{+\infty}$

$= \frac{1}{(p-j\omega)} \times \frac{(p+j\omega)}{(p+j\omega)} = \frac{p}{p^2+\omega^2} + j\frac{\omega}{p^2+\omega^2}$ — ②

① = ② $\Rightarrow \mathcal{L}[\cos\omega t] = \frac{p}{p^2+\omega^2}$ et $\mathcal{L}[\sin\omega t] = \frac{\omega}{p^2+\omega^2}$

d'où $\mathcal{L}[f_7(t)] = \frac{\omega}{p^2+\omega^2} + \frac{p}{p^2+\omega^2} = \frac{\omega+p}{p^2+\omega^2}$.

h) $\mathcal{L}[f_8(t)] = \mathcal{L}[e^{-at}\sin\omega t]$ Selon le théorème de décalage fréquentielle on a : $\mathcal{L}[f(t).e^{-\alpha t}] = F(p+\alpha)$ où $F(p) = \mathcal{L}[f(t)]$

donc $\mathcal{L}[e^{-at}\sin\omega t] = F(p+a)$ où $F(p) = \mathcal{L}[\sin\omega t] = \frac{\omega}{p^2+\omega^2}$

alors $\mathcal{L}[f_8(t)] = \mathcal{L}[e^{-at}\sin\omega t] = \frac{\omega}{(p+a)^2+\omega^2}$.

i) $\mathcal{L}[f_9(t)] = \mathcal{L}[t^2 e^{-at}]$ selon le Th. de décalage fréquentielle

$$\mathcal{L}[t^2 e^{-at}] = F(p+a) \quad \text{où} \quad F(p) = \mathcal{L}[t^2] = \frac{2}{p^3}$$

d'où $\mathcal{L}[f_9(t)] = \dfrac{2}{(p+a)^3}$.

j) $\mathcal{L}[f_{10}(t)] = \mathcal{L}\left[e^{-3t}\sin(5t+\pi/3)\right] = F(p+3)$ où $F(p) = \mathcal{L}[\sin(\omega t+\pi/3)]$

$$\mathcal{L}[\sin(5t+\pi/3)] = \mathcal{L}\left[\sin(5t)\cdot\cos(\pi/3) + \sin(\pi/3)\cdot\cos(5t)\right]$$

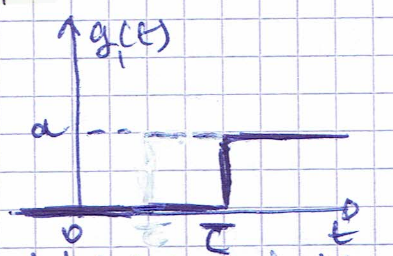
$$\cos(\pi/3)\,\mathcal{L}[\sin 5t] + \sin\pi/3\,\mathcal{L}[\cos 5t] = \cos\pi/3\left(\frac{5}{p^2+25}\right) + \sin\pi/3\left(\frac{p}{p^2+25}\right)$$

$$= \frac{5\cos\pi/3 + p\sin\pi/3}{p^2+25} \quad \text{avec} \begin{cases}\cos\pi/3 = 1/2\\ \sin\pi/3 = \sqrt{3}/2\end{cases}$$

d'où $\mathcal{L}\left[e^{-3t}\sin(5t+\pi/3)\right] = \dfrac{5\cos\pi/3 + (p+3)\sin\pi/3}{(p+3)^2+25}$

Sol. Ex 2.

a)

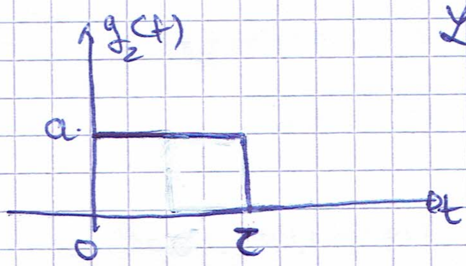


fct échelon retardée.

$$\mathcal{L}[g_1(t)] = \int_0^{+\infty} g_1(t)\,e^{-pt}\,dt = \int_\tau^{+\infty} a e^{-pt}\,dt$$

$$= -\frac{a}{p}e^{-pt}\Big|_\tau^{+\infty} = \frac{a}{p}e^{-\tau p} \quad (a = \text{cte} > 0)$$

