First series of tutorials in analysis 3

Numerical Series

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In this series of tutorials, we will explore the fundamental concepts of Numerical Series. Each exercise is designed to reinforce your understanding of the topics discussed in class.

Exercise 1

Consider an arithmetic sequence defined by an initial term $u_1 = 5$ and a common difference of $r = 3$.

1. Derive a formula for the general term u_n of the sequence.
2. Determine the sum of the first 10 terms of the sequence.

Exercise 2

Let a geometric sequence be defined by $v_3 = 2$ and a common ratio $q = 0.5$.

1. Find the expression for the general term v_n.
2. Calculate the sum of the first 8 terms of the sequence.

Exercise 3

Analyze the convergence and properties of the following series:

1. $\sum(\exp\frac{1}{n} - \exp\frac{1}{n+1})$.
2. $\sum \frac{x^n - x^{n+1}}{(1-x^n)(1-x^{n+1})}$, for $x \in]-1, 1[$.
3. $\sum \ln(1 - \frac{1}{n^2})$.
4. $\sum \frac{1}{1+2+3+...+n}$.
5. $\sum \ln(1 - \frac{1}{2n})$.
6. $\sum \ln(1 - \frac{1}{2n+1})$.
7. $\sum \frac{n^2+1}{n^2}$.
8. $\sum \frac{(2n+1)^4}{(7n^2+1)^3}$.
9. $\sum n \sin\left(\frac{1}{n}\right)$.
10. $\sum \frac{n}{n^3+1}$.
11. $\sum \frac{\sqrt{n}}{n^2+\sqrt{n}}$.
12. $\sum \frac{1}{\sqrt{n}} \ln(1 + \frac{1}{\sqrt{n}})$.

Exercise 3 continued

13. $\sum \frac{(-1)^n+n}{n^2+1}$.
14. $\sum \frac{1}{n!}$.
15. $\sum(-1)^n$.
16. $\sum \frac{(-1)^n}{n}$.
17. $\sum \frac{(-1)^n}{n^2+1}$.
18. $\sum \frac{(-1)^n}{n+1}$.
19. $\sum \frac{(-1)^n}{\sqrt{n}}$.
20. $\sum(-1)^n \sin(\frac{1}{n})$.
21. $\sum \frac{(-1)^n}{\ln n}$.
22. $\sum \frac{(-1)^n}{\ln n+(-1)^n}$.
23. $\sum \sin(3^{-n})$.
24. $\sum \ln(\frac{n^2}{n^2+1})$.
25. $\sum \sqrt{n+1}-\sqrt{n}$.
26. $\sum(\sqrt{n+1}-\sqrt{n})^2$.
27. $\sum \frac{1\times4\times7\times....\times(3n-2)}{3\times6\times9\times....\times3n}$.
28. $\sum u_n^2$, where: $u_n=\frac{1\times4\times7\times....\times(3n-2)}{3\times6\times9\times....\times3n}$.
29. $\sum \frac{(2n)!}{2^{2n}(n!)^2}$.
30. $\sum \left(\frac{1}{2^n}+5\frac{1}{3^n}\right)$.
31. $\sum \frac{\sin(n\theta)}{n^\alpha}$.
32. $\sum_{n\geq1} \frac{\cos(n)}{n}$.
33. $\sum_{n\geq1}(\alpha+\frac{1}{n})^n$ where $\alpha\geq0$.
34. $\sum_{n\geq0}(n^2+1)e^{-3n}$.
35. $\sum_{n\geq2} \frac{1}{(\ln n)^{\ln n}}$.
36. $\sum(\frac{n}{n+1})^n$.
37. $\sum(\frac{n}{(n+1)})^{n^2}$.
38. $\sum \frac{n!}{n^n}$.
39. $\sum \ln(\frac{n^2}{n^2+1})$.
40. $\sum(\sqrt{n+1}-\sqrt{n})$.
41. $\sum(\sqrt{n+1}-\sqrt{n})^2$.
42. $\sum \frac{1}{\sqrt{n-1}}-\frac{2}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}$.
43. $\sum_{n\geq1}(\alpha-\frac{1}{n})^n$ where $\alpha\geq0$.
44. $\sum_{n\geq1}(ne^{\frac{1}{n}}-n)$.
45. $\sum \frac{1}{n\cos^2 n}$.

Exercise 4

1. Calculate the following limits as x approaches 0:
   - $\frac{\sin x}{x}$.
   - $\frac{ln(1+x)}{x}$.
   - $\frac{x+\sin x}{x}$.
   - $\frac{x+\sin x}{2x}$.
2. Determine the nature of the following series with their general terms u_n:
   - $u_n=\sin(\frac{1}{n})$.
   - $u_n=\ln(1+\frac{1}{n^2})$.
   - $u_n=1+n^2\sin(\frac{1}{n^2})$.
   - $u_n=\frac{1}{\sqrt{n}}+\sin(\frac{1}{\sqrt{n}})$.

