République Algerienne Démocratique et Populaire
MINISTÉRE DE L'ENSEIGNEMENT SUPÉRIEURE
ET DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITÉ MOSTAPHA BEN BOULAID BATNA 2
FACULTÉ DES MATHÉMATIQUES ET D'INFORMATIQUE
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**Cours**

Déstiné aux étudiants de la troixième année LMD

**Module**

# Équations de la physique mathématiques

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Année Universitaire 2020-2021

# 1 Preliminaries about partial differential equations

## 1.1 Introduction

A partial differential equation (**PDE**) describes a relation between an unknown function and its partial derivatives. PDEs appear frequently in all areas of physics and engineering. Moreover, recently PDEs take an important attention in other areas such as biology, chemistry, computer sciences and in economics. In fact, in each area where there is an interaction between a number of independent variables, we attempt to define functions in these variables and to model a variety of processes by constructing equations for these functions. When the value of the unknown functions at a certain point depend only on what happens in the neighborhood of this point, we shall, in general, obtain a **PDE**.

**Definition 1.1.1.** *The general form of a* **PDE** *for a function* $u(x_1, x_2, \cdots, x_n)$ *is a relation that takes the general form*

$$F(x_1, x_2, \cdots, x_n, u, u_{x_1}, u_{x_2}, \cdots, u_{x_{11}}, \cdots) = 0, \quad \text{(Eq1)}$$

*where* $x_1, x_2, \cdots, x_n$ *are the* **independent variables***,* $u$ *is* **the unknown function***,* $u_{x_i}$ *is* **the partial derivative** $\frac{\partial u}{\partial x_i}$.

**Remark 1.1.2.** *The equation* (Eq1) *is, in general, supplemented by additional conditions such as* **initial conditions** *or* **boundary conditions**.

The analysis of PDEs has many features:

**(1)** The classical approach was to develop methods for finding explicit solutions, for example the **characteristic method** enables us to determine solutions for**transport equations** which are PDEs of first order, and **the separation method** helps to define solutions for example to **heat equation, wave equation, Laplace equation** whose are PDEs of second order.

**(2)** The technical advances were followed by theoretical progress aimed at **understanding the solution's structure**. The goal is to discover some of the solution's properties before actually computing it, and sometimes even without a complete solution.

**(3)** The theoretical analysis of PDEs is not merely of academic interest, but rather has many applications. It should be stressed that there exist very complex equations that cannot be solved even with the aid of supercomputers. All we can do in these cases is to attempt to obtain **qualitative information on the solution**.

**(4)** In addition, a deep important question relates to the formulation of the equation and its associated side conditions. In general, the equation **originates** from **a model of a physical or engineering problem**.

**(5)** It is not automatically obvious that the model is indeed consistent in the sense that it leads to a solvable PDE. Furthermore, it is desired in most cases that the solution will be unique, and that it will be stable under small perturbations of the data. A theoretical understanding of the equation enables us to check whether these conditions are satisfied.

As we shall see in what follows, there are many ways to solve PDEs, each way applicable to a certain class of equations. Therefore it is important to have a thorough analysis of the equation before (or during) solving it.
The French mathematician Jacques Hadamard $(1865-1963)$ coined the notion of **well-posedness** for a PDEs problem, that we have the following definition.

**Definition 1.1.3.** *A problem of PDEs is called* **well-posed** *if it satisfies all of the following criteria*

**(1)** *Existence: the problem has a solution.*

**(2)** *Uniqueness: there is no more than one solution.*

**(3)** *Stability: a small change in the equation or in the side conditions gives rise to a small change in the solution.*

*If one or more of the conditions above does not hold, we say that the problem is* **ill-posed**.

# 1.2 Classification

We pointed out in the introduction that PDEs are often classified into different types. In fact, there exist several such classifications. Some of them will be described here.

## 1.2.1 The order of an equation

The first classification is according to the order of the equation.

