University of Batna 2 \Department of Mathematics \2024-2025

Exercise Sheet 1 : Metric space

Exercise 1. *Are the following functions metrics on X ?*

(1) $d(x, y) = |x^3 - y^3|$, $X = \mathbb{R}$.

(2) $d(x, y) = |x^2 - y^2|$, $X = \mathbb{R}$.

(3) $d(x, y) = e^{x-y}$, $X = \mathbb{R}$.

(4) $d(x, y) = |x - 3y|$, $X = \mathbb{R}$.

Exercise 2. *(the discrete metric). Let X be a non-empty set. Define a function on $X \times X$ by*

$$d(x, y) = \begin{cases} 1, & if \quad x \neq y \\ 0, & if \quad x = y. \end{cases}$$

(1) Show that d is a metric on X.

(2) Let $x \in X$ and $r > 0$. Find the open ball $B(x, r)$ and the closed ball $\overline{B(x, r)}$.

(a) *Find the sphere $S(x, r)$.*

(b) *Deduce that every subset in a discrete metric is bounded.*

(c) *Show that every subset in a discrete metric space is open.*

(d) *Deduce that every subset in a discrete metric space is closed.*

(e) *Show that the discrete metric on $\mathbb{R}$ is not equivalent to the usual metric on $\mathbb{R}$.*

Exercise 3. *(1) Show that the closed ball $\overline{B}(x, r)$ and the sphere $S(x, r)$ are colsed in the metric space (X, d).*

(2) Show that $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 2y\}$ is open in Euclidean metric $\mathbb{R}^2$.

(3) Show that $B = \{x \in \mathbb{R} : x^3 + 2x^2 - 3x < 0\}$ is closed in usual metric $\mathbb{R}$.

Exercise 4. *Let (x, d) be a metric space. Show that both*

(a) $\delta(x, y) = \dfrac{d(x, y)}{d(x, y) + 1}$

(b) $\rho(x, y) = min\{1, d(x, y)\}$.

define metrics on X.

Exercise 5. *Let $X = C([a, b])$ be the space of real-valued continuous functions on the interval $[a, b]$,$(a < b)$.*

(1) Show that

$$d_1(f, g) = \int_a^b |f(x) - g(x)|dx$$

$$d_2(f, g) = (\int_a^b (f(x) - g(x))^2 dx)^{\frac{1}{2}}$$

$$d_\infty(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

are metrics on X.

(2) *Compute the distances* $d_1(x, x^3)$ *and* $d_\infty(x, x^3)$ *in the space* $C([0, 2])$.

(3) *Show that the two metrics* d_1 *and* d_∞ *are not equivalent.*

Exercise 6. *Let* (X, d) *a metric space and* A *be a non-empty subset of* X.

(1) *Show that*

$$\forall x, y \in X : |d(x, A) - d(y, A)| \leq d(x, y).$$

(2) *Deduce that the function* $f(x) = d(x, A)$ *is uniformly continuous on* X.