Corrigé-type TD N° 5

Variables aléatoires

EX01 : X : N^{bre} de Kg de tommates récoltés

$X = x_i$	0	1	2	3	
$P(X \leq x)$	0,1	0,6	0,9	1	
$P(X = x_i) = P_i$	0,1	0,5	0,3	0,1	$\sum P_i = 1$

$E(X) = \sum x_i P_i = 1,4$

$V(X) = E(X^2) - [E(X)]^2$ tel que : $E(X^2) = \sum x_i^2 P_i = 2,6$

On obtient : $V(X) = 0,64$.

EX2 : X : N^{bre} de doses d'herbicide.

N^{bre} de cas possibles $|\Omega| = C_{20}^{6} = 38760$

$X = x_i$	0	1	2	3	4	5	6
P_i	$C_{12}^{0} C_8^6 / C_{20}^6$	$C_{12}^{1} C_8^5 / C_{20}^6$	$C_{12}^{2} C_8^4 / C_{20}^6$	$C_{12}^{3} C_8^3 / C_{20}^6$	$C_{12}^{4} C_8^2 / C_{20}^6$	$C_{12}^{5} C_8^1 / C_{20}^6$	$C_{12}^{6} C_8^0 / C_{20}^6$

$E(X) = 3,6$

$V(X) = 1,06$

EX03 : X : dose ingérée par l'animal.

$\Omega = \{ \{0, 100_1\} ; \{0, 100_2\}, \{0, 200_1\} \{0, 200_2\} \{100, 100\}$
$\{0, 300_1\} \{0, 300_2\} \{100_1, 200_1\} \{100_1, 200_2\} \{100_2, 200_1\} \{100_2, 200_2\}.$
$\{200, 200\} \{100_1, 300_1\} \{100_1, 300_2\}. \{100_2, 300_1\}. \{100_2, 300_2\}$
$\{200_1, 300_1\} \{200_2, 300_2\} \{200_2, 300_1\}. \{200_2, 300_2\}$
$\{300, 300\} \}$

$|\Omega| = C_7^2 = 21$

x_i	100	200	300	400	500	600
P_i	$\frac{C_1^1 C_2^1}{C_7^2}$	$\frac{C_1^1 C_2^1 + C_2^2}{C_7^2}$	$\frac{C_1^1 C_2^1 + C_2^1 C_2^1}{C_7^2}$	$\frac{C_2^2 + C_2^1 C_2^1}{C_7^2}$	$\frac{C_2^1 C_2^1}{C_7^2}$	$\frac{C_2^2}{C_7^2}$
	$2/21$	$3/21$	$6/21$	$5/21$	$4/21$	$1/21$

$$\sum P_i = 1.$$

$E(X) = 342{,}86$ $\quad V(X) = 17687{,}08$

$$F_X(x) = \begin{cases} 0 & x < 100 \\ 2/21 & 100 \leq x < 200 \\ 5/21 & 200 \leq x < 300 \\ 11/21 & 300 \leq x < 400 \\ 16/21 & 400 \leq x < 500 \\ 20/21 & 500 \leq x < 600 \\ 21/21 = 1 & x \geq 600 \end{cases}$$

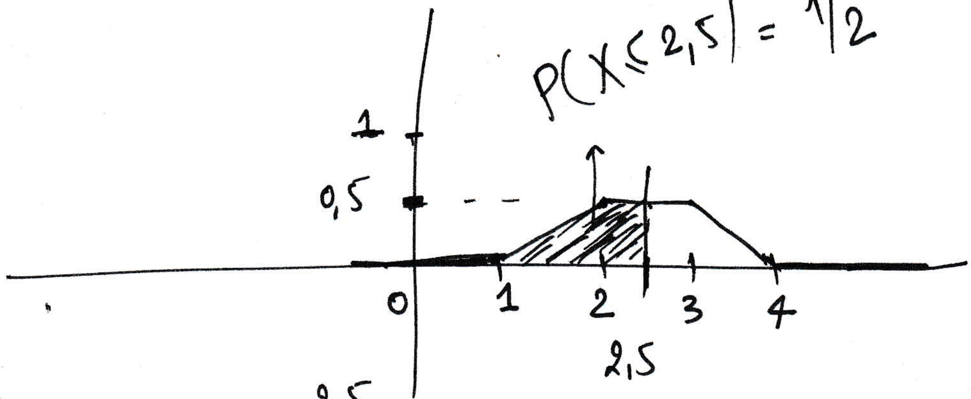
Fonction de répartitions (en escalier).