Exercise 5

1. Evaluate the integral of the function $\frac{1}{x \ln x}$ over the interval $x \geq 2$.
2. Determine the convergence or divergence of the infinite series with general term u_n, where $u_n = \frac{1}{n \ln n}$ for all $n \geq 2$.

Solution to Exercise 1

1. **Expression for the general term:**
   An arithmetic sequence follows the formula:
   $$u_n = u_1 + (n-1) \times r$$
   Therefore, in this case:
   $$u_n = 5 + (n-1) \times 3$$
   $$u_n = 5 + 3n - 3 = 3n + 2$$
2. **Sum of the first 10 terms:**
   The sum of the first n terms of an arithmetic sequence is given by the formula:
   $$S_n = \frac{n}{2} \times (u_1 + u_n)$$
   For $n = 10$:
   $$u_{10} = 3 \times 10 + 2 = 32$$
   So, the sum of the first 10 terms is:
   $$S_{10} = \frac{10}{2} \times (5 + 32) = 5 \times 37 = 185.$$

Solution to Exercise 2

1. **Expression for the general term:**
   A geometric sequence follows the formula:
   $$v_n = v_3 \times q^{n-3}$$
   Therefore, in this case:
   $$v_n = 2 \times (0.5)^{n-3}$$
2. **Sum of the first 8 terms:**
   The sum of the first n terms of a geometric sequence is given by the formula:
   $$S_n = v_3 \times \frac{1 - q^{n-3+1}}{1-q} \quad \text{for} \quad q \neq 1$$
   For $n = 10$:

$$S_{10} = 2 \times \frac{1 - (0.5)^{10}}{1 - 0.5}$$
$$S_{10} = 2 \times \frac{1 - 0.0009765625}{0.5}$$
$$S_{10} = 3.99609376$$

Therefore, the sum of the first 10 terms is approximately 4.

Solution to Exercise 3

The nature of the series:

1. The first proposed series is clearly telescoping, with the partial sum given by:

$$\sum_{k=1}^{n} \left(\exp \frac{1}{k} - \exp \frac{1}{k+1} \right) = e - e^{\frac{1}{n+1}}.$$

Thus,

$$\lim_{n \to +\infty} \left(e - e^{\frac{1}{n+1}} \right) = e - 1,$$

which confirms the convergence of the series. Therefore, the sum is:

$$\sum_{n=1}^{+\infty} \left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}} \right) = e - 1.$$

2. We observe that:

$$\frac{x^n - x^{n+1}}{(1 - x^n)(1 - x^{n+1})} = \frac{1}{1 - x^n} - \frac{1}{1 - x^{n+1}}.$$

This shows that our series is telescoping, so we have:

$$\sum_{k=1}^{n} \left(\frac{1}{1 - x^k} - \frac{1}{1 - x^{k+1}} \right) = \frac{1}{1 - x} - \frac{1}{1 - x^{n+1}}.$$

Taking the limit as $n \to +\infty$:

$$\lim_{n \to +\infty} \left(\frac{1}{1 - x} - \frac{1}{1 - x^{n+1}} \right) = \frac{1}{1 - x} - 1,$$

which confirms the convergence of the series. Therefore, the sum is:

$$\sum_{n=1}^{+\infty} \frac{x^n - x^{n+1}}{(1 - x^n)(1 - x^{n+1})} = \frac{1}{1 - x} - 1.$$

3. We have:

$$\ln \left(1 - \frac{1}{n^2} \right) = \ln(n - 1) + \ln(n + 1) - 2 \ln n,$$

so

$$\sum_{k=1}^{n} (\ln(k - 1) + \ln(k + 1) - 2 \ln k) = -\ln 2 + \ln \left(\frac{n + 1}{n} \right).$$

solution to exercise 3 continued

Taking the limit as $n \to +\infty$:

$$\lim_{n\to+\infty} \left(-\ln 2 + \ln\left(\frac{n+1}{n}\right)\right) = -\ln 2,$$

which confirms the convergence of the series. Therefore, the sum is:

$$\sum_{n=1}^{+\infty} \ln\left(1 - \frac{1}{n^2}\right) = -\ln 2.$$

4. We know that:

   $$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

   so

   $$\frac{1}{1 + 2 + \cdots + n} = \frac{2}{n(n+1)} \sim \frac{2}{n^2}.$$

   Therefore,

   $$\sum \frac{1}{1+2+\cdots+n}, \quad \sum \frac{2}{n(n+1)}, \quad \sum \frac{1}{n(n+1)}, \quad \text{and} \quad \sum \frac{1}{n^2}$$