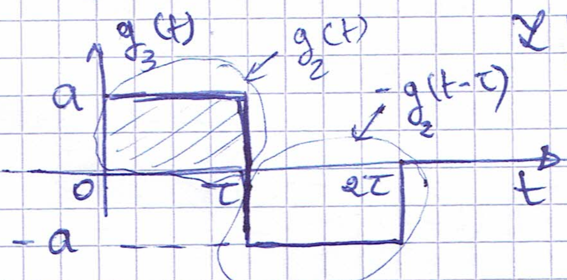
b)



$$\mathcal{L}[g_2(t)] = \int_0^{+\infty} g_2(t)\,e^{-pt}\,dt = \int_0^{\tau} a e^{-pt}\,dt$$

$$= -\frac{a}{p}e^{-pt}\Big|_0^{\tau} = \frac{a}{p}\left(1 - e^{-\tau p}\right)$$

c)

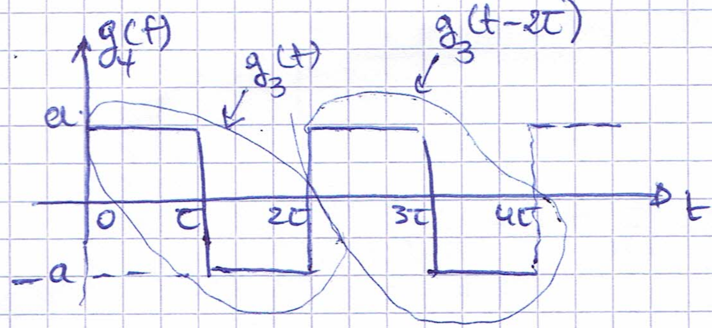


$$\mathcal{L}[g_3(t)] = \mathcal{L}[g_2(t)] - \mathcal{L}[g_2(t-\tau)]$$

selon le Th. de retard.

$$\mathcal{L}[g_2(t-\tau)] = e^{-\tau p}\cdot\mathcal{L}[g_2(t)]$$

alors $\mathcal{L}[g_3(t)] = \mathcal{L}[g_2(t)]\left(1 - e^{-\tau p}\right) = \dfrac{a}{p}\left(1 - e^{-\tau p}\right)^2$.



$$g_4(t) = g_3(t) + g_3(t-2\tau) + g_3(t-4\tau) + \cdots + g_3(t-2n\tau) + \cdots$$

$$\mathcal{L}[g_4(t)] = \mathcal{L}[g_3(t)]\left(1 + e^{-2\tau p} + e^{-4\tau p} + \cdots + e^{-2n\tau p} + \cdots\right)$$

$$= \mathcal{L}[g_3(t)]\left(\lim_{n\to\infty} \frac{1-r^n}{1-r}\right) = \mathcal{L}[g_3(t)]\left(\frac{1-\cancel{e^{-2n\tau p}}^{\,0}}{1-e^{-2\tau p}}\right) \quad \text{où } r = e^{-2\tau p}$$

$$= \frac{a}{p}\,\frac{(1-e^{-\tau p})^2}{(1-e^{-2\tau p})} = \frac{a}{p}\,\frac{(1-e^{-\tau p})}{(1+e^{-\tau p})}$$

de la même manière on calcule la T.L de $g_5(t)$.

$$\mathcal{L}[g_5(t)] = \mathcal{L}[\delta(t)] + \mathcal{L}[\delta(t-\tau)] + \mathcal{L}[\delta(t-2\tau)] + \cdots$$

$$= \mathcal{L}[\delta(t)]\left(1 + e^{-\tau p} + e^{-2\tau p} + \cdots + e^{-n\tau p} + \cdots\right)$$

$$= \lim_{n\to+\infty} \frac{1-r^n}{1-r} = \frac{1-\cancel{e^{-n\tau p}}^{\,0}}{1-e^{-\tau p}} = \frac{1}{1-e^{-\tau p}}$$

$\left\{ \text{où } r = e^{-\tau p},\ n \to +\infty \right.$

<u>Sol. Exo 3.</u>

a) $$\mathcal{L}^{-1}[F_1(p)] = \mathcal{L}^{-1}\left[\frac{1}{(p+3)(p+4)}\right] = \mathcal{L}^{-1}\left[\frac{a_1}{p+3}\right] + \mathcal{L}^{-1}\left[\frac{a_2}{p+4}\right]$$

où $a_1 = (p+3)F_1(p)\big|_{p=-3} = \frac{1}{p+4}\Big|_{p=-3} = 1.$

$a_2 = (p+4)F_1(p)\big|_{p=-4} = -1.$

d'où $\boxed{\mathcal{L}^{-1}[F_1(p)] = (e^{-3t} - e^{-4t}).u(t)}$

b) $$\mathcal{L}^{-1}[F_2(p)] = \mathcal{L}^{-1}\left[\frac{p+2}{(p+1)^2(p+3)}\right] = \mathcal{L}^{-1}\left[\frac{b_2}{(p+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{b_1}{p+1}\right] + \mathcal{L}^{-1}\left[\frac{a}{p+3}\right]$$

où $b_2 = (p+1)^2 \cdot F_2(p)\big|_{p=-1} = 1.$

$b_1 = \frac{d}{dp}\left[(p+1)^2 F_2(p)\right]_{p=-1} = \frac{d}{dp}\left[\frac{p+2}{p+3}\right]_{p=-1} = +1.$