**Definition 1.2.1.** *The order is defined to be* **the order of the highest derivative** *in the equation. If the highest derivative is of order $n$, then the equation is said to be of order $n$.*

**Example 1.2.2.** **(1)***Let us taking the wave equation (or vibration equation)*

$$u_{tt} - c^2 u_{xx} = f(x,t),$$

*with $u(x,t)$ is the unknown function with independent variables $(t,x)$ and $f$ is a known function. This PDE is of order 2.*
**(2)** *The transport equation given by*

$$u_t + uu_x = 0$$

*is a PDE of an unknown $u(t,x)$ of order 1.*
**(3)** *The minimal surface equation given by*

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0$$

*is a PDE of unknown $u(x,y)$ of order 2.*

## 1.2.2 Linear equations

Another classification is into two groups: linear or nonlinear equations.

**Definition 1.2.3.** *An equation is called* **linear** *if in (Eq1), $F$ is a linear function of the unknown function $u$ and its derivatives.*

**Example 1.2.4.** **(1)** *The PDE of unknown $u(x,y)$ given by*

$$x^3 u_x + e^{xy} u_y - sin(x^2+y^2)u = x^3$$

*is linear.*
**(2)** *Transport equation given by*

$$u_t + uu_x = 0$$

*is non linear.*
**(3)** *Eikonal equation given by*

$$u_x^2 + u_y^2 = c^2$$

*is non linear.*

**Remark 1.2.5.** *The nonlinear equations are often further classified into sub-classes according to the type of the nonlinearity. Generally speaking, the nonlinearity is more pronounced when it appears in* **a higher derivative**. *For example, the following two PDEs are both nonlinear:*

$$u_{xx} + u_{yy} = u^3$$

$$u_{xx} + u_{yy} = (u_x^2 + u_y^2)u$$

*We observe that in both of the previous equations, the terms of* **high derivatives** *are linear, but, in first equation, the non linearity appears only in the* **unknown** *$u$,and so such equations are called* **semilinear**. *While in the second equation, the non linearity appears the terms of derivatives less than the order of the PDE, and so such equation s are called* **quasilinear**.

### 1.2.3 Homogeneity of equations

**Definition 1.2.6.** *The PDE* (Eq1) *is said to be* **homogeneous**, *if there is no term that contains only independent variables.*

**Example 1.2.7.** **(1)** *The heat equation given by*

$$u_t - \alpha u_{xx} = f(x,t)$$

*is non homogeneous.*

**(2)** *The Laplace equation*

$$u_{xx} + u_{yy} = 0$$

*is homogeneous.*

**(3)** *The telegraph equation*

$$u_t + u_{tt} - u_{xx} = 0$$

*is homogeneous.*

**(4)** *Eikonal equation*

$$u_x^2 + u_y^2 = c^2$$

*is non homogeneous.*

### 1.2.4 Scalar equations and systems of equations

**Definition 1.2.8.** *A single PDE with just one unknown function is called a scalar equation. In contrast, a set of $m$ equations with $\ell$ unknown functions is called a system of $m$ equations.*

## 1.3 Solving equations (solutions of PDE)

**Definition 1.3.1.** *A function in the set $\mathcal{C}^n$ that satisfies a PDE (**??**) of order $n$, will be called a* **classical (or strong) solution** *of the PDE.*

**Example 1.3.2.**

**(1)** *Let us taking the PDE*

$$u_{xx} = 0$$

*for an unknown function $u(x,y)$ of independent variables $(x,y)$. We can consider the equation as an ordinary differential equation in the variable $x$, with $y$ being a parameter. To do this, let us use the change of unknown*

$$v(x,y) == \partial_x u(x,y).$$

*Then the PDE $u_{xx} = 0$ becomes*

$$\partial_x v = 0$$

*This obtained equation is an ODE in $x$. Thus, by integration with respect to $x$, we have*