   all share the same convergence behavior. By conclusion, the series are convergent.
5. The series

   $$\sum \ln\left(\frac{1}{2n}\right) \quad \text{and} \quad \sum \ln\left(\frac{1}{n}\right)$$

   have the same convergence behavior. We know that

   $$\sum \ln\left(\frac{1}{n}\right)$$

   is a Riemann series with $a = 1 \leq 1$, and thus it diverges.
6. The series

   $$\sum \ln\left(\frac{1}{2n+1}\right) \quad \text{and} \quad \sum \ln\left(\frac{1}{2n}\right)$$

   have the same convergence behavior. From the previous solution, we know that $\sum \ln\left(\frac{1}{2n}\right)$ diverges.
   We also know that

   $$\lim_{n\to+\infty} \frac{n^2+1}{n^2} = 1 \neq 0.$$

   Thus, this is a divergent series.
8. We have:

   $$\frac{(2n+1)^4}{(7n^2+1)^3} \sim \frac{2^4}{7^3 n^2}.$$

   Since the general term behaves asymptotically like $\frac{2^4}{7^3 n^2}$, the series converges by comparison with a convergent Riemann series where $p = 2 > 1$.

solution to exercise 3 continued

9. We have
$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = 1$$
(recall that $\sin x \sim x$ as $x \to 0$). Since the general term does not tend to 0, the series diverges.
10. We have
$$u_n \sim \frac{n}{n^3} = \frac{1}{n^2}.$$
By comparison with a convergent Riemann series (where $p = 2 > 1$), the series converges.
11. The reasoning is the same: $u_n \sim \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ and by comparison to a convergent Riemann series, the series is convergent.
12. Since $\ln(1+x) \sim x$ as $x \to 0$, we have
$$\ln\left(1+\frac{1}{\sqrt{n}}\right) \sim \frac{1}{\sqrt{n}} \quad \text{as} \quad n \to +\infty.$$
Therefore,
$$\frac{1}{\sqrt{n}}\ln\left(1+\frac{1}{\sqrt{n}}\right) \sim \frac{1}{n} \quad \text{as} \quad n \to +\infty.$$
Since the series $\sum \frac{1}{n}$ diverges (a divergent Riemann series), the given series also diverges by comparison.
13. **First Method:** We have
$$(-1)^n + n \sim n \quad \text{and} \quad n^2 + 1 \sim n^2 \quad \text{as} \quad n \to +\infty.$$
Therefore,
$$\frac{(-1)^n + n}{n^2+1} \sim \frac{1}{n}.$$
By comparison with the harmonic series $\sum \frac{1}{n}$, which is divergent, we conclude that the series $\sum u_n$ is divergent.
Second Method: We express the general term as
$$\frac{(-1)^n + n}{n^2+1} = \frac{(-1)^n}{n^2+1} + \frac{n}{n^2+1}.$$
Now, we analyze each term: - $\left|\frac{(-1)^n}{n^2+1}\right| \sim \frac{1}{n^2}$ as $n \to +\infty$, - $\frac{n}{n^2+1} \sim \frac{1}{n}$ as $n \to +\infty$.
Since $\sum \frac{1}{n^2}$ is a convergent series and $\sum \frac{1}{n}$ is a divergent harmonic series, we conclude that the overall series is divergent, as the sum of a convergent and a divergent series is divergent:
$$\text{CV} + \text{DV} = \text{DV}.$$

solution to exercise 3 continued

14. We use **D'Alembert's Criterion** (the ratio test), which states that for a series $\sum u_n$, if

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = L,$$

then: - If $L<1$, the series is convergent. - If $L>1$, the series is divergent. - If $L=1$, the test is inconclusive.
Now, let's apply this criterion to the given series. We compute the limit of the ratio:

$$\lim_{n\to+\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to+\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to+\infty}\frac{n!}{(n+1)!} = \lim_{n\to+\infty}\frac{n!}{n!(n+1)} = 0.$$

15. For the series

$$\sum(-1)^n,$$

we have

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty}(-1)^n = \pm 1 \neq 0.$$

Since the general term does not tend to 0, the series is divergent by the necessary condition for convergence. Therefore, it is a **divergent** series.