$a = (p+3)F_2(p)\big|_{p=-3} = -1.$

d'où $\boxed{\mathcal{L}^{-1}[F_2(p)] = (te^{-t} + e^{-t} - e^{-3t}).u(t)}$

c) $\mathcal{L}^{-1}[F_3(p)] = \mathcal{L}^{-1}\left[\frac{p+1}{p^2+4p+16}\right] = \mathcal{L}^{-1}\left[\frac{p+1+1-1}{(p+2)^2+12}\right]$

$= \mathcal{L}^{-1}\left[\frac{p+2}{(p+2)^2+(\sqrt{12})^2}\right] - \frac{1}{\sqrt{12}}\mathcal{L}^{-1}\left[\frac{\sqrt{12}}{(p+2)^2+(\sqrt{12})^2}\right]$

$= e^{-2t}\cdot\cos\sqrt{12}\,t - \frac{1}{\sqrt{12}}e^{-2t}\sin\sqrt{12}\,t$, pour $t \geq 0$.

d) $\mathcal{L}^{-1}[F_5(p)] = \mathcal{L}^{-1}\left[\frac{4p^3+p^2-22p+16}{p(p+2)(p-2)^2}\right] = \mathcal{L}^{-1}\left[\frac{a_1}{p}\right] + \mathcal{L}^{-1}\left[\frac{a_2}{p+2}\right] + \mathcal{L}^{-1}\left[\frac{b_2}{(p-2)^2}\right]$

$+ \mathcal{L}^{-1}\left[\frac{b_1}{p-2}\right]$

où $a_1 = pF(p)\big|_{p=0} = \frac{16}{8} = 2$; $a_2 = (p+2)F_5(p)\big|_{p=-2} = -1$.

$b_2 = (p-2)^2F_5(p)\big|_{p=+2} = 1$; $b_1 = \frac{d}{dp}\left[\frac{4p^3+p^2-22p+16}{p(p+2)}\right]_{p=+2} = b_1 = 2$

d'où $\boxed{f_5(t) = 2u(t) - e^{-2t}u(t) + te^{-2t}.u(t) + b_1e^{-2t}.u(t)}$.

Sol. Exp 4.

1) $\underbrace{\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = 0}$ avec $x(0)=0$, $\dot{x}(0)=2$.

$\downarrow$ T.Laplace

$p^2X(p) - p\underset{0}{\cancel{x(0)}} - \underset{2}{\cancel{\dot{x}(0)}} + 3(pX(p) - \underset{0}{\cancel{x(0)}}) + 2X(p) = 0.$

$\Rightarrow p^2X(p) + 3pX(p) + 2X(p) = 2 \Rightarrow X(p) = \frac{2}{p^2+3p+2} = \frac{2}{(p+1)(p+2)}$

$\Rightarrow X(p) = \frac{a_1}{p+1} + \frac{a_2}{p+2}$ où $\begin{cases} a_1 = (p+1)X(p)\big|_{p=-1} = 2 \\ a_2 = (p+2)X(p)\big|_{p=-2} = -2 \end{cases}$

d'où $x(t) = \mathcal{L}^{-1}\left[\frac{2}{p+1}\right] + \mathcal{L}^{-1}\left[\frac{-2}{p+2}\right] = (2e^{-t} - 2e^{-2t}).u(t)$.

2) $\underbrace{\dddot{x}(t) - \ddot{x}(t) = 0}$ avec $x(0)=2$, $\dot{x}(0)=0$ et $\ddot{x}(0)=1$

$\downarrow$ T.Laplace

$p^3X(p) - p^2\underset{2}{\cancel{x(0)}} - p\underset{0}{\cancel{\dot{x}(0)}} - \underset{1}{\cancel{\ddot{x}(0)}} - p^2X(p) - p\underset{2}{\cancel{x(0)}} - \underset{0}{\cancel{\dot{x}(0)}} = 0$.

$\Rightarrow X(p) = \frac{2p^2+2p+1}{p^3-p^2} = \frac{2p^2+2p+1}{p^2(p-1)} = \frac{b_2}{p^2} + \frac{b_1}{p} + \frac{a}{p-1}$