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1 Preliminaries about partial differential equations

1.1 Introduction

A partial differential equation (**PDE**) describes a relation between an unknown function and its partial derivatives. PDEs appear frequently in all areas of physics and engineering. Moreover, recently PDEs take an important attention in other areas such as biology, chemistry, computer sciences and in economics. In fact, in each area where there is an interaction between a number of independent variables, we attempt to define functions in these variables and to model a variety of processes by constructing equations for these functions. When the value of the unknown functions at a certain point depend only on what happens in the neighborhood of this point, we shall, in general, obtain a **PDE**.

Definition 1.1.1. *The general form of a* **PDE** *for a function* $u(x_1, x_2, \cdots, x_n)$ *is a relation that takes the general form*

$$F(x_1, x_2, \cdots, x_n, u, u_{x_1}, u_{x_2}, \cdots, u_{x_{11}}, \cdots) = 0, \tag{Eq1}$$

where $x_1, x_2, \cdots, x_n$ *are the* **independent variables**, u *is* **the unknown function**, u_{x_i} *is* **the partial derivative** $\frac{\partial u}{\partial x_i}$.

Remark 1.1.2. *The equation* (Eq1) *is, in general, supplemented by additional conditions such as* **initial conditions** *or* **boundary conditions**.

The analysis of PDEs has many features:

(1) The classical approach was to develop methods for finding explicit solutions, for example the **characteristic method** enables us to determine solutions for**transport equations** which are PDEs of first order, and **the separation method** helps to define solutions for example to **heat equation, wave equation, Laplace equation** whose are PDEs of second order.

(2) The technical advances were followed by theoretical progress aimed at **understanding the solution's structure**. The goal is to discover some of the solution's properties before actually computing it, and sometimes even without a complete solution.

(3) The theoretical analysis of PDEs is not merely of academic interest, but rather has many applications. It should be stressed that there exist very complex equations that cannot be solved even with the aid of supercomputers. All we can do in these cases is to attempt to obtain **qualitative information on the solution**.

(4) In addition, a deep important question relates to the formulation of the equation and its associated side conditions. In general, the equation **originates** from **a model of a physical or engineering problem**.

(5) It is not automatically obvious that the model is indeed consistent in the sense that it leads to a solvable PDE. Furthermore, it is desired in most cases that the solution will be unique, and that it will be stable under small perturbations of the data. A theoretical understanding of the equation enables us to check whether these conditions are satisfied.

As we shall see in what follows, there are many ways to solve PDEs, each way applicable to a certain class of equations. Therefore it is important to have a thorough analysis of the equation before (or during) solving it.
The French mathematician Jacques Hadamard $(1865 - 1963)$ coined the notion of **well-posedness** for a PDEs problem, that we have the following definition.

Definition 1.1.3. *A problem of PDEs is called* **well-posed** *if it satisfies all of the following criteria*

(1) *Existence: the problem has a solution.*

(2) *Uniqueness: there is no more than one solution.*

(3) *Stability: a small change in the equation or in the side conditions gives rise to a small change in the solution.*

If one or more of the conditions above does not hold, we say that the problem is **ill-posed**.

1.2 Classification

We pointed out in the introduction that PDEs are often classified into different types. In fact, there exist several such classifications. Some of them will be described here.

1.2.1 The order of an equation

The first classification is according to the order of the equation.

Definition 1.2.1. *The order is defined to be* **the order of the highest derivative** *in the equation. If the highest derivative is of order n, then the equation is said to be of order n.*

Example 1.2.2. **(1)***Let us taking the wave equation (or vibration equation)*

$$u_{tt} - c^2 u_{xx} = f(x,t),$$

with $u(x,t)$ is the unknown function with independent variables (t,x) and f is a known function. This PDE is of order 2.
(2) *The transport equation given by*

$$u_t + uu_x = 0$$

is a PDE of an unknown $u(t,x)$ of order 1.
(3) *The minimal surface equation given by*

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0$$

is a PDE of unknown $u(x,y)$ of order 2.

1.2.2 Linear equations

Another classification is into two groups: linear or nonlinear equations.