$$\int \partial_x v(x,y) dx = 0$$

*Thus a general solution of this ODE is given by*

$$v(x,y) = A(y) + B,$$

*where $A(\cdot)$ is a function of the variable $y$ and $B$ is a constant. Then*

$$v(x,y) = \partial_x u(x,y) = A(y) + B.$$

*By integration in $x$ once again, we deduce a general solution to the PDE as follows*

$$u(x, y) = C(y)x + D(y),$$

*where $C(\cdot)$ and $D(\cdot)$ are arbitrary function in the variable $y$.*

## 1.4 Exercises

### Exercise 1.

*Show that each of the following equations has a solution of the form $u(x, y) = f(ax + by)$ for a proper choice of the constants $a$ and $b$, and find the constants for each example:*

**(1)** $u_x + 3u_y = 0,$

**(2)** $3u_x - 7u_y = 0,$

**(3)** $2u_x + \pi u_y = 0.$

### Exercise 2.

*Let $u(x, y) = \sqrt{x^2 + y^2}$ be a solution to the minimal surface equation*

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0$$

**(1)** *Prove that $h(r)$ with $r = \sqrt{x^2 + y^2}$ satisfies the ordinary differential equation*

$$rh''(r) + h'(r)\big(1 + (h'(r))^2\big) = 0$$

**(2)** *Determine a general solution to this ordinary differential equation.*

# 2 First order equations and the characteristic method

## 2.1 Definitions

**Definition 2.1.1.** *A first order partial differential equation for an unknown function $u(x_1, x_2, \cdots, x_n)$ has the following general form*

$$F\big(x_1, x_2, \cdots, x_n, u, u_{x_1}, u_{x_2}, \cdots, u_{x_n}\big) = 0. \tag{Eq2}$$

*First-order equations appear in a variety of physical and engineering processes, such as the transport of material in a fluid flow and propagation of wave fronts in optics. Nevertheless they appear less frequently than second-order equations.*
*For simplicity, we will limit our study in this chapter to first order equations with* **unknown of two independent variables***. So, let $u$ be an unknown function of two independent variables $(x, y) \in \Omega$, where $\Omega$ is an open subset of $\mathbb{R}^2$. We consider the* **first order equation of two variables in the general form**

$$F\big(x, y, u, u_x, u_y\big) = 0. \tag{Eq3}$$

**Remark 2.1.2.** *We observe that the triplet $(x, y, u(x, y))$ is a surface of $\mathbb{R}^3$ whose graph is given by $u(x, y)$, then such surface satisfies the equation (Eq3). So, to solve (Eq3), the main solution method will be a direct construction of the solution surface.*

**Definition 2.1.3.** *The first order equation (Eq3) is said to be*

**(1)** **linear***, if it has the following general form*

$$a(x, y)u_x + b(x, y)u_y = c_0(x, y)u + c_1(x, y),$$

*where $a(x, y), b(x, y), c_0(x, y), c_1(x, y)$ are known functions in the variables $(x, y)$.*

**(2)** **quasi- linear***, if it has the following general form*

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u),$$

**(3) non linear**, *if* (Eq3) *is non linear.*

**Example 2.1.4.**

$$u_x = \alpha u + g(x, y)$$

*is a linear and non homogeneous first order equation with*

$$a(x, y) = 1, \quad b(x, y) = 0, \quad c_0(x, y) = \alpha, c_1(x, y) = g(x, y).$$

*We observe that in this PDE, there is only a derivative in $x$, then we can regard the variable $y$ as a* **parameter**. *Let us solving this PDE with the* **initial condition**

$$u(0, y) = y.$$

*Since we are actually dealing with an ODE, the solution (using* **Duhamel's formula**) *is given by*

$$u(x, y) = e^{\alpha x}\Big(y + \int_0^x e^{-\alpha s} b(s, y) ds\Big)$$

## 2.2 Characteristic method

**We aim to solving first-order PDE by the characteristic method. This method was developed by Hamilton who investigated the propagation of light. He work to derive the rules governing this propagation from a purely geometric theory, like to Euclidean geometry.**
**We shall first develop the method of characteristic heuristically. Later we shall present a precise theorem that guarantees that, under suitable assumptions, the equation together with its associated condition has a bf unique solution. The characteristic method is based on âknittingâ the solution surface with a one parameter family of curves that intersect a given curve in space.**