16. For the series

$$\sum\frac{(-1)^n}{n},$$

we will apply **Leibniz's Rule** for alternating series. According to Leibniz's criterion, if the terms of the series alternate in sign, are decreasing in absolute value, and approach zero as $n\to\infty$, the series converges.
- The general term is $\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$, which is a decreasing sequence. - We also have $\lim_{n\to+\infty}\frac{1}{n}=0$.
Since both conditions of Leibniz's criterion are satisfied, the series

$$\sum\frac{(-1)^n}{n}$$

is **convergent**.

17. For the series

$$\sum\frac{(-1)^n}{n^2+1},$$

we have

$$\frac{1}{(n+1)^2}\sim\frac{1}{n^2}\quad\text{as}\quad n\to\infty.$$

Thus, the series

$$\sum\left|\frac{(-1)^n}{n^2+1}\right|$$

is equivalent to the series

$$\sum\frac{1}{n^2},$$

which is a **convergent series** (since $\sum\frac{1}{n^2}$ is a convergent p-series with $p=2>1$). Since the absolute series converges, the original series

$$\sum\frac{(-1)^n}{n^2+1}$$

solution to exercise 3 continued

is **absolutely convergent**, and thus the series is convergent.

18. For the series

$$\sum \frac{(-1)^n}{n+1},$$

we apply the **same procedure** as for $\sum \frac{(-1)^n}{n}$. Specifically, we check if the series satisfies the conditions of **Leibniz's Rule** for alternating series.
- The general term $\left|\frac{(-1)^n}{n+1}\right| = \frac{1}{n+1}$ is a decreasing sequence. - We also have $\lim_{n\to+\infty} \frac{1}{n+1} = 0$.
Since both conditions are satisfied, by **Leibniz's Rule**, the series

$$\sum \frac{(-1)^n}{n+1}$$

is **convergent**.

19. For the series

$$\sum \frac{(-1)^n}{\sqrt{n}},$$

we apply the **same procedure** as for $\sum \frac{(-1)^n}{n}$ using **Leibniz's Rule** for alternating series.
- The general term $\left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}}$ is a decreasing sequence because $\frac{1}{\sqrt{n}}$ decreases as n increases. - We also have $\lim_{n\to+\infty} \frac{1}{\sqrt{n}} = 0$.
Since both conditions are satisfied, by **Leibniz's Rule**, the series

$$\sum \frac{(-1)^n}{\sqrt{n}}$$

is **convergent**.

20. For the series

$$\sum (-1)^n \sin\left(\frac{1}{n}\right),$$

we know that

$$\sin\left(\frac{1}{n}\right) \sim \frac{1}{n} \quad \text{as} \quad n \to \infty.$$

Thus, the series

$$\sum (-1)^n \sin\left(\frac{1}{n}\right)$$

and

$$\sum \frac{(-1)^n}{n}$$

are of the same nature. Since we know that the series

$$\sum \frac{(-1)^n}{n}$$

is **convergent** by Leibniz's criterion for alternating series, it follows that the series

$$\sum (-1)^n \sin\left(\frac{1}{n}\right)$$

is also **convergent**.

solution to exercise 3 continued

21. For the series
$$\sum \frac{(-1)^n}{\ln n},$$
we apply the **same procedure** as for $\sum \frac{(-1)^n}{n}$, using **Leibniz's Rule** for alternating series.
- The general term $\left|\frac{(-1)^n}{\ln n}\right| = \frac{1}{\ln n}$ is a decreasing sequence because $\ln n$ increases as n increases, so $\frac{1}{\ln n}$ decreases. - We also have $\lim_{n\to+\infty} \frac{1}{\ln n} = 0$.
Since both conditions are satisfied, by **Leibniz's Rule**, the series
$$\sum \frac{(-1)^n}{\ln n}$$
is **convergent**.

22. For the series
$$\sum \frac{(-1)^n}{\ln n + (-1)^n},$$
we first rewrite the general term as:
$$\frac{(-1)^n}{\ln n + (-1)^n} = \frac{(-1)^n}{\ln n} \cdot \frac{1}{1 + \frac{(-1)^n}{\ln n}}.$$
Next, we perform a **Taylor expansion** for $\frac{1}{1+\frac{(-1)^n}{\ln n}}$. Using the expansion for small x, we have:
$$\frac{1}{1 + \frac{(-1)^n}{\ln n}} = 1 - \frac{(-1)^n}{\ln n} + \frac{1}{(\ln n)^2} + o\left(\frac{1}{(\ln n)^2}\right).$$
Thus, we get:
$$\frac{(-1)^n}{\ln n} \cdot \left(1 - \frac{(-1)^n}{\ln n} + \frac{1}{(\ln n)^2} + o\left(\frac{1}{(\ln n)^2}\right)\right) =$$
$$\frac{(-1)^n}{\ln n} - \frac{1}{(\ln n)^2} + \frac{(-1)^n}{(\ln n)^3} + o\left(\frac{1}{(\ln n)^3}\right).$$
We define:
$$u_n = \frac{(-1)^n}{\ln n}, \quad v_n = \frac{(-1)^n}{(\ln n)^3}, \quad w_n = \frac{(-1)^n}{(\ln n)^3} + o\left(\frac{1}{(\ln n)^3}\right).$$
- The series $\sum u_n$ and $\sum v_n$ are convergent, as $\sum \frac{1}{\ln n}$ is an alternating series and $\sum \frac{1}{(\ln n)^3}$ is a convergent Bertrand series. - Since $|w_n| \sim \frac{1}{(\ln n)^3}$ and the series $\sum \frac{1}{(\ln n)^3}$ is convergent, we conclude that the series $\sum w_n$ is also convergent.
Therefore, the sum of these convergent series is **convergent**.