Definition 1.2.3. *An equation is called* **linear** *if in (Eq1), F is a linear function of the unknown function u and its derivatives.*

Example 1.2.4. **(1)** *The PDE of unknown $u(x,y)$ given by*

$$x^3 u_x + e^{xy} u_y - sin(x^2+y^2)u = x^3$$

is linear.
(2) *Transport equation given by*

$$u_t + uu_x = 0$$

is non linear.
(3) *Eikonal equation given by*

$$u_x^2 + u_y^2 = c^2$$

is non linear.

Remark 1.2.5. *The nonlinear equations are often further classified into sub-classes according to the type of the nonlinearity. Generally speaking, the nonlinearity is more pronounced when it appears in* **a higher derivative**. *For example, the following two PDEs are both nonlinear:*

$$u_{xx} + u_{yy} = u^3$$

$$u_{xx} + u_{yy} = (u_x^2 + u_y^2)u$$

We observe that in both of the previous equations, the terms of **high derivatives** *are linear, but, in first equation, the non linearity appears only in the* **unknown** *u,and so such equations are called* **semilinear**. *While in the second equation, the non linearity appears the terms of derivatives less than the order of the PDE, and so such equation s are called* **quasilinear**.

1.2.3 Homogeneity of equations

Definition 1.2.6. *The PDE* (Eq1) *is said to be* **homogeneous**, *if there is no term that contains only independent variables.*

Example 1.2.7. **(1)** *The heat equation given by*

$$u_t - \alpha u_{xx} = f(x,t)$$

is non homogeneous.

(2) *The Laplace equation*

$$u_{xx} + u_{yy} = 0$$

is homogeneous.

(3) *The telegraph equation*

$$u_t + u_{tt} - u_{xx} = 0$$

is homogeneous.

(4) *Eikonal equation*

$$u_x^2 + u_y^2 = c^2$$

is non homogeneous.

1.2.4 Scalar equations and systems of equations

Definition 1.2.8. *A single PDE with just one unknown function is called a scalar equation. In contrast, a set of m equations with ℓ unknown functions is called a system of m equations.*

1.3 Solving equations (solutions of PDE)

Definition 1.3.1. *A function in the set $\mathcal{C}^n$ that satisfies a PDE (**??**) of order n, will be called a* **classical (or strong) solution** *of the PDE.*

Example 1.3.2.

(1) *Let us taking the PDE*

$$u_{xx} = 0$$

for an unknown function $u(x,y)$ of independent variables (x,y). We can consider the equation as an ordinary differential equation in the variable x, with y being a parameter. To do this, let us use the change of unknown

$$v(x,y) == \partial_x u(x,y).$$

Then the PDE $u_{xx} = 0$ becomes

$$\partial_x v = 0$$

This obtained equation is an ODE in x. Thus, by integration with respect to x, we have

$$\int \partial_x v(x,y) dx = 0$$

Thus a general solution of this ODE is given by

$$v(x,y) = A(y) + B,$$

where $A(\cdot)$ is a function of the variable y and B is a constant. Then

$$v(x,y) = \partial_x u(x,y) = A(y) + B.$$

By integration in x once again, we deduce a general solution to the PDE as follows

$$u(x,y) = C(y)x + D(y),$$

where $C(\cdot)$ and $D(\cdot)$ are arbitrary function in the variable y.

1.4 Exercises

Exercise 1.

Show that each of the following equations has a solution of the form $u(x,y) = f(ax+by)$ for a proper choice of the constants a and b, and find the constants for each example:

(1) $u_x + 3u_y = 0,$

(2) $3u_x - 7u_y = 0,$

(3) $2u_x + \pi u_y = 0.$

Exercise 2.

Let $u(x,y) = \sqrt{x^2+y^2}$ be a solution to the minimal surface equation

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0$$

(1) *Prove that $h(r)$ with $r = \sqrt{x^2+y^2}$ satisfies the ordinary differential equation*

$$rh''(r) + h'(r)\big(1 + (h'(r))^2\big) = 0$$

(2) *Determine a general solution to this ordinary differential equation.*