EX 04 :

$$\begin{cases} f(x) \geq 0 \\ \int_{-\infty}^{+\infty} f(x)\,dx = 1 \end{cases}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \cancel{\int_{-\infty}^{1} 0\,dx}^{\,0} + \int_1^2 0{,}5(x-1)\,dx + \int_2^3 0{,}5\,dx + \int_3^4 2 - \frac{x}{2}\,dx + \cancel{\int_4^{+\infty} 0\,dx}^{\,0}$$

$$= 0{,}5\left(\frac{x^2}{2} - x\right|_1^2 + \left(0{,}5\,x\right|_2^3 + \left(2x - \frac{x^2}{4}\right|_2^3 = 1.$$



② $P(X \leq 2{,}5) = \int_{-\infty}^{2{,}5} f(x)\,dx$

$= \cancel{\int_{-\infty}^{1} 0\,dx}^{\,0} + \int_{1}^{2} 0{,}5(x-1)\,dx + \int_{2}^{2{,}5} 0{,}5\,dx$

$= 0{,}5\left(\frac{x^2}{2} - x\right)\Big|_1^2 + 0{,}5x\Big|_2^{2{,}5} = 1/2$

Oui, ce résultat est prévisible car l'aire totale est de 1 ; 2.5 point de symétrie

donc $P(X \leq 2{,}5) = 1/2$

③ $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

i) si $x < 1$ $\quad F_X(x) = 0$

ii) si $1 \leq x < 2$ $\quad F_X(x) = \cancel{\int_{-\infty}^{1} 0\,dt}^{\,0} + \int_{1}^{x} 0{,}5(t-1)\,dt$

$= 0{,}5\left(\frac{t^2}{2} - t\right)\Big|_1^x = \frac{x^2}{4} - \frac{x}{2} + \frac{1}{4}$

iii) $2 \leq x < 3$ $\quad F_X(x) = \cancel{\int_{-\infty}^{1} 0\,dt} + \int_{1}^{2} 0{,}5\left(\frac{t^2}{2} - t\right)dt + \int_{2}^{x} 0{,}5\,dt$

$= \frac{1}{4} + (0{,}5t\Big|_2^x = \frac{x}{2} - \frac{3}{4}$

iii) $3 \leq x < 4$,

$$F_X(x) = \cancel{\int_{-\infty}^{1} 0\,dt} + 0{,}5\int_1^2 \left(\frac{t^2}{2} - t\right) dt + \int_2^3 0{,}5\,dt + \int_3^x \left(2 - \frac{t}{2}\right) dt$$

$$= \frac{1}{4} + \frac{1}{2} + \left(2t - \frac{t^2}{4}\right)\Big|_3^x = \frac{3}{4} + 2x - \frac{x^2}{4} - 6 + \frac{9}{4}$$

$$= -\frac{x^2}{4} + 2x - 3$$

iv) $x \geq 4$ $\quad F_X(x) = 1$

Finalement, on obtient :

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{x^2}{4} - \frac{x}{2} + \frac{1}{4} & 1 \leq x < 2 \\ \frac{x}{2} - \frac{3}{4} & 2 \leq x < 3 \\ -\frac{x^2}{4} + 2x - 3 & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