23. For the series
$$\sum \sin(3^{-n}),$$
we know that:
$$\sin(3^{-n}) \sim_\infty \frac{1}{3^n}.$$

solution to exercise 3 continued

The general term $\frac{1}{3^n}$ corresponds to the terms of a geometric series with the ratio $q = \frac{1}{3} < 1$, which is known to converge. Therefore, the series $\sum \sin(3^{-n})$ also converges.

24. For the series

$$\sum \ln\left(\frac{n^2}{n^2+1}\right),$$

we have:

$$\ln\left(\frac{n^2}{n^2+1}\right) = -\ln\left(1+\frac{1}{n^2}\right) \sim_\infty -\frac{1}{n^2}.$$

Since the series $\sum \frac{1}{n^2}$ converges (it is a Riemann series with $p > 1$), it follows that the series $\sum \ln\left(\frac{n^2}{n^2+1}\right)$ also converges.

25. For the series

$$\sum(\sqrt{n+1}-\sqrt{n}),$$

we observe that this is a telescoping series. Specifically, we have:

$$\sum_{n=1}^{N}(\sqrt{n+1}-\sqrt{n}) = \sum_{n=1}^{N}\sqrt{n+1} - \sum_{n=1}^{N}\sqrt{n}.$$

Simplifying the sums, we obtain:

$$\sqrt{N+1}-\sqrt{1}.$$

Since $\sqrt{N+1} \to \infty$ as $N \to \infty$, the series diverges.

26. We begin with the sum:

$$\sum(\sqrt{n+1}-\sqrt{n})^2$$

First, we expand the expression:

$$(\sqrt{n+1}-\sqrt{n})^2 = \frac{1}{(\sqrt{n+1}+\sqrt{n})^2}$$

For large n, we approximate this as:

$$\frac{1}{(\sqrt{n+1}+\sqrt{n})^2} \sim \frac{1}{4(n+1)}$$

This is the general term of a divergent Riemann series. Hence, the sum

$$\sum(\sqrt{n+1}-\sqrt{n})^2$$

also diverges.

27. We are given the condition:

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = 1,$$

so we apply Raabe-Duhamel's test, which leads to the following expression:

$$\lim_{n\to+\infty} n\left(1-\frac{u_{n+1}}{u_n}\right) = \beta.$$

solution to exercise 3 continued

1. **First method:** The sequence u_n is given by:

$$u_n = \frac{1 \times 4 \times 7 \times \cdots \times (3n-2)}{3 \times 6 \times 9 \times \cdots \times 3n}.$$

We use the binomial expansion for $(1+x)^\alpha$:

$$(1+x)^\alpha = 1 + \alpha x - \frac{\alpha(\alpha-1)}{2}x^2 + \cdots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n + o(x^n).$$

Now, calculate the ratio:

$$\frac{u_{n+1}}{u_n} = \frac{3n+1}{3n+3} = \left(1 + \frac{1}{3n}\right)\left(1 + \frac{1}{n}\right)^{-1}.$$

Using the expansion for $\left(1 + \frac{1}{n}\right)^{-1}$, we get:

$$\frac{u_{n+1}}{u_n} = \left(1 + \frac{1}{3n}\right)\left(1 - \frac{1}{n} + o\left(\frac{1}{n}\right)\right) = 1 - \frac{2}{3n} + o\left(\frac{1}{n^2}\right).$$

Thus, the series diverges (DV).

2. **Second method:** Applying Raabe-Duhamel's test, we compute:

$$\lim_{n\to+\infty} n\left(1 - \frac{3n+1}{3n+3}\right) = \frac{2}{3} < 1.$$

This confirms that the series diverges as well.