### 2.2.1 Characteristic method for linear equations

**Let us consider the general linear equation**

$$a(x, y)u_x + b(x, y)u_y = c_0(x, y)u + c_1(x, y), \tag{2.1}$$

**with initial conditions**

$$u(x_0, y_0) = u_0. \tag{2.2}$$

**Let $s \in I$ be parameter with $I$ is an interval of $\mathbb{R}$, and denote by $\Gamma$ a parametrization over the variables $(x, y)$. Thus, one can write conditions**

(2.2) by this parametrization as follows

$$\Gamma(s) = (x_0(s), y_0(s), u_0(s)), \quad s \in I \tag{2.3}$$

The curve $\Gamma$ is called the initial curve.
We observe that the PDE (2.1) can be written as a scalar product, that is to say

$$\Big(a(x,y), b(x,y), c_0(x,y)u + c_1(x,y)\Big) \cdot \Big(u_x, u_y, -1\Big) = 0. \tag{2.4}$$

Since $\Big(u_x, u_y, -1\Big)$ is normal to the surface $u$, the vector

$$\Big(a(x,y), b(x,y), c_0(x,y)u + c_1(x,y)\Big)$$

is in the tangent plane. Then, one can write

$$\frac{dx}{dt}(t) = a(x(t), y(t)) \tag{2.5}$$

$$\frac{dy}{dt}(t) = b(x(t), y(t)) \tag{2.6}$$

$$\frac{du}{dt}(t) = c_0(x(t), y(t))u(t) + c_1(x(t), y(t)) \tag{2.7}$$

This system of equations defines spatial curves lying on the solution surface (conditioned so that the curves start on the surface). This is a system of first-order ordinary differential equations. They are called the system of characteristic equations. The solutions are called characteristic curves of the equation. Notice that equations (2.1) are autonomous, that is to say there is no explicit dependence on the parameter $t$. In order to determine a characteristic curve we need an initial condition. We shall require the initial point to lie on the initial curve $\Gamma$.

Since each curve $(x(t), y(t), u(t))$ derives from a different point $\Gamma(s)$, we shall write the curves explicitly from $(x(t,s), y(t,s), u(t,s))$. Then, the initial conditions are writing as follows

$$x(0,s) = x_0(s), \quad y(0,s) = y_0(s), \quad u(0,s) = u_0(s). \tag{2.8}$$

**Remark 2.2.1.** *Above, we have selected the parameter $t$ such that the characteristic curve is located on $\Gamma$*

*when $t = 0$. Thus, we can select any other parameterization. We also notice that, in general, the parameterization*

$$(x(t,s), y(t,s), u(t,s))$$

*represents a surface in* $\mathbb{R}^3$*.*

**Example 2.2.2.** *Let us taking the first order linear equation*

$$u_x + u_y = 2$$

*under the initial conditions* $u(x,0) = x^2$*.*
*The characteristic equations and the parametric initial conditions are*

$$\frac{d}{dt}x(t,s) = 1 \quad \frac{d}{dt}y(t,s) = 1, \quad \frac{d}{dt}u(t,s) = 2,$$
$$x(0,s) = s, \quad y(0,s) = 0, \quad u(0,s) = s^2.$$