4) $E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx$

$$= \cancel{\int_{-\infty}^{1} 0\,dx} + \int_1^2 0{,}5\,x(x-1)\,dx + \int_2^3 0{,}5\,x\,dx + \int_3^4 2x - \frac{x^2}{2}\,dx + \cancel{\int_4^{+\infty} 0\,dx}$$

$$= 0{,}5\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_1^2 + 0{,}5 \cdot \frac{x^2}{2}\Big|_2^3 + \left(\frac{2x^2}{2} - \frac{x^3}{6}\right)\Big|_3^4 = \frac{5}{2} = 2{,}5$$

$$V(X) = E(X^2) - [E(X)]^2.$$

$$E(X^2) = \int_{-\infty}^{1} 0\,dx + \int_1^2 0{,}5\,x^2(x-1)\,dx + \int_2^3 0{,}5\,x^2\,dx + \int_3^4 2x^2 - \frac{x^3}{2}\,dx + \int_4^{+\infty} 0\,dx$$

$$= \frac{160}{24}$$

Donc : $V(X) = 5/12$.

Soit $Y = X - 2{,}5$

$E(Y) = E(X - 2{,}5) = E(X) - 2{,}5 = 2{,}5 - 2{,}5 = 0$

$V(Y) = V(X - 2{,}5) = V(X) = 5/12$.

EXO5 :

① $\int_{-\infty}^{+\infty} f(x)\,dx = 1 = \int_2^4 k(4-x)(x-2)\,dx$

$= k\int_2^4 (6x - x^2 - 8)\,dx = k.\left[6.\frac{x^2}{2} - \frac{x^3}{3} - 8x\right|_2^4 = 1$

$\Rightarrow k = 3/4$.

② $F_X(x) = \int_{-\infty}^{x} f(t)\,dt$.

i) $t < 2$

$F(x) = 0$

ii) $2 \leq t < 4$

$F_X(t) = \cancel{\int_{-\infty}^{2} 0\,dt} + \int_2^x \frac{3}{4}(6t - t^2 - 8)\,dt = \frac{3}{4}\left(3t^2 - \frac{t^3}{3} - 8t\right)\Big|_2^x$

$= \frac{9}{4}x^2 - \frac{x^3}{4} - 6x + 5$

iii) $t \geq 4$ $F_X(t) = 1$

Alors

$$F_X(x) = \begin{cases} 0 & x < 2 \\ \frac{9}{4}x^2 - \frac{1}{4}x^3 - 6x + 5 & 2 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

$E(X) = 3$ $\quad V(X) = 0,2.$

(4) $P(2,2 \leq X \leq 3,8) = F_X(3,8) - F_X(2,2)$

$= 0,944.$

Donc : La proportion des pièces inutilisables est de : $(1 - 0,944 = 0,056)$

donc $\underline{5,6\,\%}$

<u>EX06</u> : (1) $\int_0^4 c(4x - x^2)\,dx = 1 \Rightarrow C = 3/32.$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{32}(6x^2 - x^3) & 0 \leq x < 4 \\ 0 & \geq 4 \end{cases}$$

$P(1,5 \leq X \leq 3,95) = F(3,95) - F(1,5)$

$= 0,683.$

<u>EX07</u> : $\lim\limits_{x \to -\infty} F(x) = 0 = \lim\limits_{x \to -\infty} a = a$

$\lim\limits_{x \to +\infty} F(x) = 1 = \lim\limits_{x \to +\infty} d = d$

F est <u>continue</u> $\lim\limits_{x \to -1} F(x) =$ $\quad : -b + c =$

$= \lim$ $\quad F(-1) = 0$

$-b + c = 1$ —— (*)

$$\lim_{x \to +1} F(x) = \qquad = b + c = F(1) = 1$$

$$b + c = 1 \quad \text{——} (**)$$

Alors : $b = c = 1/2$

$a = 0 \qquad d = 1$

$$F(x) = \begin{cases} 0 & x \leq -1 \\ 1/2\, x + 1/2 & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$f(x) = F'(x) = \begin{cases} 1/2 & -1 \leq x < 1 \\ 0 & \text{Ailleurs} \end{cases}$$