28. We are given the condition:

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = 1,$$

so we apply Raabe-Duhamel's test, which gives:

$$\lim_{n\to+\infty} n\left(1 - \frac{u_{n+1}}{u_n}\right) = \beta.$$

(a) **First method:** Let $v_n = (u_n)^2$. We compute the ratio:

$$\frac{v_{n+1}}{v_n} = 1 - \frac{4}{3n} + o\left(\frac{1}{n^2}\right).$$

Since this ratio tends to 1, Raabe-Duhamel's test confirms that the series converges (CV).

(a) **Second method:** We compute:

$$\lim_{n\to+\infty} n\left(1 - \left(\frac{3n+1}{3n+3}\right)^2\right) = \frac{4}{3} \geq 1.$$

Since this limit is greater than or equal to 1, it confirms that the series converges.

solution to exercise 3 continued

29. We are given that:

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = 1,$$

so we apply Raabe-Duhamel's rule, which leads to the following expression:

$$\lim_{n\to+\infty} n\left(1 - \frac{u_{n+1}}{u_n}\right) = \beta.$$

30. The given series is:

$$\sum \left(\frac{1}{2^n} + 5\frac{1}{3^n}\right)$$

We can examine this series by breaking it down into two distinct geometric series:

$$\sum \left(\frac{1}{2^n} + 5\frac{1}{3^n}\right) = \sum \frac{1}{2^n} + 5\sum \frac{1}{3^n}.$$

Checking the convergence of each series

1. Series $\sum \frac{1}{2^n}$:

This is a geometric series with ratio $r = \frac{1}{2}$. A geometric series of the form $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$. In this case, $r = \frac{1}{2}$, which is less than 1, so the series converges.
The sum of this geometric series is given by:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2.$$

2. Series $\sum \frac{1}{3^n}$:

This is also a geometric series with ratio $r = \frac{1}{3}$. Since $\frac{1}{3} < 1$, the series also converges.
The sum of this series is:

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

Conclusion

Since both geometric series converge, the sum of these two series will also converge. In fact, the sum of convergent series is convergent.
Therefore, the given series $\sum \left(\frac{1}{2^n} + 5\frac{1}{3^n}\right)$ is **convergent**.

solution to exercise 3 continued

Remark

For real series, we use the absolute value, while for complex series, we use the modulus.

31. To study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^{\alpha}},$$

The Abel criterion states that a series of the form $\sum a_n b_n$ converges if:

(a) $a_n = \frac{1}{n^{\alpha}}$ is a decreasing sequence tending to zero.

(b) To prove that the sequence of partial sums of $b_n = \sin(n\theta)$ is bounded, consider the partial sum:

$$S_N = \sum_{n=1}^{N} \sin(n\theta).$$

We will use a technique based on trigonometric series analysis and summation formulas for sines.

$$\begin{aligned}
\sum_{n=1}^{N} e^{in\theta} = \sum_{n=1}^{N} (e^{i\theta})^n &= \frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}} \\
&= \frac{e^{\frac{i(N+1)\theta}{2}}}{e^{\frac{i\theta}{2}}} \frac{e^{\frac{-i(N+1)\theta}{2}} - e^{\frac{i(N+1)\theta}{2}}}{e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}} \\
&= e^{\frac{iN\theta}{2}} \frac{e^{\frac{-i(N+1)\theta}{2}} - e^{\frac{i(N+1)\theta}{2}}}{e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}} \\
&= e^{\frac{iN\theta}{2}} \left(\frac{-2i\sin(\frac{(N+1)\theta}{2})}{-2i\sin(\frac{\theta}{2})} \right)
\end{aligned}$$

So:

$$\begin{aligned}
|S_N| = |\sum_{n=1}^{N} Im(e^{in\theta})| &= \left| e^{\frac{iN\theta}{2}} \left(\frac{-2i\sin(\frac{(N+1)\theta}{2})}{-2i\sin(\frac{\theta}{2})} \right) \right| \\
&\leq \left| \frac{1}{\sin(\frac{\theta}{2})} \right|
\end{aligned}$$

Since $\sin\left(\frac{\theta}{2}\right) \neq 0$ (in other words, θ is not a multiple of 2π), the term $\frac{1}{\left|\sin\left(\frac{\theta}{2}\right)\right|}$ is a finite constant, and thus, the sequence of partial sums S_N is bounded.

Therefore, the sequence $\sum_{n=1}^{N} \sin(n\theta)$ is bounded for any θ that is not a multiple of 2π.