*To solve each of the previous EDO, we integrate with respect to* $t$*. We get*

$$x(t,s) = t + f_1(s), \quad y(t,s) = t + f_2(s), \quad u(t,s) = 2t + f_3(s).$$

*Using the initial conditions, one get*

$$x(t,s) = t + s, \quad y(t,s) = t, \quad u(t,s) = 2t + s^2.$$

*We have thus obtained a parametric representation of the integral surface. To find an explicit representation of the surface* $u$ *as a function of variables* $(x,y)$*, we need to invert the transformation* $(x(t,s), y(t,s))$*, and to express it in the form*

$$t(x,y), s(t,y))$$

*That is to say, we will give* $(t,s)$ *as functions of* $(x,y)$*. Here, the inverse of the last equations is given by*

$$t = y, \quad s = x - y.$$

*Hence, the explicit solution is given by*

$$u(x,y) = 2y + (x-y)^2.$$

**Remark 2.2.3.** *Let us recall that the* **Jacobian** *associated to the characteristic equations at points located on the initial curve* $\Gamma$ *is given by*

$$J = \begin{vmatrix} a & b \\ (x_0)_s & (y_0)_s \end{vmatrix} = a(y_0)_s - b(x_0)_s, \tag{2.9}$$

*where* $(x_0)_s = \frac{dx_0}{ds}$ *and* $(y_0)_s = \frac{dy_0}{ds}$*. Thus the Jacobian vanishes at some point if and only if the vectors* $(a,b)$ *and* $((x_0)_s, (y_0)_s)$ *are linearly dependent. Hence* **the geometrical meaning of a vanishing Jacobian** *is that* **the projection of** $\Gamma$

**on the** $(x,y)-$ **plane is tangent at this point to the projection of the characteristic curve on that plane***. Thus, a first order equation admits a unique* *solution near the initial curve, if*

$$J \neq 0. \tag{2.10}$$

*This condition is called* **the transversality condition**.

**Example 2.2.4.** *We aim to solve the linear first order equations*

$$u_x + u_y + u = 1,$$

*with initial conditions*

$$u(x,y) = \sin x, \quad \text{on} \quad y = x + x^2, \quad x > 0.$$

*The characteristic equations and the associated initial conditions are given by*

$$\frac{d}{dt}x(t,s) = 1, \quad \frac{d}{dt}y(t,s) = 1, \quad \frac{d}{dt}u(t,s) + u(t,s) = 1$$

$$x(0,s) = s, \quad y(0,s) = s + s^2, \quad u(0,s) = \sin s.$$

*The associated Jacobian with respect to initial curve is reads as follows*

$$J = \begin{vmatrix} 1 & 1 \\ 1 & 1+2s \end{vmatrix} = 2s.$$

*Hence, the problem admits a unique solution if $J \neq 0$, that is to say $s \neq 0$. Since we are limited with the case $x > 0$, then $s \neq 0$, so the uniqueness of solution. Furthermore, the solutions of the characteristic equations under the initial conditions are*

$$\big(x(t,s), y(t,s), u(t,s)\big) = \big(s + t, s + t^2, 1 - (1 - \sin s)e^{-t}\big).$$

*To obtain $s$ and $t$ in terms of $x$ and $y$, we substitute the equation of $x$ into the equation of $y$, then we get*

$$s = \sqrt{y - x}.$$

*The sign of the square root is selected according to the condition $x > 0$. Hence*

$$t = x - \sqrt{y - x}.$$

*Thus, an explicit solution is*

$$u(x,y) = 1 - \Big(1 - \sin\sqrt{y-x}\Big)e^{-x-\sqrt{y-x}},$$

*and such solution exists on the domain*

$$\Omega = \{(x,y) : 0 < x < y\} \cup \{(x,y) : x \leq 0, x + x^2 < y\}$$

### 2.2.2 Characteristic method for quasi-linear equations

One can readily verify that the method of characteristics applies to the quasi-linear equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \tag{2.11}$$

as well. Namely, each point on the initial curve

is a starting point for a characteristic curve. The characteristic equations in this case are reads as follows

$$\frac{dx}{dt}(t) = a(x(t), y(t), u(t)) \tag{2.12}$$

$$\frac{dy}{dt}(t) = b(x(t), y(t), u(t)) \tag{2.13}$$

$$\frac{du}{dt}(t) = c(x(t), y(t), u(t)) \tag{2.14}$$

with the initial conditions

$$x(0, s) = x_0(s), \quad y(0, s) = y_0(s), \quad u(0, s) = u_0(s). \tag{2.15}$$

**Remark 2.2.5.** **(1)** *The difference between the characteristic method equations (2.5) and (2.12) is that in the former case the first two equations of (2.5) are independent of the third equation and of the initial conditions.*