Conclusion By applying Abel's criterion, we were able to establish the convergence of our series.

solution to exercise 3 continued

33. $\sum_{n\geq 1}(\alpha+\frac{1}{n})^n$ where $\alpha \geq 0$:
We apply Cauchy's criterion, obtaining:

$$\lim_{n\to+\infty}\sqrt[n]{u_n}=\lim_{n\to+\infty}\sqrt[n]{\left(\alpha+\frac{1}{n}\right)^n}=\alpha$$

From this, we conclude:
   - If $\alpha > 1$, the series diverges.
   - If $\alpha < 1$, the series converges.
   - If $\alpha = 1$, we cannot draw a conclusion directly.

For the case $\alpha = 1$, we find:

$$\lim_{n\to+\infty}u_n=\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n=\lim_{n\to+\infty}e^{n\ln\left(1+\frac{1}{n}\right)}=\lim_{n\to+\infty}e^{\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}}=e\neq 0$$

Thus, we conclude that the series converges only when $\alpha < 1$.

34. $\sum_{n\geq 0}(n^2+1)e^{-3n}$: We will use d'Alembert's criterion, i.e.,

$$\lim_{n\to+\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to+\infty}\frac{((n+1)^2+1)e^{-3(n+1)}}{(n^2+1)e^{-3n}}=e^{-3}<1,$$

so the series converges.

35. $\sum_{n\geq 2}\frac{1}{(\ln n)^{\ln n}}$: We will use Riemann's criterion, i.e.,

$$\begin{aligned}\lim_{n\to+\infty}n^2u_n&=\lim_{n\to+\infty}n^2\frac{1}{(\ln n)^{\ln n}}=\lim_{n\to+\infty}e^{\ln n^2\frac{1}{(\ln n)^{\ln n}}}=\lim_{n\to+\infty}e^{2\ln n-\ln(\ln n)^{\ln n}}\\&=\lim_{n\to+\infty}e^{2\ln n-\ln n\ln(\ln n)}=\lim_{n\to+\infty}e^{\ln n(2-\ln(\ln n))}=0,\end{aligned}$$

so the series converges.

36. $\sum(\frac{n}{n+1})^n$:
We have:

$$\lim_{n\to+\infty}u_n=\lim_{n\to+\infty}\left(\frac{n}{n+1}\right)^n=e^{-1}\neq 0,$$

so this is a series that diverges.

37. $\sum(\frac{n}{n+1})^{n^2}$:
We observe that:

$$\sum\left(\frac{n}{n+1}\right)^{n^2}=e^{-n^2\ln\left(1+\frac{1}{n}\right)}\sim_\infty e^{-n}.$$

Thus, $\sum\left(\frac{n}{n+1}\right)^{n^2}$ and $\sum e^{-n}$ are two series of the same nature. Additionally, $\sum e^{-n}$ is a geometric series with ratio $\frac{1}{e}<1$, which is convergent. This proves the convergence of $\sum\left(\frac{n}{n+1}\right)^{n^2}$.

38. $\sum\frac{n!}{n^n}$:
We will use d'Alembert's criterion, i.e.,

$$\lim_{n\to+\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to+\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}=e^{-1}<1,$$

solution to exercise 3 continued

so the series converges.

39. $\sum \ln(\frac{n^2}{n^2+1})$:
We have:
$$\ln\left(\frac{n^2}{n^2+1}\right) = -\ln\left(1+\frac{1}{n^2}\right) \sim_\infty -\frac{1}{n^2},$$
which is the general term of a convergent series.

40. $\sum(\sqrt{n+1}-\sqrt{n})$:
It is a telescoping series, and we have:
$$\sum_{n=1}^{N}(\sqrt{n+1}-\sqrt{n}) = \sum_{n=1}^{N}\sqrt{n+1} - \sum_{n=1}^{N}\sqrt{n} = \sqrt{N+1}-1 \to +\infty.$$
Since $(\sqrt{N+1}-1)_n$ is a divergent sequence, the series is also divergent.

41. $\sum(\sqrt{n+1}-\sqrt{n})^2$:
We have:
$$(\sqrt{n+1}-\sqrt{n})^2 = \frac{1}{(\sqrt{n+1}+\sqrt{n})^2} \sim_\infty \frac{1}{4n},$$
which is the general term of a divergent Riemann series. Therefore, the series $\sum(\sqrt{n+1}-\sqrt{n})^2$ is also divergent.

42. $\sum \frac{1}{\sqrt{n-1}} - \frac{2}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}$.

44. $\sum_{n\geq 1}(ne^{\frac{1}{n}}-n)$:
We compute: $ne^{\frac{1}{n}}-n = n(e^{\frac{1}{n}}-1) = n(1+\frac{1}{n}+O(\frac{1}{n})-1) = 1+O(1) \to 1 \neq 0$, which implies that the series is convergent.

45. $\sum \frac{1}{n\cos^2 n}$: We have: $|\frac{1}{n\cos^2 n}| > \frac{1}{n}$. This is a series with positive terms, all greater than $\frac{1}{n}$, which corresponds to the general term of a divergent Riemann series with a parameter less than 1. Therefore, the series diverges.

Correction to Exercise 4

1. We have:
   a. $\lim_{x\to 0}\frac{\sin x}{x} = 1$.
   b. $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$.
   c. $\lim_{x\to 0}\frac{x+\sin x}{x} = 2$.
   d. $\lim_{x\to 0}\frac{x+\sin x}{2x} = 1$.
   (a) Based on the previous question, we have:
      i. $\sin\left(\frac{1}{n}\right) \sim_\infty \frac{1}{n}$ so $\sum\sin\left(\frac{1}{n}\right)$ and $\sum\frac{1}{n}$ are of the same nature, and since $\sum\frac{1}{n}$ is divergent, $\sum\sin\left(\frac{1}{n}\right)$ is also divergent.
      ii. $\ln\left(1+\frac{1}{n^2}\right) \sim_\infty \frac{1}{n^2}$ so $\sum\ln\left(1+\frac{1}{n^2}\right)$ and $\sum\frac{1}{n^2}$ are of the same nature, and since $\sum\frac{1}{n^2}$ is convergent, $\sum\ln\left(1+\frac{1}{n^2}\right)$ is also convergent.
      iii. $\lim_{n\to+\infty} 1+n^2\sin\frac{1}{n^2} = 2 \neq 0$ so $\sum\left(1+n^2\sin\frac{1}{n^2}\right)$ is grossly divergent.
      iv. $\left(\frac{1}{\sqrt{n}}+\sin\frac{1}{\sqrt{n}}\right) \sim_\infty \frac{1}{2\sqrt{n}}$ so $\sum\left(\frac{1}{\sqrt{n}}+\sin\frac{1}{\sqrt{n}}\right)$ and $\sum\frac{1}{2\sqrt{n}}$ are of the same nature, and since $\sum\frac{1}{2\sqrt{n}}$ is divergent, $\sum\left(\frac{1}{\sqrt{n}}+\sin\frac{1}{\sqrt{n}}\right)$ is also divergent.

is **absolutely convergent**, and thus the series is convergent.

Correction to Exercise 5

1. To evaluate the integral of $\frac{1}{x \ln x}$ for $x \geq 2$, we proceed as follows:

$$\int_{2}^{+\infty} \frac{1}{x \ln x} \, dx.$$

We first compute the integral from $x = 2$ to an arbitrary upper bound $x = t$, and then take the limit as $t \to +\infty$.

Step 1: Setting up the Integral

Consider the integral

$$\int_{2}^{t} \frac{1}{x \ln x} \, dx.$$

To simplify this, let $u = \ln x$, so that $du = \frac{1}{x} \, dx$. Substituting, we get:

$$\int_{2}^{t} \frac{1}{x \ln x} \, dx = \int_{\ln 2}^{\ln t} \frac{1}{u} \, du.$$

Step 2: Integrate with respect to u

The integral of $\frac{1}{u}$ with respect to u is $\ln |u|$, so we have:

$$\int_{\ln 2}^{\ln t} \frac{1}{u} \, du = [\ln |u|]_{\ln 2}^{\ln t} = \ln(\ln t) - \ln(\ln 2).$$

Step 3: Taking the Limit as $t \to +\infty$

To determine the behavior of the original improper integral, take the limit as $t \to +\infty$:

$$\int_2^{+\infty} \frac{1}{x \ln x}\, dx = \lim_{t \to +\infty} \left(\ln(\ln t) - \ln(\ln 2)\right).$$

As $t \to +\infty$, $\ln(\ln t) \to +\infty$. Therefore,

$$\int_2^{+\infty} \frac{1}{x \ln x}\, dx = +\infty.$$

Conclusion

Since the integral diverges, we conclude that

$$\int_2^{+\infty} \frac{1}{x \ln x}\, dx = +\infty.$$

Thus, the series diverges to infinity.

2. The nature of the series $\sum \frac{1}{n \ln n}$:
   Let $f(x) = \frac{1}{x \ln x}$, which is a continuous, decreasing, and positive function, and we have:
   $$\lim_{x \to \infty} \frac{1}{x \ln x} = 0.$$
   Thus, $\int_2^{+\infty} \frac{1}{x \ln x}\, dx$ and $\sum \frac{1}{n \ln n}$ are of the same nature. From the previous question, we know that our series diverges.