**(2)** *We can observe the special role played by the* **projection of the characteristic curves** *on the $(x, y)-$plane. Hence, in the linear case, the equation for this* **projection** *is given by*

$$\frac{dx}{dt}(t) = a(x(t), y(t)), \frac{dy}{dt}(t) = b(x(t), y(t)). \tag{2.16}$$

*In the quasi-linear case, this uncoupling of the characteristic equations is no longer possible, since the coefficients $a$ and $b$ depend on $u$. We also point out that in the linear case, the equation for $u$ is always linear, and thus it is guaranteed to have a global solution (provided that the solutions $x(t)$ and $y(t)$ exist globally).*

**(3)** *We recall also that in the* **linear case***, the equation for $u$ is* **always linear***, and thus it is* **guaranteed** *to have a* **global solution** *(that is to say the solutions $x(t), y(t)$ exist globally with respect to $t$).*

**Example 2.2.6.** *We aim to solve the quasi-linear first order equation*

$$(y + u)u_x + yu_y = x - y,$$

*under the initial condition*

$$u(x,1) = 1 + x.$$

*The characteristic equations and the parametric initial conditions are given by*

$$\frac{d}{dt}x(t,s) = y(t,s) + u(t,s), \quad \frac{d}{dt}y(t,s) = y(t,s), \quad \frac{d}{dt}u(t,s) = x(t,s) - y(t,s),$$

$$x(0,s) = s, \quad y(0,s) = 1, \quad u(0,s) = 1 + s.$$

*Firstly, we observe that the solution of the second characteristic equation* $y_t = y$ *is*

$$y(t,s) = e^t$$

*Adding the first equation and the third one gives*

$$(x+u)_t = x + u$$

*Thus, we get*

$$x(t,s) + u(t,s) = (1+2s)e^t.$$

*Then by using the first equation, we get*

$$x(t,s) = (1+s)e^t - e^{-t}.$$

*and*

$$u(t,s) = se^t + e{-}t.$$

*Also, we observe that*

$$x(t,s) - y(t,s) = se^t - e{-}t.$$

*Finally, we get the solution*

$$u(x,y) = \frac{2}{y} + (x-y).$$

## Exercises

### Exercise 1.

*Use the characteristic method to solve the following first order equations with initial conditions*

- $xu_x + yu_y = 2u, \quad u(x,1) = e^x,$
- $uu_x + u_y = 1, \quad u(x,y) = \frac{1}{2}x$
- $-yu_x + xu_y = u, \quad x > 0, y > 0 \quad \text{and} \quad u(x,0) = f(x).$
- $u_x + u_y = u^2, \quad u(x,0) = 1.$
- $(y + 2ux)u_x - (x + 2uy)u_y = \frac{1}{2}(x^2 - y^2), \quad u(x,y) = 0, \quad \text{on} \quad x - y = 0$

**Exercise 2.**

*Let us taking the first order PDE*

$$(x+y^2)u_x + yu_y + (\frac{x}{y} - y)u = 1, \quad u(x,1) = 0, x \in \mathbb{R}.$$

**(1)** *Find a general solution to this problem.*

**(2)** *Study the condition of transversality of the solution and deduce about the uniqueness.*

**(3)** *Draw the projections on the* $(x,y)-$*plane of the initial conditions.*

**(4)** *Is the solution obtained defined at the origin* $(0,0)$.

### 2.2.3 Characteristic method for non linear equations